Generation of J-Grassmannians of buildings of type A_n and D_n with J a non-connected set of types

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Abstract

Let $X_n(\mathbb{K})$ be a building of Coxeter type $X_n = A_n$ or $X_n = D_n$ defined over a given division ring $\mathbb{K}$ (a field when $X_n = D_n$). For a non-connected set J of nodes of the diagram X_n, let $\Gamma(\mathbb{K}) = \mathrm{Gr}_J(X_n(\mathbb{K}))$ be the J-Grassmannian of $X_n(\mathbb{K})$. We prove that $\Gamma(\mathbb{K})$ cannot be generated over any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$. As a consequence, the generating rank of $\Gamma(\mathbb{K})$ is infinite when $\mathbb{K}$ is not finitely generated. In particular, if $\mathbb{K}$ is the algebraic closure of a finite field of prime order then the generating rank of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ is infinite, although its embedding rank is either $(n+1)^2 - 1$ or $(n+1)^2$.

1 Introduction

We presume that the reader has some acquaintance of buildings and is familiar with basics of point-line geometry, such as the notions of subspace and generation. In case, we refer to Tits [14] for buildings and Shult [13] for point-line geometries.

1.1 Basic definitions and known results

Let $\mathfrak{X}$ be a class of buildings such that, for every division ring $\mathbb{K}$, at most one (up to isomorphisms) member of $\mathfrak{X}$ is defined over $\mathbb{K}$. For instance, $\mathfrak{X}$ can be the class of buildings belonging to a given Coxeter diagram with no multiple strokes or a given Dynkin diagram, possibly of twisted type. With $\mathfrak{X}$ as above, let $\Delta(\mathbb{K})$ be the member of $\mathfrak{X}$ defined over $\mathbb{K}$ (provided it exists) and, for a nonempty subset J of the type-set of $\Delta(\mathbb{K})$, let $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ be the J-Grassmannian of $\Delta(\mathbb{K})$, regarded as a point-line geometry. For a sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$, suppose that $\mathfrak{X}$ also contains a member $\Delta(\mathbb{K}_0)$ defined over $\mathbb{K}_0$ and $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ contains $\mathrm{Gr}_J(\Delta(\mathbb{K}_0))$ as a subgeometry (as it is always the case for the geometries to be considered in this paper). We say that $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ is *generated over* $\mathbb{K}_0$ ($\mathbb{K}_0$-*generated* for short) if $\mathrm{Gr}_J(\Delta(\mathbb{K}_0))$ generates $\mathrm{Gr}_J(\Delta(\mathbb{K}))$.

Clearly, if $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ is $\mathbb{K}_0$-generated and $\mathrm{Gr}_J(\Delta(\mathbb{K}_0))$ is $\mathbb{K}_1$-generated for a division ring $\mathbb{K}_1 < \mathbb{K}_0$, then $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ is $\mathbb{K}_1$-generated too. It is also clear that if $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ is $\mathbb{K}_0$-generated then the generating rank of $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ cannot be larger than that of $\mathrm{Gr}_J(\Delta(\mathbb{K}_0))$. On the other hand, suppose that every finite set of points of $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ belongs to a subgeometry of $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ isomorphic to $\mathrm{Gr}_J(\Delta(\mathbb{K}_0))$ for a finitely generated sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (as it is often the case). Suppose moreover that $\mathbb{K}$ is not finitely generated and $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ is not $\mathbb{K}_0$-generated, for any $\mathbb{K}_0 < \mathbb{K}$. Then $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ has infinite generating rank, as we prove in Lemma 1.4. In short, obvious links exist between the $\mathbb{K}_0$-generation problem and the computation of generating ranks. Less obviously, some relations also seem to exist between $\mathbb{K}_0$-generability and the existence of the absolutely universal embedding. For instance, a number of Grassmannians $\mathrm{Gr}_J(\Delta(\mathbb{K}))$ for which the existence of the absolutely universal embedding is still an open problem, cannot be generated over any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (see Section 1.3, Remark 1.6).

We shall now briefly survey what is currently known on $\mathbb{K}_0$-generation. For X_n a Coxeter diagram of rank n with no multiple strokes or a Dynkin diagram of rank n (but not of twisted type) and a division ring $\mathbb{K}$ (a field if $X_n \neq A_n$), let $X_n(\mathbb{K})$ be the unique building of type X_n defined over $\mathbb{K}$. In particular, $B_n(\mathbb{K})$ and $C_n(\mathbb{K})$ are the buildings associated to the orthogonal group $\mathrm{O}(2n+1,\mathbb{K})$ and the symplectic group $\mathrm{Sp}(2n,\mathbb{K})$ respectively.

Suppose firstly that $\mathrm{Gr}_J(X_n(\mathbb{K}))$ is spanned by $\mathrm{Gr}_J(A)$ for an apartment A of $X_n(\mathbb{K})$ (for short, $\mathrm{Gr}_J(X_n(\mathbb{K}))$ *is spanned by an apartment*). For every sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$, the geometry $\mathrm{Gr}_J(A)$ is contained in a subgeometry of $\mathrm{Gr}_J(X_n(\mathbb{K}))$ isomorphic to $\mathrm{Gr}_J(X_n(\mathbb{K}_0))$. Hence $\mathrm{Gr}_J(X_n(\mathbb{K}))$ is $\mathbb{K}_0$-generated for any $\mathbb{K}_0 \leq \mathbb{K}$. In particular, $\mathrm{Gr}_J(X_n(\mathbb{K}))$ is generated over the prime subfield of $\mathbb{K}$.

It is known (Cooperstein and Shult [8], Blok and Brouwer [1]) that the following grassmannians are generated by apartments, where we take the integers $1, 2, ..., n$ as types as usual but when $X_n = D_n$, according to the notation adopted in Section 1.2, we replace $n-1$ and n with $+$ and $-$: $\mathrm{Gr}_k(A_n(\mathbb{K}))$ for $1 \leq k \leq n$; $\mathrm{Gr}_1(D_n(\mathbb{K}))$ and $\mathrm{Gr}_+(D_n(\mathbb{K}))$ as well as $\mathrm{Gr}_-(D_n(\mathbb{K}))$; $\mathrm{Gr}_1(C_n(\mathbb{K}))$ and $\mathrm{Gr}_n(B_n(\mathbb{K}))$ but with $\mathrm{char}(\mathbb{K}) \neq 2$ in both cases; $\mathrm{Gr}_1(E_6(\mathbb{K}))$, $\mathrm{Gr}_6(E_6(\mathbb{K})))$ and $\mathrm{Gr}_1(E_7(\mathbb{K}))$ (the nodes of the E_7-diagram being labelled as in [8]). Therefore, all above mentioned Grassmannians are generated over the prime subfield of $\mathbb{K}$. It is easily seen that the same holds for $\mathrm{Gr}_1(B_n(\mathbb{K}))$, even if this geometry is not spanned by any apartment. It is likely that if $\mathrm{char}(\mathbb{K}) \neq 2$ then, for every $i \leq n$, the i-Grassmannian $\mathrm{Gr}_i(C_n(\mathbb{K}))$ is generated over the prime subfield of $\mathbb{K}$, but we are not aware of any explicit proof of this claim.

We now turn to $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$. This geometry is interesting in its own. When $\mathbb{K}$ is a field it is known as the *long root geometry* for $\mathrm{SL}(n+1,\mathbb{K})$. In [2] it is proved that if $n > 2$ then $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ is not $\mathbb{K}_0$-generated, for any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (see also [6, Theorem 5.10] for an alternative proof in the special case where $n = 3$ and $\mathbb{K}$ is a field). However, when $\mathbb{K}$ is a field and is generated by $\mathbb{K}_0 \cup \{a_1, ..., a_t\}$ for suitable elements $a_1, ..., a_t \in \mathbb{K} \setminus \mathbb{K}_0$, then $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ can be generated by adding at most t elements to $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}_0))$ (Blok and Pasini [2]). In particular, when $\mathbb{K}$ is finite, $(n+1)^2$ points are enough to generate $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}_0))$. Indeed in this case $\mathbb{K}$ is a simple extension of its prime subfield $\mathbb{K}_0$ and the generating rank of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}_0))$ is equal to $(n+1)^2 - 1$ (Cooperstein [7]).

Not so much is known on $\mathrm{Gr}_k(B_n(\mathbb{K}))$ for $1 < k < n$ and $\mathrm{Gr}_k(D_n(\mathbb{K}))$ for $1 < k \leq n-2$. Probably, what makes these cases so difficult is the fact that the special case $\mathrm{Gr}_{1,3}(A_3(\mathbb{K})) \cong \mathrm{Gr}_{+,-}(D_3(\mathbb{K}))$ of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ somehow enters the game in any attempt to compute the generating rank of $\mathrm{Gr}_k(B_n(\mathbb{K}))$ or $\mathrm{Gr}_k(D_n(\mathbb{K}))$ and, as we have seen above, as far as generation is concerned, $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ can behave wildly. Nevertheless, in [6] we have shown that for $\mathbb{K} = \mathbb{F}_4, \mathbb{F}_8$ or $\mathbb{F}_9$ the Grassmannians $\mathrm{Gr}_2(B_n(\mathbb{K}))$ ($n \geq 3$) and $\mathrm{Gr}_2(D_n(\mathbb{K}))$ ($n > 3$) are generated over the corresponding prime subfields $\mathbb{F}_2$ or $\mathbb{F}_3$. The generating ranks of $\mathrm{Gr}_2(B_n(\mathbb{K}_0))$ and $\mathrm{Gr}_2(D_n(\mathbb{K}_0))$, for $\mathbb{K}_0$ a finite field of prime order, are known to be equal to $\binom{2n+1}{2}$ and $\binom{2n}{2}$ respectively (Cooperstein [7]). Hence $\binom{2n+1}{2}$ and $\binom{2n}{2}$ are the generating ranks of $\mathrm{Gr}_2(B_n(\mathbb{K}))$ and $\mathrm{Gr}_2(D_n(\mathbb{K}))$ respectively, with $\mathbb{K}$ as above.

1.2 Setting and main results

We refer to [11, Chapter 5] for the definition of the J-Grassmannian $\mathrm{Gr}_J(\Delta)$ of a geometry Δ. We recall that when Δ satisfies the so-called Intersection Property (which is always the case when Δ is a building) then $\mathrm{Gr}_J(\Delta)$ is the same as the J-shadow space of Δ as defined by Tits [14, Chapter 12]. According to [11] (and [14]), the J-Grassmannian of a geometry Δ is a geometry with a string-shaped diagram graph and the same rank as Δ, but in this paper, following Buekenhout and Cohen [4, §2.5], we shall mostly regard it as a point-line geometry, with the J-flags of Δ taken as points, while the lines are the flags of Δ of type $(J \setminus \{j\}) \cup \mathrm{fr}(j)$ for $j \in J$, where $\mathrm{fr}(j)$ stands for the set of types adjacent to j in the diagram of Δ; a point and a line of $\mathrm{Gr}_J(\Delta)$ are incident precisely when they are incident as flags of Δ.

So, the lines of $\mathrm{Gr}_J(\Delta)$ are particular flags of Δ. This setting will indeed be helpful in some respects but it forces to distinguish between a line and its set of points and this distinction often ends in a burden for the exposition; we will often neglect it. This is a harmless abuse. Indeed only Grassmannians of buildings are considered in this paper; buildings satisfy the Intersection Property and, if that property holds in a geometry Δ, then no two lines of $\mathrm{Gr}_J(\Delta)$ have the same points (even better: no two lines of $\mathrm{Gr}_J(\Delta)$ have two points in common).

As in Section 1.1, given a division ring $\mathbb{K}$, we denote by $A_n(\mathbb{K})$ the building of type A_n defined over $\mathbb{K}$. Similarly, if the division ring $\mathbb{K}$ is a field (namely, is commutative) then $D_n(\mathbb{K})$ stands for the building of type D_n defined over $\mathbb{K}$. We allow $n = 3$ in D_n. So, $D_3 = A_3$. Nevertheless, when writing $D_3(\mathbb{K})$ we always understand that $\mathbb{K}$ is a field, for consistency of notation.

Let X_n stand for either A_n or D_n. It is well known that the elements of $X_n(\mathbb{K})$ can be identified with suitable vector subspaces of a vector space V over $\mathbb{K}$ of dimension either $n+1$ or $2n$ according to whether $X_n = A_n$ or $X_n = D_n$. Similarly, given a proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$, the building $X_n(\mathbb{K}_0)$ is realized in a vector space V_0 over $\mathbb{K}_0$, of the same dimension as V. We can always assume that V_0 is the set of $\mathbb{K}_0$-linear combinations of the vectors of a selected basis E of V, so that V is obtained from V_0 by scalar extension from $\mathbb{K}_0$ to $\mathbb{K}$. Thus, with E suitably selected when $X_n = D_n$, the building $X_n(\mathbb{K}_0)$ is turned into a subgeometry of $X_n(\mathbb{K})$ (see Sections 2.3 and 2.4 for more details). Accordingly, for every subset J of the set of nodes of the diagram X_n, the J-Grassmannian $\mathrm{Gr}_J(X_n(\mathbb{K}_0))$ can be regarded as a subgeometry of $\mathrm{Gr}_J(X_n(\mathbb{K}))$. Our main goal in this paper is to show that, if J consists of extremal nodes of X_n and $|J| > 1$ then $\mathrm{Gr}_J(X_n(\mathbb{K}_0))$ does not generate $\mathrm{Gr}_J(X_n(\mathbb{K}))$.

We firstly consider the $\{1,n\}$-Grassmannian $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ of $A_n(\mathbb{K})$; see Fig. 1.

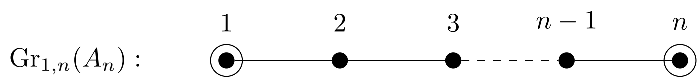

Figure 1: The $\{1,n\}$-Grassmannian of A_n

The points of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ are flags of type $\{1,n\}$ in $A_n(\mathbb{K})$; its lines are flags of type either $\{2,n\}$ or $\{1,n-1\}$; a point p and a line ℓ are incident if and only if $p \cup \ell$ is a flag of $A_n(\mathbb{K})$.

Turning to D_n, we label the nodes of this diagram as in Fig. 2.

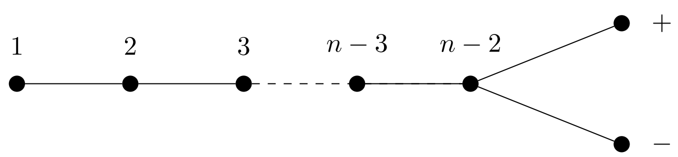

Figure 2: Labeling of types for buildings of type D_n

We are interested in the J-Grassmannians $\mathrm{Gr}_J(D_n(\mathbb{K}))$, where $J = \{+,-\}$ or $J = \{1,+,-\}$ or $J = \{1,-\}$ (we can omit the case $J = \{1,+\}$ since $\mathrm{Gr}_{1,+}(D_n(\mathbb{K})) \cong \mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$); see Fig. 3.

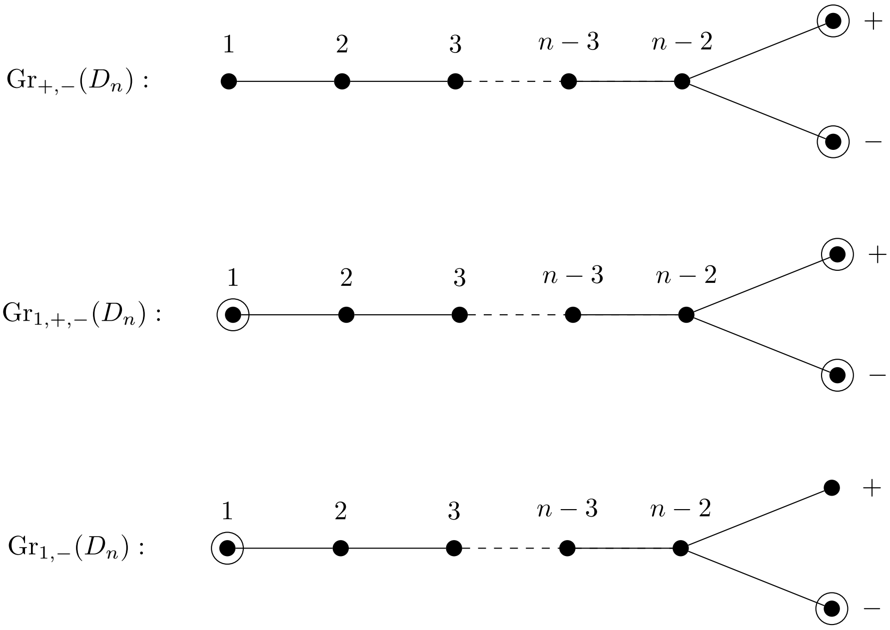

Figure 3: Geometries associated to buildings of type D_n

Explicitly, the points of $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$ are the flags of $D_n(\mathbb{K})$ of type $\{+,-\}$ while the lines are the flags of types $\{n-2,+\}$ and $\{n-2,-\}$ with incidence between a point p and a line ℓ given by the condition that $p \cup \ell$ must be a flag of $D_n(\mathbb{K})$. As for $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$, its points are the flags of type $\{1,+,-\}$, and the lines are the flags of type $\{2,+,-\}$, $\{1,n-2,+\}$ or $\{1,n-2,-\}$; incidence is defined as above. Finally, the points of $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$ are the flags of type $\{1,-\}$ and the lines are the flags of type either $\{2,-\}$ or $\{1,n-2\}$.
Note that when $n=3$, since $D_3(\mathbb{K}) \cong A_3(\mathbb{K})$, we have $\mathrm{Gr}_{1,3}(A_3(\mathbb{K})) \cong \mathrm{Gr}_{+,-}(D_3(\mathbb{K}))$. In any case, $\mathrm{Gr}_{+,-}(D_n(\mathbb{K})) \cong \mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$, where $B_n^+(\mathbb{K}) := \mathrm{Gr}_1(D_n(\mathbb{K}))$ is the 1-Grassmannian of $D_n(\mathbb{K})$ (but regarded as a geometry of rank n), namely the top-thin polar space associated to the group $\mathrm{O}^+(2n,\mathbb{K})$; see Fig. 4.

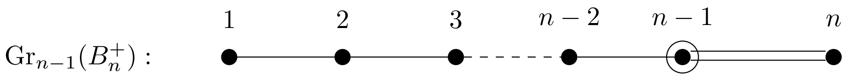

Figure 4: Geometry $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$

The following, to be proved in Section 3, is our first main result in this paper:

Theorem 1.1. *For a division ring $\mathbb{K}$, let $\Gamma(\mathbb{K})$ be one of the following: $Gr_{1,n}(A_n(\mathbb{K}))$ for $n \geq 3$; $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$, $n \geq 3$; $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ with $n \geq 4$; $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$ for $n \geq 4$. Then $\Gamma(\mathbb{K})$ is not $\mathbb{K}_0$-generated for any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$.*

As said in Section 1.1, the case of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ has been already considered in [2], but the proof we shall give in this paper is different and simpler than that of [2]. Theorem 1.1 also contains a proof of a conjecture presented in [6, Conjecture 5.11].

Corollary 1.2. *The $(n-1)$-Grassmannian $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ of the top-thin polar space $B_n^+(\mathbb{K}) = \mathrm{Gr}_1(D_n(\mathbb{K}))$ is not $\mathbb{K}_0$-generated for any proper subfield $\mathbb{K}_0$ of $\mathbb{K}$.*

Apparently, this corollary is an obvious consequence of Theorem 1.1 and the isomorphism $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. However its proof is not so trivial as one might believe; we will give it in Section 3.2.

As we shall see in Section 3.3, Theorem 1.1 admits the following far reaching generalization:

Theorem 1.3. *Let $\Gamma(\mathbb{K})$ be either $\mathrm{Gr}_J(A_n(\mathbb{K}))$ or $\mathrm{Gr}_J(D_n(\mathbb{K}))$, with J a non-connected set of nodes of the diagram A_n or D_n respectively. Then $\Gamma(\mathbb{K})$ is not $\mathbb{K}_0$-generated, for any proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$.*

1.3 A result on generation and embeddings

We recall that the *generating rank* of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is the number

$$\mathrm{gr}(\Gamma) := \min\{|X|\colon X \subseteq \mathcal{P}, \langle X\rangle_\Gamma = \mathcal{P}\},$$

where, for a subset X of the point-set $\mathcal{P}$ of Γ, we denote by $\langle X\rangle_\Gamma$ the subspace of Γ generated by X. Turning to embeddings, a (*full*) *projective embedding* $e : \Gamma \to \mathrm{PG}(V)$ of Γ (henceforth often called an *embedding* of Γ, for short) is an injective map $e : \mathcal{P} \to \mathrm{PG}(V)$ from the point-set $\mathcal{P}$ of Γ to the set of points of the projective space $\mathrm{PG}(V)$ of a vector space V, such that for every line $\ell \in \mathcal{L}$ of Γ the set $e(\ell) := \{e(p)\colon p \in \ell\}$ is a projective line of $\mathrm{PG}(V)$ and $e(\mathcal{P})$ spans $\mathrm{PG}(V)$. We put $\dim(e) := \dim(V)$, calling $\dim(e)$ the *dimension* of e. If $\mathbb{K}$ is the underlying division ring of V, we say that e is *defined over* $\mathbb{K}$, also that e is a $\mathbb{K}$-*embedding*. If Γ admits a projective embedding we say that Γ is *projectively embeddable* (also *embeddable*, for short).

If $e : \Gamma \to \mathrm{PG}(V)$ and $e' : \Gamma \to \mathrm{PG}(V')$ are two $\mathbb{K}$-embeddings of Γ we say that e *dominates* e' if there is a $\mathbb{K}$-semilinear mapping $\varphi : V \to V'$ such that $e' = \varphi \cdot e$. If φ is an isomorphism then we say that e and e' are *isomorphic*. Following Tits [14], we say that an embedding e is *dominant* if, modulo isomorphisms, it is not dominated by any embedding other than itself. Every $\mathbb{K}$-embedding e of Γ admits a *hull* $\tilde{e}$, uniquely determined modulo isomorphisms and characterized by the following property: $\tilde{e}$ dominates all $\mathbb{K}$-embeddings of Γ which dominate e (see Ronan [12]). Accordingly, an embedding is dominant if and only if it is the hull of at least one embedding; equivalently, it is its own hull. Finally, an embedding $\tilde{e}$ of Γ is *absolutely universal* (henceforth called just *universal*, for short) if it dominates all embeddings of Γ. In other words, Γ admits the universal embedding if and only if all of its embeddings have the same hull, that common hull being the universal embedding of Γ. Note that this forces all embeddings of Γ to be defined over the same division ring. Note also that the universal embedding, if it exists, is homogeneous, an embedding e of Γ being *homogeneous* if $eg \cong e$ for every automorphism g of Γ.

The *embedding rank* $\mathrm{er}(\Gamma)$ of an embeddable geometry Γ is defined as follows:

$$\mathrm{er}(\Gamma) := \sup\{\dim(\varepsilon) : \varepsilon \text{ projective embedding of } \Gamma\}.$$

Obviously, if Γ admits the universal embedding $\tilde{e}$ then $\mathrm{er}(\Gamma) = \dim(\tilde{e})$, but $\mathrm{er}(\Gamma)$ is defined even if no embedding of Γ is universal.

If $e : \Gamma \to \mathrm{PG}(V)$ is an embedding of $\Gamma = (\mathcal{P}, \mathcal{L})$ then stretching a line in Γ through two collinear points $p, q \in \mathcal{P}$ corresponds to forming the span $\langle v, w\rangle \subseteq V$ of any two non-zero vectors $v \in e(p)$ and $w \in e(q)$. If $X \subseteq \mathcal{P}$ generates Γ then $\mathcal{P} = \cup_{n=0}^{\infty} X_n$ where $X_0 := X$ and

$X_{n+1} := \cup_{p,q\in X_n}\langle p,q\rangle_\Gamma$. Consequently, if we select a non-zero vector $v_p \in e(p)$ for every point $p \in X$ then $\{v_p\}_{p\in X}$ spans V. This makes it clear that $|X| \geq \dim(e)$. Accordingly,

$$\dim(e) \;\leq\; \mathrm{gr}(\Gamma). \tag{1}$$

Therefore, if $\mathrm{gr}(\Gamma)$ is finite and $\dim(e) = \mathrm{gr}(\Gamma)$ then e is dominant (hence universal, if Γ admits the universal embedding). In any case, (1) implies the following:

$$\mathrm{er}(\Gamma) \;\leq\; \mathrm{gr}(\Gamma). \tag{2}$$

In fact the equality $\mathrm{er}(\Gamma) = \mathrm{gr}(\Gamma)$ holds for many embeddable geometries but not for all of them. For instance Heiss [9] gives an example where $\mathrm{gr}(\Gamma) = \mathrm{er}(\Gamma) + 1$. The example of [9] looks fairly artificial. A more natural example, where $\mathrm{er}(\Gamma)$ is finite but $\mathrm{gr}(\Gamma)$ is infinite is given by Theorem 1.5, to be stated below. That theorem will be obtained in Section 4 with the help of the following lemma. In order to properly state it, we recall that a division ring $\mathbb{K}$ is *finitely generated* if it is generated by a finite subset $X \subseteq \mathbb{K}$. For instance, an algebraic extension of a finite field of prime order $\mathbb{F}_p$ is finitely generated if and only if it is finite, in which case it is a simple extension of $\mathbb{F}_p$. On the other hand, no algebraically closed field is finitely generated.

Lemma 1.4. *Let $\Gamma(\mathbb{K})$ be either $\mathrm{Gr}_J(A_n(\mathbb{K}))$ or $\mathrm{Gr}_J(D_n(\mathbb{K}))$ for a set of types J non-connected as a set of nodes of A_n or D_n. Suppose that $\mathbb{K}$ is not finitely generated. Then the generating rank of $\Gamma(\mathbb{K})$ is infinite.*

Lemma 1.4 will be obtained in Section 4 as a consequence of Theorem 1.3. By exploiting it we will obtain the following:

Theorem 1.5. *Let $\mathbb{F}_p$ be a finite field of prime order and $\overline{\mathbb{F}}_p$ its algebraic closure. Then, for $n \geq 3$, the geometry $\mathrm{Gr}_{1,n}(A_n(\overline{\mathbb{F}}_p))$ has infinite generating rank but its embedding rank is equal to either $(n+1)^2 - 1$ or $(n+1)^2$.*

Remark 1.6. It is well known that if $\mathbb{K}$ is a field then $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ admits an $(n+1)^2 - 1$ dimensional embedding, say e_{Lie}, in (the projective space of) the space of square matrices of order $n+1$ with entries in $\mathbb{K}$ and null trace (see e.g. Blok and Pasini [3]; the choice of the symbol e_{Lie} for this embedding is motivated by the fact that it affords the representation of the group $\mathrm{SL}(n+1,\mathbb{K})$ in its action on its own Lie algebra). However $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ does not satisfy the sufficient conditions of Kasikova and Shult [10] for the existence of the universal embedding. So, we do not know if it always admits the universal embedding, let alone if e_{Lie} is universal. A complete answer is known only when $\mathbb{K}$ is a prime field. In this case e_{Lie} is indeed universal (Blok and Pasini [3, Section 3]). A bit less is known when $\mathbb{K}$ is a number field or a perfect field of positive characteristic; in this case e_{Lie} dominates all homogeneous embeddings of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ (Völklein [15]).

As for the remaining geometries of Theorem 1.1, namely $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$, $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ and $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$, they too are embeddable (see [3]) and, when $\mathbb{K}$ is a prime field, they admit the universal embedding (Blok and Pasini [3, Section 4]), even if none of them satisfies the conditions of Kasikova and Shult [10].

Remark 1.7. The geometry Δ_2^+ of [5] with $n = 3$ is the same as $\mathrm{Gr}_{1,3}(A_3(\mathbb{F}))$. According to the above, Lemma 4.8 of [5], which deals with that geometry and its Weyl embedding ε_2^+ (which is the same as e_{Lie}), might possibly be wrong as stated. It should be corrected as follows: when $n = 3$ and $\mathbb{F}$ is a perfect field of positive characteristic or a number field, then $\tilde{\varepsilon}_2^+$ dominates all homogeneous embeddings of Δ_2^+.

Remark 1.8. In our survey of embeddings we have stuck to full projective embeddings, but in the proof of Theorem 1.5 we shall deal with lax embeddings too. *Lax projective embeddings* are defined in the same way as full projective embeddings but for replacing the condition that $e(\ell)$ is a line of $\mathrm{PG}(V)$ with the weaker condition that $e(\ell)$ spans a line of $\mathrm{PG}(V)$, for every line ℓ of Γ. Many authors also require that no two lines of Γ span the same line of $\mathrm{PG}(V)$, but in view of our needs in this paper we can safely renounce that requirement. The only fact relevant for us is that inequality (1) holds true even if e is lax, as it is clear from the way we have obtained it.

2 Preliminaries

We have already defined the Grassmannians $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$, $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$, $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ and $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$ in Section 1.2. In this section we shall turn back to them, adding more details. We will also better fix our notation and terminology for $A_n(\mathbb{K})$ and $D_n(\mathbb{K})$. Finally, we shall better explain in which sense $\mathrm{Gr}_J(A_n(\mathbb{K}))$ and $\mathrm{Gr}_J(D_n(\mathbb{K}))$ contain $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ and $\mathrm{Gr}_J(D_n(\mathbb{K}_0))$ for a sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$.

2.1 The geometry $A_n(\mathbb{K})$ and its Grassmannian $\mathrm{Gr}_{1.n}(A(\mathbb{K}))$

Let $A_n(\mathbb{K})$ be a geometry of type A_n defined over a division ring $\mathbb{K}$, with $n \geq 3$. Explicitly, $A_n(\mathbb{K}) \cong \mathrm{PG}(V_{n+1}(\mathbb{K}))$ for a $(n+1)$-dimensional right $\mathbb{K}$-vector space $V_{n+1}(\mathbb{K})$. For $i = 1, 2, \ldots, n$ the elements of $A_n(\mathbb{K})$ of type i are the i-dimensional subspaces of $V_{n+1}(\mathbb{K})$, with symmetrized inclusion as the incidence relation. As customary we call the elements of $A_n(\mathbb{K})$ of type 1, 2 and n *points*, *lines* and *hyperplanes* respectively. The elements of type $n-1$ will be called *sub-hyperplanes.* Note that, when $n = 3$, lines and sub-hyperplanes are the same objects.

Turning to $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$, its points are the point-hyperplane flags (p, H) of $A_n(\mathbb{K})$. Its lines, regarded as sets of points, are of either of the following two types:

$$\begin{array}{ll} (a) & \ell_{p,S} := \{(p, X)\colon X \text{ hyperplane, } X \supset S\} \text{ for a (point, sub-hyperplane) flag } (p, S). \\ (b) & \ell_{L,H} := \{(x, H)\colon x \text{ a point, } x \subset L\}, \text{ for a line-hyperplane flag } (L, H); \end{array} \qquad (3)$$

2.2 $D_n(\mathbb{K})$ and $\mathrm{Gr}_J(D_n(\mathbb{K}))$ for $J = \{+,-\}, \{1,-\}$ or $\{1,+,-\}$

Let $\mathbb{K}$ be a field and $V_{2n}(\mathbb{K})$ a vector space of dimension $2n$ over $\mathbb{K}$, with $n \geq 3$. Consider a non-degenerate quadratic form q on $V_{2n}(\mathbb{K})$ of Witt index n. As in Section 1.2, let $B_n^+(\mathbb{K})$ be the polar space associated to q, namely the (weak) building of rank n whose elements are the vector subspaces of $V_{2n}(\mathbb{K})$ that are totally singular with respect to q, with their dimensions taken as types. The elements of $B_n^+(\mathbb{K})$ of dimension 1 are called *points* and those of dimension 2 *lines.*

It is well kown that we can 'unfold' $B_n^+(\mathbb{K})$ so that to obtain a building $D_n(\mathbb{K})$ of type D_n (see e.g. Tits [14, Chapter 7]). Explicitly, let $\sim$ be the equivalence relation on the set of all n-dimensional subspaces of $B_n^+(\mathbb{K})$ defined as follows: $X \sim Y$ if and only if $X \cap Y$ has even codimension in X (equivalently, in Y). Let $\mathfrak{S}^+$ and $\mathfrak{S}^-$ be the two equivalence classes of $\sim$. Take $\{1, 2, \ldots, n-2, +, -\}$ as the set of types. For $1 \leq i \leq n-2$ the i-elements of $B_n^+(\mathbb{K})$ are the elements of $D_n(\mathbb{K})$ of type i and the elements of $\mathfrak{S}^+$ and $\mathfrak{S}^-$ are given types $+$ and $-$ respectively. The $(n-1)$-elements of $B_n^+(\mathbb{K})$ are dropped (but we can recover them as flags of type $\{+,-\}$). Incidence between elements of different types $\{i, j\}$ with $\{i, j\} \neq \{+,-\}$ is symmetrized inclusion; if $X \in \mathfrak{S}^+$ and $Y \in \mathfrak{S}^-$ then X is incident with Y if and only if $\dim(X \cap Y) = n-1$.

It is clear from the way $D_n(\mathbb{K})$ is defined that the 1-Grassmannian $\mathrm{Gr}_1(D_n(\mathbb{K}))$ of $D_n(\mathbb{K})$, regarded as a geometry of rank n, is just the same as $B_n^+(\mathbb{K})$. So, we can go back and forth from

$D_n(\mathbb{K})$ to $B_n^+(\mathbb{K})$ as if they were the same object. In the sequel we will sometimes avail of this opportunity, if profitable.

Turning to Grassmannians, $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$ is the point-line geometry where the points are the flags (M_1, M_2) of $D_n(\mathbb{K})$ of type $(+,-)$ and the lines are of the following two forms:

$$\begin{array}{lll} (a) & \ell_{U,M_1} & := \{(M_1, X)\colon X \in \mathfrak{S}^-,\ M_1 \cap X \supset U\} \\ & & \text{ with } (U, M_1) \text{ a flag of type } (n-2,+); \\ (b) & \ell_{U,M_2} & := \{(X, M_2)\colon X \in \mathfrak{S}^+,\ X \cap M_2 \supset U\} \\ & & \text{ with } (U, M_2) \text{ a flag of type } (n-2,-). \end{array} \qquad (4)$$

Recall that the points of the Grassmannian $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ of $B_n^+(\mathbb{K})$ are the $(n-1)$-dimensional subspaces of $V_{2n}(\mathbb{K})$ totally singular for the quadratic form q and the lines are the sets $\ell_{X,M} := \{Y : X \subset Y \subset M\}$ where $\dim(X) = n-2$, $\dim(M) = n$, $X \subset M$ and M is totally singular. Every point X of $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ is the intersection $X = M_1 \cap M_2$ of a unique pair $\{M_1, M_2\}$ of n-dimensional totally singular subspaces, which necessarily form a $(+,-)$-flag of $D_n(\mathbb{K})$. Conversely, for every $(+,-)$-flag (M_1, M_2) of $D_n(\mathbb{K})$, the intersection $X = M_1 \cap M_2$ is a point of $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$. A bijecive mapping ι is thus naturally defined from the set of points of $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ onto the set of points of $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. The mapping ι induces a bijection from the set of lines of $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ onto the set of lines of $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. In fact, if $\ell_{X,M}$ is a line of $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ then $\iota(\ell_{X,M})$ is the line of $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$ denoted by the very same symbol $\ell_{X,M}$ and it has form (a) or (b) according to whether M belongs to $\mathfrak{S}^+$ or $\mathfrak{S}^-$. To sum up, $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K})) \cong \mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$.

The Grassmannian $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$ is the point-line geometry where the points are the flags (p, M) of $D_n(\mathbb{K})$ of type $(1,-)$ and the lines are as follows:

$$\begin{array}{ll} (a) & \ell_{p,U} := \{(p, X)\colon X \in \mathfrak{S}^-, X \supset U\} \text{ with } (p, U) \text{ a flag of type } (1, n-2); \\ (b) & \ell_{L,M} := \{(x, M)\colon \dim(x) = 1, x \subset L\} \text{ with } (L, M) \text{ a flag of type } (2,-). \end{array} \qquad (5)$$

The Grassmannian $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ is the point-line geometry where the points are the flags (p, M_1, M_2) of $D_n(\mathbb{K})$ of type $(1,+,-)$; the lines are as follows:

$$\begin{array}{lll} (a) & \ell_{L,M_1,M_2} & := \{(p, M_1, M_2)\colon \dim(p) = 1, p \subset L\} \\ & & \text{ with } (L, M_1, M_2) \text{ a flag of type } (2,+,-); \\ (b) & \ell_{p,U,M_1} & := \{(p, M_1, X)\colon X \in \mathfrak{S}^-, X \supset U\} \\ & & \text{ with } (p, U, M_1) \text{ a flag of type } (1, n-2, +); \\ (c) & \ell_{p,U,M_2} & := \{(p, X, M_2)\colon X \in \mathfrak{S}^+, X \supset U\} \\ & & \text{ with } (p, U, M_2) \text{ a flag of type } (1, n-2, -). \end{array} \qquad (6)$$

2.3 The subgeometry $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ of $\mathrm{Gr}_J(A_n(\mathbb{K}))$ for $\mathbb{K}_0 \leq \mathbb{K}$

With $V_{n+1}(\mathbb{K})$ as in Section 2.1, let $\mathbb{K}_0$ be a sub-division ring of $\mathbb{K}$. Given a basis E of $V_{n+1}(\mathbb{K})$, we say that a vector of $V_{n+1}(\mathbb{K})$ is $\mathbb{K}_0$-*rational* (with respect to E) if it is a linear combination of vectors of E with coefficients in $\mathbb{K}_0$. A subspace of $V_{n+1}(\mathbb{K})$ is $\mathbb{K}_0$-*rational* (with respect to E) if it admits a basis formed by $\mathbb{K}_0$-rational vectors. In other words, if $V_{n+1,E}(\mathbb{K}_0)$ is the $\mathbb{K}_0$-vector space formed by the $\mathbb{K}_0$-rational vectors, the $\mathbb{K}_0$-rational subspaces of $V_{n+1}(\mathbb{K})$ are the spans in $V_{n+1}(\mathbb{K})$ of the subspaces of $V_{n+1,E}(\mathbb{K}_0) \subseteq V_{n+1}(\mathbb{K})$.

Clearly, the sum of two $\mathbb{K}_0$-rational subspaces of $V_{n+1}(\mathbb{K})$ is still $\mathbb{K}_0$-rational. Moreover:

Lemma 2.1. *If X and Y are two $\mathbb{K}_0$-rational subspaces, then $X + Y$ has the same dimension in $V_{n+1}(\mathbb{K})$ as in $V_{n+1,E}(\mathbb{K}_0)$.*

Proof. This statement is just a rephrasing of the following well known fact from linear algebra: the rank of a finite set of vectors of a $\mathbb{K}_0$-vector space does not change if we replace $\mathbb{K}_0$ with a larger division ring $\mathbb{K}$. □

Corollary 2.2. *The intersection of two $\mathbb{K}_0$-rational subspaces is still $\mathbb{K}_0$-rational.*

Proof. Let X_0, Y_0 be two subspaces of $V_{n+1,E}(\mathbb{K}_0)$ and X, Y theirs spans in $V_{n+1}(\mathbb{K})$. Then $X \cap Y$ contains the span Z of $X_0 \cap Y_0$ in $V_{n+1}(\mathbb{K})$. We must prove that $X \cap Y = Z$. Clearly, $\dim(X) = \dim(X_0)$ and $\dim(Y) = \dim(Y_0)$. Moreover $\dim(X + Y) = \dim(X_0 + Y_0)$ by Lemma 2.1. Hence $\dim(X \cap Y) = \dim(X) + \dim(Y) - \dim(X + Y) = \dim(X_0) + \dim(Y_0) - \dim(X_0 + Y_0) = \dim(X_0 \cap Y_0) = \dim(Z)$. Therefore $X \cap Y = Z$. □

The following is now obvious:

Proposition 2.3. *The $\mathbb{K}_0$-rational elements of $A_n(\mathbb{K})$ form a geometry $A_{n,E}(\mathbb{K}_0) \cong A_n(\mathbb{K}_0)$.*

In view of Proposition 2.3, we can freely identify $A_n(\mathbb{K}_0)$ with $A_{n,E}(\mathbb{K}_0)$, thus regarding $A_n(\mathbb{K}_0)$ as a subgeometry of $A_n(\mathbb{K})$. The flags of $A_n(\mathbb{K}_0)$ are thus identified with the $\mathbb{K}_0$-*rational* flags of $A_n(\mathbb{K})$, namely the flags of $A_n(\mathbb{K})$ all elements of which are $\mathbb{K}_0$-rational (with respect to the selected basis E of $V_{n+1}(\mathbb{K})$). Accordingly, for $\emptyset \neq J \subseteq \{1, 2, ..., n\}$ the J-Grassmannian $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ of $A_n(\mathbb{K}_0)$ is identified with the subgeometry $\mathrm{Gr}_{J,E}(A_n(\mathbb{K}_0))$ of $\mathrm{Gr}_J(A_n(\mathbb{K}))$ formed by the $\mathbb{K}_0$-*rational* points and lines of $\mathrm{Gr}_J(A_n(\mathbb{K}))$, namely the points and lines of $\mathrm{Gr}_J(A_n(\mathbb{K}))$ which are $\mathbb{K}_0$-rational as flags of $A_n(\mathbb{K})$.

Henceforth, by a harmless little abuse, we will always regard $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ as the same as $\mathrm{Gr}_{J,E}(A_n(\mathbb{K}_0))$, thus referring to the span of $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ in $\mathrm{Gr}_J(A_n(\mathbb{K}))$, as we have done in the Introduction, while in fact we mean the span of $\mathrm{Gr}_{J,E}(A_n(\mathbb{K}_0))$.

The next proposition states that, regarded $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ as a subgeometry of $\mathrm{Gr}_J(A_n(\mathbb{K}))$, the collinearity graph of $\mathrm{Gr}_J(A_n(\mathbb{K}_0))$ is just the graph induced on its point-set by the collinearity graph of $\mathrm{Gr}_J(A_n(\mathbb{K}))$.

Proposition 2.4. *A line of $\mathrm{Gr}_J(A_n(\mathbb{K}))$ is $\mathbb{K}_0$-rational if and only if at least two of its points are $\mathbb{K}_0$-rational.*

Proof. The 'only if' part of this claim easily follows from the isomorphism $\mathrm{Gr}_{J,E}(A_n(\mathbb{K}_0)) \cong \mathrm{Gr}_J(A_n(\mathbb{K}_0))$. Turning to the 'if' part, given $j_0 \in J$, let L be a flag of $A_n(\mathbb{K})$ of type $(J \setminus \{j_0\}) \cup \mathrm{fr}(j_0)$ and let P and P' be two distinct J-flags of $A_n(\mathbb{K})$ incident with L. We must prove that if both P and P' are $\mathbb{K}_0$-rational then L too is $\mathbb{K}_0$-rational. There are three cases to examine: J contains elements $j < j_0$ as well elements $j' > j_0$; $j_0 \leq j$ for every $j \in J$; $j_0 \geq j$ for every $j \in J$. We shall examine only the first case, leaving the remaining two (easier) cases to the reader.

With j_0 as in the first case, the flag L has type $(J \setminus \{j_0\}) \cup \{j_0 - 1, j_0 + 1\}$ and contains $Q := P \cap P'$, which is a flag of type $J \setminus \{j_0\}$. Moreover, there are distinct j_0-subspaces S, S' of $V_{n+1}(\mathbb{K})$ incident with L such that $P = Q \cup \{S\}$ and $P' = Q \cup \{S'\}$. As S and S' are incident with L, the elements of L of type $j_0 - 1$ and $j_0 + 1$ coincide with $S \cap S'$ and $S + S'$ respectively, namely $L = Q \cup \{S \cap S', S + S'\}$. By assumption, P and P' are $\mathbb{K}_0$-rational. Hence $Q = P \cap P'$ as well as S and S' are $\mathbb{K}_0$-rational. If $j_0 - 1 \in J$ then $S \cap S' \in Q$, hence $S \cap S'$ is $\mathbb{K}_0$-rational. Otherwise $S \cap S'$ is $\mathbb{K}_0$-rational by Corollary 2.2. Similarly, $S + S'$ is $\mathbb{K}_0$-rational. Thus, all elements of L are $\mathbb{K}_0$-rational, namely L is $\mathbb{K}_0$-rational. □

2.4 The subgeometry $\mathrm{Gr}_J(D_n(\mathbb{K}_0))$ of $\mathrm{Gr}_J(D_n(\mathbb{K}))$ for $\mathbb{K}_0 \leq \mathbb{K}$

Let $\mathbb{K}_0$ be a subfield of $\mathbb{K}$. Let $q : V_{2n}(\mathbb{K}) \to \mathbb{K}$ be the quadratic form considered in Section 2.2. Without loss of generality we can assume to have chosen the basis $E = (e_1, \ldots, e_{2n})$ of $V_{2n}(\mathbb{K})$ in such a way that q admits the following canonical expression with respect to E:

$$q(x_1, \ldots, x_{2n}) \;=\; x_1x_2 + \cdots + x_{2n-1}x_{2n}. \tag{7}$$

As in Section 2.3, we can consider the $\mathbb{K}_0$-vector space $V_{2n,E}(\mathbb{K}_0)$ formed by the $\mathbb{K}_0$-rational vectors (with respect to E). The form q induces a quadratic form q_0 on $V_{2n,E}(\mathbb{K}_0)$. Clearly, a $\mathbb{K}_0$-rational subspace X of $V_{2n}(\mathbb{K})$ is totally singular for q if and only if $X \cap V_{2n,E}(\mathbb{K}_0)$ is totally singular for q_0. Hence the polar space $B^+_n(\mathbb{K}_0)$ associated to q_0 can be identified with the subgeometry $B^+_{n,E}(\mathbb{K}_0)$ of $B^+_n(\mathbb{K})$ formed by the $\mathbb{K}_0$-rational subspaces of $V_{2n}(\mathbb{K})$ which are totally singular for q. Similarly, $D_n(\mathbb{K}_0)$ can be identified with the subgeometry $D_{n,E}(\mathbb{K}_0)$ of $D_n(\mathbb{K})$ formed by the $\mathbb{K}_0$-rational elements of $D_n(\mathbb{K})$.

A flag of $D_n(\mathbb{K})$ is $\mathbb{K}_0$-*rational* if all of its elements are $\mathbb{K}_0$-rational. Given a nonempty subset J of the type-set $\{1, 2, ..., n-2, +, -\}$ of $D_n(\mathbb{K})$, a point or a line of $\mathrm{Gr}_J(D_n(\mathbb{K}))$ are said to be $\mathbb{K}_0$-*rational* if they are $\mathbb{K}_0$-rational as flags of $D_n(\mathbb{K})$. The $\mathbb{K}_0$-rational points and lines of $\mathrm{Gr}_J(D_n(\mathbb{K}))$ form a subgeometry $\mathrm{Gr}_{J,E}(D_n(\mathbb{K}_0))$ of $\mathrm{Gr}_J(D_n(\mathbb{K}))$ isomorphic to $\mathrm{Gr}_J(D_n(\mathbb{K}_0))$. An analogue of Proposition 2.4 also holds:

Proposition 2.5. *A line of* $\mathrm{Gr}_J(D_n(\mathbb{K}))$ *is* $\mathbb{K}_0$*-rational if and only if at least two of its points are* $\mathbb{K}_0$*-rational.*

Proof. This statement can be proved in the same way as Proposition 2.4 but for a couple of cases in the proof of the 'only if' part, which we shall now discuss.

1. Suppose that J contains at least one of the types $+$ and $-$, say $+ \in J$. Suppose moreover that $n-2 \notin J$. Let L be a flag of $D_n(\mathbb{K})$ of type $(J \setminus \{+\}) \cup \mathrm{fr}(+) = (J \setminus \{+\}) \cup \{n-2\}$ and let P, P' be distinct $\mathbb{K}_0$-rational flags of type J, both incident with L. Then $Q = P \cap P'$ is a $\mathbb{K}_0$-rational flag, $P = Q \cup \{M\}$ and $P' = Q \cup \{M'\}$ for distinct $\mathbb{K}_0$-rational element $M, M' \in \mathfrak{S}^+$. Also, $L = Q \cup S$ for an $(n-2)$-element S incident with Q. We have $S \subseteq M \cap M'$ since P and P' are incident with L. However, $\dim(M \cap M')$ has even codimension in M and M', since M and M' belong to the same family of n-elements of $B^+_n(\mathbb{K})$, namely $\mathfrak{S}^+$. Therefore $S = M \cap M'$. Hence S is $\mathbb{K}_0$-rational by Corollary 2.2. Thus, L is $\mathbb{K}_0$-rational.

2. The set J contains none of the types $+$ or $-$ but it contains $n-2$. To fix ideas, suppose that $n > 3$. Let L be a flag of $D_n(\mathbb{K})$ of type $(J \setminus \{n-2\}) \cup \mathrm{fr}(n-2) = (J \setminus \{n-2\}) \cup \{n-3, +, -\}$ and let P, P' be distinct $\mathbb{K}_0$-rational flags of type J, both incident with L. Then $Q = P \cap P'$ is a $\mathbb{K}_0$-rational flag, $P = Q \cup \{S\}$ and $P' = Q \cup \{S'\}$ for distinct $\mathbb{K}_0$-rational $(n-2)$-elements S, S' of $D_n(\mathbb{K})$ and $L = Q \cup \{R, M_1, M_2\}$ for an $(n-3, +, -)$-flag (R, M_1, M_2) incident with Q. As both P and P' are incident with L, the sum $S + S'$ is contained in $M \cap M'$. However $\dim(M \cap M) = n-1$ while $\dim(S + S') \geq n-1$ since $S \neq S'$. Consequently, $M \cap M' = S + S'$. On the other hand, $S + S'$ is a $\mathbb{K}_0$-rational subspace of $V_{2n}(\mathbb{K})$, since both S and S' are $\mathbb{K}_0$-rational. Hence $M \cap M'$ is $\mathbb{K}_0$-rational. Therefore $M \cap M'$ is an $(n-1)$-element of $B^+_{n,E}(\mathbb{K}_0) = \mathrm{Gr}_1(D_{n,E}(\mathbb{K}_0))$. Accordingly, $M \cap M' = M_0 \cap M'_0$ for a $(+, -)$-flag (M_0, M'_0) of $D_{n,E}(\mathbb{K}_0)$. On the other hand, all $(+, -)$-flags of $D_{n,E}(\mathbb{K}_0)$ are $(+, -)$-flags of $D_n(\mathbb{K})$ too and two $(+, -)$-flags (M, M') and (M_0, M'_0) of $D_n(\mathbb{K})$ coincide if $M \cap M' = M_0 \cap M'_0$. It follows that $M = M_0$ and $M' = M_0$, namely both M and M' are $\mathbb{K}_0$-rational. It remains to prove that R too is $\mathbb{K}_0$-rational. If $n-3 \in J$ then $R \in Q$ and there is nothing to prove. Otherwise $R = S \cap S'$. Hence R is $\mathbb{K}_0$-rational by Corollary 2.2. Therefore L is $\mathbb{K}_0$-rational.

We have assumed that $n > 3$. When $n = 3$ we have $J = \{n-2\}$ and $L = (M_1, M_2)$, of type $(+, -)$; we get the conclusion as above, but now with no R to care of. $\square$

It goes without saying that all we have said for $D_n(\mathbb{K}_0)$ and $\mathrm{Gr}_J(D_n(\mathbb{K}_0))$ in this section holds for $B_n^+(\mathbb{K}_0)$ and $\mathrm{Gr}_J(B_n^+(\mathbb{K}_0))$ as well.

3 Proof of Theorems 1.1 and 1.3

For X_n equal to A_n or D_n and a nonempty set of types J, let $\Gamma(\mathbb{K}) := \mathrm{Gr}_J(X_n(\mathbb{K}))$ and $\Gamma(\mathbb{K}_0) := \mathrm{Gr}_J(X_n(\mathbb{K}_0))$ be its $\mathbb{K}_0$-rational subgeometry for a proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$ (Sections 2.3 and 2.4).

Definition 1. We say that a node t of X_n *splits* J if $t \notin J$ and J is not contained in one single connected component of $X_n \setminus \{t\}$. In other words, t separates at least two of the types of J.

Definition 2. We say that a J-flag F (point of $\Gamma(\mathbb{K})$) is *nearly* $\mathbb{K}_0$*-rational* if either at least one of its elements is $\mathbb{K}_0$-rational or there exists a $\mathbb{K}_0$-rational element of $X_n(\mathbb{K})$ incident with F and such that its type splits J. We denote by $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ the set of all nearly $\mathbb{K}_0$-rational points of $\Gamma(\mathbb{K})$.

Obviously, $\Gamma(\mathbb{K}_0) \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$. We shall prove the following:

Theorem 3.1. *If* $\Gamma(\mathbb{K})$ *is* $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$, $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$, $\mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ *or* $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$ *then* $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ *is a proper subspace of* $\Gamma(\mathbb{K})$.

Theorem 1.1 immediately follows from Theorem 3.1 and the inclusion $\Gamma(\mathbb{K}_0) \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

3.1 Proof of Theorem 3.1

We need a preliminary result from linear algebra, to be exploited later, when discussing the case $\Gamma(\mathbb{K}) = \mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$.

Lemma 3.2. *Suppose that* $\mathbb{K}$ *is a field and let* $V := V_4(\mathbb{K})$. *Given a basis* $E = (e_1, e_2, e_3, e_4)$ *of* V, *let* $E \wedge E = (e_i \wedge e_j)_{i<j}$ *be the corresponding basis of the second exterior power* $V \wedge V$ *of* V. *Then all the following hold:*

(1) *The span* $\langle v, w\rangle$ *of two independent vectors* $v, w \in V$ *is* $\mathbb{K}_0$*-rational with respect to* E *if and only if, modulo proportionality,* $v \wedge w$ *is* $\mathbb{K}_0$*-rational with respect to* $E \wedge E$.

(2) *A non-zero vector* $v \in V$ *is proportional to a* $\mathbb{K}_0$*-rational vector if and only if the subspace* $S_v := \langle v \wedge x\rangle_{x\in V}$ *of* $V \wedge V$ *is* $\mathbb{K}_0$*-rational (with respect to* $E \wedge E$*).*

(3) *The span* $\langle u, v, w\rangle$ *of three independent vectors* $u, v, w \in V$ *is* $\mathbb{K}_0$*-rational if and only if* $\langle u \wedge v, u \wedge w, v \wedge w\rangle$ *is* $\mathbb{K}_0$*-rational.*

Proof. (1) Without loss, we can assume that $v = e_1 + e_3a_3 + e_4a_4$ and $w = e_2 + e_3b_3 + e_4b_4$ for $a_3, a_4, b_3, b_4 \in \mathbb{K}$. Hence $v \wedge w = e_{1,2} + e_{1,3}b_3 + e_{1,4}b_4 - e_{2,3}a_3 - e_{2,4}a_4 + e_{3,4}(a_3b_4 - a_4b_3)$, where we write $e_{i,j}$ for $e_i \wedge e_j$. Both parts of (1) are equivalent to the following single claim: $a_3, a_4, b_3, b_4 \in \mathbb{K}_0$. Hence they are mutually equivalent.

(2) Without loss of generality, we can assume that $v = e_1 + e_2a_2 + e_3a_3 + e_4a_4$. Hence $S_v := \langle v \wedge e_2, v \wedge e_3, v \wedge e_4\rangle$. We have

$$v \wedge e_2 = e_{1,2} - e_{2,3}a_3 - e_{2,4}a_4, \quad v \wedge e_3 = e_{1,3} + e_{2,3}a_2 - e_{3,4}a_4, \quad v \wedge e_4 = e_{1,4} + e_{2,4}a_2 + e_{3,4}a_3,$$

with $e_{i,j} := e_i \wedge e_j$, as above. Both parts of claim (2) are thus equivalent to this: $a_2, a_3, a_4 \in \mathbb{K}_0$. Claim (2) is proved.

(3) Without loss of generality of generality, we can assume that $u = e_1 + e_4 a$, $v = e_2 + e_4 b$ and $w = e_3 + e_4 c$. Hence $u \wedge v = e_{1,2} + e_{1,4} b - e_{2,4} a$, $u \wedge w = e_{1,3} - e_{1,4} c - e_{3,4} a$ and $v \wedge w = e_{2,3} + e_{2,4} c - e_{3,4} b$. Both parts of (3) are equivalent to this: $a, b, c \in \mathbb{K}_0$. Claim (3) follows. $\square$

Lemma 3.3. *If* $\Gamma(\mathbb{K})$ *is as in the hypotheses of Theorem* 3.1 *then the set* $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ *is a subspace of* $\Gamma(\mathbb{K})$.

Proof. We must show that, for any two nearly $\mathbb{K}_0$-rational collinear points F, F' of $\Gamma(\mathbb{K})$, the line $\langle F, F' \rangle_{\Gamma(\mathbb{K})}$ is fully contained in $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Case 1. $\Gamma(\mathbb{K}) = \mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$. Let $F = (p, H)$ and $F' = (p', H')$ be two distinct collinear points of $\Gamma(\mathbb{K})$, namely two point-hyperplane flags with either $p \neq p'$ and $H = H'$ or $p = p'$ but $H \neq H'$. Suppose moreover that F and F' are nearly $\mathbb{K}_0$-rational.

(a) Let $p = p'$ and $H \neq H'$. Now $\langle F, F' \rangle_{\Gamma(\mathbb{K})} = \ell_{p,S} = \{(p, X) \colon X \supset S, \dim(X) = n\}$ where $S = H \cap H' \supset p$ is a sub-hyperplane containing p. By assumption, there exist $\mathbb{K}_0$-rational subspaces U_0, U_0' of $V_{n+1}(\mathbb{K})$ such that $p \subseteq U_0 \subseteq H$ and $p \subseteq U_0' \subseteq H'$. The subspace $U_0 \cap U_0'$ is $\mathbb{K}_0$-rational (Corollary 2.2), it contains p and is contained in S. Hence it is contained in every hyperplane $X \supset S$. As $U_0 \cap U_0'$ is $\mathbb{K}_0$-rational, the flag (p, X) is nearly $\mathbb{K}_0$-rational for every hyperplane $X \subset S$, namely $\ell_{p,S} \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

(b) Let $H = H'$ but $p \neq p'$. Then $\langle F, F' \rangle_{\Gamma(\mathbb{K})} = \ell_{L,H} = \{(x, H) \colon x \subset L, \dim(x) = 1\}$ where $L = p + p' \subset H$ is the span of $p \cup p'$ in $V_{n+1}(\mathbb{K})$. The argument used in case (a) can be dualized as follows. By assumption, there exist $\mathbb{K}_0$-rational subsaces U_0, U_0' of $V_{n+1}(\mathbb{K})$ such that $p \subseteq U_0 \subseteq H$ and $p' \subseteq U_0' \subseteq H'$. Clearly, $L \subseteq U_0 + U_0' \subseteq H$. Hence $x \subseteq U_0 + U_0' \subseteq H$ for every 1-subspace x of L. However $U_0 + U_0'$ is $\mathbb{K}_0$-rational. Therefore (x, H) is nearly $\mathbb{K}_0$-rational. It follows that $\ell_{L,H} \subseteq \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Case 2. $\Gamma(\mathbb{K}) = \mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$. Let $F = (p, M)$ and $F' = (p', M')$ be two collinear points of $\mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$. Since F and F' are collinear, either $p = p'$ or $M = M'$. The line $\langle F, F' \rangle_{\Gamma(\mathbb{K})}$ is as in (a) or (b) of (5) according to whether $p = p'$ or $M = M'$. When $p = p'$ then the same argument as in (a) of Case 1 does the job, with the only change that $M \cap M'$, which now plays the role of $H \cap H'$, has dimension $n-2$ instead of $n-1$. If $M = M'$ then an argument similar to that used for (b) of Case 1 yields the conclusion. We leave the details to the reader.

Case 3. $\Gamma(\mathbb{K}) = \mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$. Let $F = (p, M_1, M_2)$ and $F' = (p', M_1', M_2')$ be two collinear points of $\Gamma(\mathbb{K})$ and suppose they both are nearly $\mathbb{K}_0$-rational. Two subcases can occur.

(a) $M_i = M_i'$ for $i = 1, 2$. If at least one of the n-spaces M_1 and M_2 is $\mathbb{K}_0$-rational, there is nothing to prove. Suppose that neither of them is $\mathbb{K}_0$-rational. Then, since F and F' are nearly $\mathbb{K}_0$-rational by assumption, there are $\mathbb{K}_0$-rational subspaces U_0 and U_0' with $p \subseteq U_0 \subset M_1 \cap M_2$ and $p' \subseteq U_0' \subset M_1 \cap M_2$. We have $\langle F, F' \rangle_{\Gamma(\mathbb{K})} = \ell_{L,M_1,M_2}$ as in (a) of (6) with $L = p + p'$. The sum $U_0 + U_0'$ is a $\mathbb{K}_0$-rational subspace of $V_{2n}(\mathbb{K})$ and contains L. If $\dim(U_0 + U_0') < n-1$ then $U_0 + U_0'$ is a $\mathbb{K}_0$-rational element of $D_n(\mathbb{K})$ incident with the flag (L, M_1, M_2), which corresponds to the line ℓ_{L,M_1,M_2}. As in (b) of Case 1, it follows that all points of ℓ_{L,M_1,M_2} are nearly $\mathbb{K}_0$-rational.

On the other hand, let $\dim(U_0 + U_0') > n-2$. Then necessarily $U_0 + U_0' = M_1 \cap M_2$. In this case $U_0 + U_0'$ is not an element of $D_n(\mathbb{K})$, but it is a $\mathbb{K}_0$-rational $(n-1)$-element of $B_n^+(\mathbb{K})$, hence an $(n-1)$-element of the subgeometry $B_n^+(\mathbb{K}_0)$ of $B_n^+(\mathbb{K})$. As such, $U_0 + U_0'$ is contained in just two n-elements N_1 and N_2 of $B_n^+(\mathbb{K}_0)$. However N_1 and N_2 also belong to $B_n^+(\mathbb{K})$. In fact, they are the unique two n-elements of $B_n^+(\mathbb{K})$ which contain $U_0 + U_0'$. On the other hand, $U_0 + U_0'$ is contained in M_1 and M_2. Therefore $\{M_1, M_2\} = \{N_1, N_2\}$. However N_1 and N_2 are

$\mathbb{K}_0$-rational. Hence M_1 and M_2 are $\mathbb{K}_0$-rational, contrary to our assumptions. We have reached a contradiction. The proof is complete, as far as the present subcase is concerned.

(b) Let $p = p'$, $M_i = M'_i$ but $M_j \neq M'_j$, for $\{i, j\} = \{1, 2\}$. To fix ideas, assume that $M_1 = M'_1$ and $M_2 \neq M'_2$. If M_1 or p are $\mathbb{K}_0$-rational there is nothing to prove. Suppose that neither M_1 nor p are $\mathbb{K}_0$-rational. Recalling that F and F' are nearly $\mathbb{K}_0$-rational, one of the following occurs:

(b1) There are $\mathbb{K}_0$-rational subspaces U_0, U'_0 of dimension at most $n-2$ such that $p \subseteq U_0 \subset M_1 \cap M_2$ and $p \subseteq U'_0 \subset M_1 \cap M'_2$.
(b2) Just one of M_2 and M'_2 is $\mathbb{K}_0$-rational. To fix ideas, let M'_2 be the $\mathbb{K}_0$-rational one. Then there exists a $\mathbb{K}_0$-rational subspace U_0 of dimension $\dim(U_0) \leq n-2$ such that $p \subseteq U_0 \subset M_1 \cap M_2$.
(b3) Both M_2 and M'_2 are $\mathbb{K}_0$-rational.

In subcases (b1) and (b2) we can consider the element $U_0 \cap U'_0$ or $U_0 \cap M'_2$ respectively. This element contains p and is $\mathbb{K}_0$-rational by Corollary 2.2. So, we get the conclusion as in (a) of Case 1. In subcase (b3), the intersection $U_0 = M_2 \cap M'_2$ is $\mathbb{K}_0$-rational by Corollary 2.2 and has dimension $\dim(U_0) = n - 2k$ for a positive integer $k < n/2$, since M_2 and M'_2 belong to the same class $\mathfrak{S}^-$. Hence $\dim(U_0) \leq n - 2$. Moreover, $U_0 \subset M_1$, since both M_2 and M'_2 are incident with M_1 in $D_n(\mathbb{K})$. Clearly, $p \subseteq U_0$. Again, the conclusion follows as in (a) of Case 1.

Case 4. $\Gamma(\mathbb{K}) = \mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. Assume firstly that $n = 3$. We have discussed this case in [6, Theorem 5.10] but we turn back to it here, using an argument different from that of [6].

By Klein correspondence, $V_{2n}(\mathbb{K}) = V_6(\mathbb{K})$ can be regarded as the exterior square of $V_4(\mathbb{K})$, with the basis $E = (e_1, ..., e_6)$ of $V_6(\mathbb{K})$, to be chosen as in Section 2.4, realized as the exterior square $E = E' \wedge E'$ of a suitable basis E' of $V_4(\mathbb{K})$. The elements of $D_3(\mathbb{K})$ of type $+$ or $-$ correspond to 1- and 3-dimensional subspaces of $V_4(\mathbb{K})$ and the 1-elements of $D_3(\mathbb{K})$ correspond to 2-subspaces of $V_4(\mathbb{K})$. By Lemma 3.2, an element of $D_3(\mathbb{K})$ is $\mathbb{K}_0$-rational with respect to E if and only if the subspace which corresponds to it in $V_4(\mathbb{K})$ is $\mathbb{K}_0$-rational with respect to E'. Accordingly, a $(+, -)$-flag of $D_3(\mathbb{K})$ is nearly $\mathbb{K}_0$-rational if and only if the corresponding $(1, 3)$-flag of $A_3(\mathbb{K})$ is nearly $\mathbb{K}_0$-rational. Thus, we are driven back to the special case $\mathrm{Gr}_{1,3}(A_3(\mathbb{K}))$ of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$, already discussed in Case 1 of this proof. It follows that $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$, as claimed.

Let now $n > 3$. Let $F = (M_1, M_2)$ and $F' = (M'_1, M'_2)$ be two distinct nearly $\mathbb{K}_0$-rational collinear points of $\Gamma(\mathbb{K})$. As F and F' are collinear, either $M_1 = M'_1$ or $M_2 = M'_2$. To fix ideas, let $M_2 = M'_2$. Hence

$$\ell_{U,M_2} \; = \; \{(M, M_2) \colon M \in \mathfrak{S}^+, M \cap M_2 \supset U\}$$

is the line of $\Gamma(\mathbb{K})$ through F and F', where $U = M_1 \cap M'_1 \subset M_2$, $\dim(U) = n - 2$ (see (4), (b)). If M_2 is $\mathbb{K}_0$-rational, there is nothing to prove. Assuming that M_2 is not $\mathbb{K}_0$-rational, there are still a number of subcases to examine.

(a) Both M_1 and M'_1 are $\mathbb{K}_0$-rational. Hence $U = M_1 \cap M'_1$ is $\mathbb{K}_0$-rational. Accordingly, every $(+, -)$-flag $(M, M_2) \in \ell_{U,M_2}$ is nearly $\mathbb{K}_0$-rational.

(b) Neither M_1 nor M'_1 are $\mathbb{K}_0$-rational. Hence there exist $\mathbb{K}_0$-rational $(n - 2)$-elements U_0 and U'_0 such that $U_0 \subset M_1 \cap M_2$ and $U'_0 \subset M'_1 \cap M_2$. If $U_0 = U'_0$ then $U_0 = M_1 \cap M'_1$. However $M_1 \cap M'_1 = U$. Hence $U = U_0$ is $\mathbb{K}_0$-rational. In this case we are done: all $(+, -)$-flags incident to U are nearly $\mathbb{K}_0$-rational.

On the other hand, let $U_0 \neq U'_0$. Then $U_0 + U'_0$ is a $\mathbb{K}_0$-rational $(n - 1)$-dimensional subspace of M_2. Being $\mathbb{K}_0$-rational, $U_0 + U'_0$ is an $(n - 1)$-element of $B^+_n(\mathbb{K}_0)$. As such, it is contained in just two n-elements of $B^+_n(\mathbb{K}_0)$. In other words, both n-elements of $B^+_n(\mathbb{K})$ containing $U_0 + U'_0$ are $\mathbb{K}_0$-rational. However M_2 is indeed one of those two elements. Therefore M_2 is $\mathbb{K}_0$-rational. This contradicts the assumptions made on M_2. Consequently, the case we have now been considering cannot occur.

(c) Just one of M_1 and M_1' is $\mathbb{K}_0$-rational. To fix ideas, let M_1 be the $\mathbb{K}_0$-rational one. As (M_1', M_2) is nearly $\mathbb{K}_0$-rational by assumption, but neither M_1' nor M_2 are $\mathbb{K}_0$-rational, there exists a $\mathbb{K}_0$-rational $(n-2)$-element $U' \subset M_1' \cap M_2$. If $U' = U$ then U is $\mathbb{K}_0$-rational and we are done.

Suppose that $U' \neq U$. Therefore $U' \not\subseteq M_1$, otherwise $U' = M_1 \cap M_1' = U$. As both M_1 and U' are $\mathbb{K}_0$-rational, their intersection $W := M_1 \cap U'$ is $\mathbb{K}_0$-rational. Note that $\dim(W) = n-3$, as one can see by noticing that $M_1 \cap U' = (M_1 \cap U') \cap U' = (M_1 \cap M_2) \cap U'$ and recaling that $M_1 \cap M_2$ is a hyperlane of M_2.

Consider the orthogonal $W^\perp$ of W with respect to the form q of Section 2.2. Taking equation (7) into account and recalling that W is $\mathbb{K}_0$-rational, we see that $W^\perp$ is a $\mathbb{K}_0$-rational vector subspace of $V_{2n}(\mathbb{K})$. In fact $W^\perp$ is the span in $V_{2n}(\mathbb{K})$ of the orthogonal $W_0^\perp \subset V_{2n,E}(\mathbb{K}_0)$ of $W_0 := W \cap V_{2n,E}(\mathbb{K}_0)$ with respect to the form q_0 induced by q on $V_{2n,E}(\mathbb{K}_0)$. All of the spaces M_1, M_1', U, U' and M_2 contain W and are totally singular, hence they are contained in $W^\perp$. Moreover, M_1 and U' are $\mathbb{K}_0$-rational.

As $W^\perp$ is $\mathbb{K}_0$-rational, we can choose a basis $B = (w_1, \dots, w_{n+3})$ of $W^\perp$ formed by $\mathbb{K}_0$-rational vectors. We can also assume that $w_1, ..., w_{n-3}$ span W. As B consists of $\mathbb{K}_0$-rational vectors, a vector subspace of $W^\perp$ is $\mathbb{K}_0$-rational with respect to B is and only if it is $\mathbb{K}_0$-rational with respect to E. Accordingly, M_1 and U' are $\mathbb{K}_0$-rational with respect to B while M_1', U and M_2 are not.

We now switch to the quotient $W^\perp/W$, taking the cosets $\bar{w}_i := w_{n-3+i} + W$ for $i = 1, 2, ..., 6$ to form a basis $\overline{B}$ of $W^\perp/W$. Since W is totally singular, the form q induces a quadratic form $\bar{q}$ on $W^\perp/W$. Let $\overline{\Delta}$ be the D_3-building associated to $\bar{q}$ and let $\overline{\Gamma}$ be its $(+,-)$-Grassmannian. So, $(M_1/W, M_2/W)$ and $(M_1'/W, M_2/W)$ are points of the line $\ell_{U/W, M_2/W}$ of $\overline{\Gamma}$, with $U/W \neq U'/W$ and $U'/W \subset M_1'/W \cap M_2/W$. By the above, M_1/W and U'/W are $\mathbb{K}_0$-rational while $M_1'/W, M_2/W$ and U/W are not. We now switch from the D_3-building $\overline{\Delta}$ to the corresponding A_3-geometry, with elements of type $+$ and $-$ realized as points and planes of $\mathrm{PG}(V_4(\mathbb{K}))$. In this new perspective, the above situation looks as follows: we have two distinct points p and p' (corresponding to M_1/W and M_1'/W), two distinct lines L and L' (corresponding to U/W and U'/W) and a plane S (corresponding to M_2/W). Both p and p' belong to L, $p' \in L'$ but $p \notin L'$. Moreover, S contains both L and L'. Hence S is spanned by p and L'. However, M_1/W and U'/W are $\mathbb{K}_0$-rational. Therefore, in view of Lemma 3.2, both p and L' are $\mathbb{K}_0$-rational with respect to a suitable basis B' of $V_4(\mathbb{K})$. Hence S is $\mathbb{K}_0$-rational with respect to B', since it is spanned by p and L'. By exploing Lemma 3.2 once again, we obtain that M_2/W is $\mathbb{K}_0$-rational with respect to $\overline{B}$. We have reached a final contradiction, which shows that the case we have been considering cannot occur. The proof is complete. □

Lemma 3.4. *Let V be a vector space over a division ring $\mathbb{K}$ and $E = (e_1, \dots, e_n)$ a basis of V. Let $\mathbb{K}_0$ be a proper sub-division ring of $\mathbb{K}$ and take $\eta \in \mathbb{K} \setminus \mathbb{K}_0$. Suppose S is a subspace of V containing $e_1 + e_2\eta$. If S is $\mathbb{K}_0$-rational (with respect to E) then $e_1, e_2 \in S$.*

Proof. Following our conventions, we assume that V is a right vector space. Let V_0 be the $\mathbb{K}_0$-vector space of the $\mathbb{K}_0$-rational vectors of V (with respect to E). In order to avoid any confusion, we denote spans in V by the symbol $\langle ... \rangle_V$ and spans in V_0 by the symbol $\langle ... \rangle_{V_0}$.

Assuming that S is $\mathbb{K}_0$-rational, let $(v_1, \dots v_k)$ be a basis of S consisting of $\mathbb{K}_0$-rational vectors and suppose that $e_1 + e_2\eta \in S$. Then $\dim(S \cap \langle e_1, e_2 \rangle_V) \geq 1$. Note that $S_0 := \langle v_1, \dots, v_k \rangle_{V_0} = S \cap V_0$ has the same dimension as S. Thus, since $\dim(S \cap \langle e_1, e_2 \rangle_V) \geq 1$, we also have $\dim(S_0 \cap \langle e_1, e_2 \rangle_{V_0}) \geq 1$ by the well known Grassmann dimension formula. It follows that there exists a non-zero vector $w \in S_0$ which is a linear combination $w = e_1c_1 + e_2c_2$ with $c_1, c_2 \in \mathbb{K}_0$ and $(c_1, c_2) \neq (0, 0)$. If either $c_1 = 0$ or $c_2 = 0$, then we are done. So, we can assume that $c_1 \neq 0 \neq c_2$. Without loss of generality, we can put $c_1 = 1$, so that $w_1 = e_1 + e_2c_2$ with $c_2 \in \mathbb{K}_0$. Now we claim that there exists $j_0 \in \{1, \dots k\}$ such that $v_{j_0} = e_1a_{1,j_0} + e_2a_{2,j_0} + \dots + e_na_{n,j_0}$ with $c_2a_{1,j_0} \neq a_{2,j_0}$. By way of contradiction, suppose that for all $v_j \in \{v_1, \dots, v_k\}$ we have

$$v_j \;=\; e_1a_{1,j} + e_2a_{2,j} + e_3a_{3,j} + \dots + e_na_{n,j}$$

with $c_2a_{1,j} = a_{2,j}$, i.e. $(a_{1,j}, a_{2,j}) = (1, c_2)d_j$ for some $d_j \in \mathbb{K}$. This implies that for all vectors $v \in S$ we have

$$v \;=\; v_1\lambda_1 + v_2\lambda_2 + \dots + v_k\lambda_k \;=\; e_1(\sum_{i=1}^{k}\lambda_i + \sum_{i=1}^{k}d_i) + e_1c_2(\sum_{i=1}^{k}\lambda_i + \sum_{i=1}^{k}d_i) + u$$

with $u \in \langle e_3, \dots, e_n\rangle, \lambda_i, d_i \in \mathbb{K}$ and $c_2 \in \mathbb{K}_0$. In particular, taking $v = e_1 + e_2\eta \in S$ we have $\sum_{i=1}^{k}\lambda_i + \sum_{i=1}^{k}d_i = 1$ and $c_2(\sum_{i=1}^{k}\lambda_i + \sum_{i=1}^{k}d_i) = \eta$, forcing $\eta = c_2 \in \mathbb{K}_0$ which is a contradiction. The claim is proved.

Consider the ordered pair (w, v_{j_0}). As $w, v_{j_0} \in S_0$, we can complete this pair to an ordered basis B of S_0 by choosing $k-2$ suitable vectors from the $k-1$ vectors in $\{v_1, \dots, v_k\} \setminus \{v_{j_0}\}$. Without getting out of V_0, we can now apply a full Gaussian reduction to the sequence of vectors of B to obtain another basis $(v'_1, \dots, v'_k)$ of S_0 such that the $(n \times k)$-matrix M of the coefficients of the vectors $v'_1, ..., v'_k$ with respect to $e_1, \dots, e_n$ is in Column Reduced Echelon Form. (Note that, according to our convention to deal with right vector spaces, vectors should be represented as columns.) By construction, the matrix M contains the identity matrix I_k as a minor. Up to a permutation of the vectors $e_3, \dots, e_n$ we can suppose that this minor encompasses the first k rows of the matrix M. The remaining $n-k$ rows form an $((n-k) \times k)$-matrix

$$N \;=\; (b_{k+i,j})_{i,j\;=1}^{n-k,k}$$

with entries $b_{k+i,j} \in \mathbb{K}_0$. However $e_1 + e_2\eta \in S = \langle S_0\rangle_V = \langle v'_1, v'_2, \dots, v'_k\rangle_V$. Hence there exist $\alpha_1, \dots, \alpha_k \in \mathbb{K}$ such that $e_1 + e_2\eta = v'_1\alpha_1 + v'_2\alpha_2 + \dots + v'_k\alpha_k$. For every $i = 1, \dots, k$ we have $v'_i = e_i + \sum_{j=k+1}^{n} e_jb_{j,i}$. Therefore

$$e_1 + e_2\eta = \sum_{i=1}^{k} e_i\alpha_i + e_{k+1}(\sum_{j=1}^{k} b_{k+1,j}\alpha_j) + e_{k+2}(\sum_{j=1}^{k} b_{k+2.j}\alpha_j) + \dots e_n(\sum_{j=1}^{k} b_{n,j}\alpha_j),$$

which implies $\alpha_1 = 1, \alpha_2 = \eta$, $\alpha_3 = \alpha_4 = \dots = \alpha_k = 0$ and

$$\sum_{j=1}^{k} b_{k+1,j}\alpha_j \;=\; \sum_{j=1}^{k} b_{k+2,j}\alpha_j \;=\; \dots \;=\; \sum_{j=1}^{k} b_{n,j}\alpha_j \;=\; 0.$$

It follows that $e_1 + e_2\eta = v'_1 + v'_2\eta$, whence $(e_1 - v'_1) = (v'_2 - e_2)\eta$. However,

$$(e_1 - v'_1) \;=\; \sum_{i=k+1}^{n} e_i(-b_{i,1}), \qquad (v'_2 - e_2) \;=\; \sum_{i=k+1}^{n} e_i(b_{i,2}\eta),$$

whence $-b_{i,1} = b_{i,2}\eta$ for all $i \geq k+1$. Since $b_{i,j} \in \mathbb{K}_0$ for all i, j and the elements $1, \eta \in \mathbb{K}$ are linearly independent over $\mathbb{K}_0$, it follows that $b_{i,1} = b_{i,2} = 0$ for all $i \geq k+1$. So, $v'_1 = e_1$ and $v'_2 = e_2$. Since $v'_1, v'_2 \in S$, we obtain $e_1, e_2 \in S$, which proves the lemma. □

Lemma 3.5. *If* $\Gamma(\mathbb{K})$ *is as in the hypotheses of Theorem* 3.1 *then not all points of* $\Gamma(\mathbb{K})$ *belong to* $\Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Proof. We first consider the case $\Gamma(\mathbb{K}) = \mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$. Pick $\eta \in \mathbb{K} \setminus \mathbb{K}_0$. With $E = (e_1, ..., e_{n+1})$ as in Section 2.3, put $p = \langle e_1 + e_2\eta\rangle$ and $H = \langle e_1 + e_2\eta, e_3, \dots, e_n, e_{n+1}\rangle$. (Needless to say, the symbol $\langle ...\rangle$ refers to spans in $V_{n+1}(\mathbb{K})$.) The flag (p, H) is a point of $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$. Let S be a subspace of $V_{n+1}(\mathbb{K})$ such that $p \subseteq S \subseteq H$. Any such subspace contains the vector $e_1 + e_2\eta$ but neither e_1 nor e_2. Hence S cannot be $\mathbb{K}_0$-rational, by Lemma 3.4. Consequently, $(p, H) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

The case $\Gamma(\mathbb{K}) = \mathrm{Gr}_{1,+,-}(D_n(\mathbb{K}))$ is entirely analogous. With $E = (e_1, e_2, ..., e_{2n})$ as in Section 2.4 and η as above, put $p = \langle e_1 + e_3\eta\rangle$, $M_1 = \langle e_1 + e_3\eta, e_2\eta - e_4, e_5, e_7, \dots, e_{2n-1}\rangle$ and $M_2 = \langle e_1 + e_3\eta, e_2\eta - e_4, e_5, e_7, \dots, e_{2n-1}\rangle$. Taking equation (7) into account, it is straighforward to see that p, M_1 and M_2 belong to $D_n(\mathbb{K})$. It is also easy to see that they form a $(1, +, -)$-flag of $D_n(\mathbb{K})$, namely a point of $\Gamma(\mathbb{K})$. Clearly, none of p, M_1 or M_2 is $\mathbb{K}_0$-rational and Lemma 3.4 implies that none of the subspaces contained in $M_1 \cap M_2$ and containing p can be $\mathbb{K}_0$-rational. Hence $(p, M_1, M_2) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

When $\Gamma(\mathbb{K}) = \mathrm{Gr}_{1,-}(D_n(\mathbb{K}))$ we can consider the flag (p, M) where $p = \langle e_1 + e_3\eta\rangle$ and $M = \langle e_1 + e_3\eta, e_2\eta - e_4, e_5, e_7, \dots, e_{2n-1}\rangle$. The subspace M is n-dimensional and totally singular for q. We can also assume to have chosen the signs $+$ and $-$ in such a way that $\mathfrak{S}^-$ is indeed the class which M belongs to. Soo, (p, M) is a point of $\Gamma(\mathbb{K})$. Once again, by Lemma 3.4 we see that $(p, M) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

Finally, let $\Gamma(\mathbb{K}) = \mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. In view of Lemma 3.2, if $n = 3$ we are back to A_3. So, assume $n > 3$. With $\eta \in \mathbb{K} \setminus \mathbb{K}_0$ and $E = (e_1, ..., e_{2n})$ as in Section 2.4, put

$$\begin{array}{lcl} M_1 & := & \langle e_1 + e_3, e_2 - e_4, e_5 + e_7\eta, e_6\eta - e_8, e_{10}, e_{12}, ..., e_{2n}\rangle, \\ M_2 & := & \langle e_1 + e_4, e_2 - e_3, e_5 + e_7\eta, e_6\eta - e_8, e_{10}, e_{12}, ..., e_{2n}\rangle. \end{array}$$

Then M_1 and M_2 are n-dimensional totally singular subspaces of $D_n(\mathbb{K})$ but neither of them is $\mathbb{K}_0$-rational. Moreover $M_1 \cap M_2 = \langle e_1 - e_2 + e_3 + e_4, e_5 + e_7\eta, e_6\eta - e_8, e_{10}, e_{12}, ..., e_{2n}\rangle$. Hence $\{M_1, M_2\}$ is a $\{+, -\}$-flag of $D_n(\mathbb{K})$, necessarily not $\mathbb{K}_0$-rational, since neither M_1 nor M_2 is $\mathbb{K}_0$-rational. Accordingly, $M_1 \cap M_2$ is not $\mathbb{K}_0$-rational. In fact all $\mathbb{K}_0$-rational subspaces of $M_1 \cap M_2$ are contained in $\langle e_1 - e_2 + e_3 + e_4, e_{10}, e_{12}, ..., e_{2n}\rangle$, which is $(n-3)$-dimensional. Their dimensions are too small for they can split $(+, -)$. Therefore $(M_1, M_2) \notin \Omega_{\mathbb{K}_0}(\Gamma(\mathbb{K}))$.

□

Lemmas 3.3 and 3.5 yield Theorem 3.1.

3.2 Proof of Corollary 1.2

As already remarked in Section 2.2, the function ι which maps every $(n-1)$-element of $B_n^+(\mathbb{K})$ onto the pair of n-elements containing it is an isomorphism from $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ to $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. We know from Theorem 1.1 that if $\mathbb{K}_0 < \mathbb{K}$ then $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}_0))$ spans a proper subspace of $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$. In order to show that the same holds for $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}_0))$ and $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$, as claimed in Corollary 1.2, we only need to prove the following:

Proposition 3.6. *The isomorphism* ι *maps the subgeometry* $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}_0))$ *of* $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}))$ *onto the subgeometry* $\mathrm{Gr}_{+.-}(D_n(\mathbb{K}_0))$ *of* $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}))$.

Proof. It goes without saying that $\mathrm{Gr}_{n-1}(B_n^+(\mathbb{K}_0)) = \mathrm{Gr}_{n-1,E}(B_n^+(\mathbb{K}_0))$ and $\mathrm{Gr}_{+,-}(D_n(\mathbb{K}_0)) = \mathrm{Gr}_{+,-,E}(D_n(\mathbb{K}_0))$ for the same basis E of $V_{2n}(\mathbb{K})$, chosen as in Section 2.4.

Let $U = M_1 \cap M_2$ for a $(+, -)$-flag (M_1, M_2) of $D_n(\mathbb{K})$. If both M_1 and M_2 are $\mathbb{K}_0$-rational then U is $\mathbb{K}_0$-rational, by Corollary 2.2. Conversely, let U be $\mathbb{K}_0$-rational. Let M_1' and M_2' be the two n-elements of $B^+_{n,E}(\mathbb{K}_0)$ containing U. Then M'_1 and M'_2 are $\mathbb{K}_0$-rational, as they belong to $B^+_{n,E}(\mathbb{K}_0)$. However, they are the only two n-elements on U. Hence $\{M'_1, M'_2\} = \{M_1, M_2\}$. Accordingly, M_1 and M_2 are $\mathbb{K}_0$-rational. $\square$

3.3 Proof of Theorem 1.3

We are not going to give a detailed proof of this theorem. We will only offer a sketch of it, leaving the details to reader.

As stated since the beginning of this section, $\mathbb{K}_0$ is a proper sub-division ring of $\mathbb{K}$ and $\Gamma(\mathbb{K}) = \mathrm{Gr}_J(X_n(\mathbb{K}))$, where X_n stands for A_n or D_n. According to the hypotheses of Theorem 1.3, we assume that J is not connected.

Suppose firstly that J contains two types j_1 and j_2, with $j_1, j_2 \leq n-2$ when $X_n = D_n$, such that $j_1 + 1 < j_2$ and $i \notin J$ for every type $i \in \{j_1+1, j_1+2, ..., j_2-1\}$. We say that a J-flag F of $X_n(\mathbb{K})$ (point of $\Gamma(\mathbb{K})$) is *nearly* $\mathbb{K}_0$*-rational at* (j_1, j_2) if there exists a $\mathbb{K}_0$-rational element X of $X_n(\mathbb{K})$ incident to F and such that $j_1 \leq \dim(X) \leq j_2$. Let $\Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K}))$ be the set of J-flags which are nearly $\mathbb{K}_0$-connected at (j_1, j_2). Using the same argument as in Case 1 of the proof of Lemma 3.3, with the roles of 1 and n respectively taken by j_1 and j_2 we see that $\Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K}))$ is a subspace of $\Gamma(\mathbb{K})$. Next, by an argument similar to that used for $\mathrm{Gr}_{1,n}(A_n(\mathbb{K}))$ in the proof of Lemma 3.5, we obtain that $\Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K})) \neq \Gamma(\mathbb{K})$, namely $\Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K}))$ is a proper subspace of $\Gamma(\mathbb{K})$. However $\Gamma(\mathbb{K}_0) := \mathrm{Gr}_J(X_n(\mathbb{K}_0))$ is contained in $\Omega_{\mathbb{K}_0,j_1,j_2}(\Gamma(\mathbb{K}))$. Hence $\Gamma(\mathbb{K}_0)$ spans a proper subspace of $\Gamma(\mathbb{K})$, as stated in Theorem 1.3.

Two more possibilities remain to examine, which are not considered in Theorem 1.1, namely $X_n(\mathbb{K}) = D_n(\mathbb{K})$ and J as follows:

$J = \{j, j+1, ..., j+k\} \cup \{+, -\}$ for $j \geq 1$, $j+k < n-2$ and either $j > 1$ or $k > 0$. In this case we can use the same arguments as for $J = \{1, +, -\}$ in the proof of Theorem 1.1, with $j+k$ playing the role of 1.

$J = \{j, j+1, ..., j+k\} \cup \{-\}$ or $J = \{j, j+1, ..., j+k\} \cup \{+\}$, for $j \geq 1$, $j+k < n-2$ and either $j > 1$ or $k > 0$. The arguments used for $J = \{1, -\}$ work for this case as well, with 1 replaced by $j+k$.

4 Proof of Lemma 1.4 and Theorem 1.5

4.1 Proof of Lemma 1.4

Assume that J is non-connected and $\mathbb{K}$ is not finitely generated. Let Sbe a finite set of points of $\Gamma(\mathbb{K}) = \mathrm{Gr}_J(X_n(\mathbb{K}))$, where X_n stands for A_n or D_n. Each element F of S is a J-flag $F = \{U_1, U_2, \dots, U_t\}$ of vector subspaces U_i of $V_N(\mathbb{K})$, where $t := |J|$ and N is $n+1$ or $2n$ according to whether X_n is A_n or D_n. Fix a basis $B_{i,F}$ for each of the vector subspaces $U_i \in F$ and each $F \in S$ and let $C(S)$ be the set of all the coordinates of the vectors of $\cup_{F\in S} \cup_{i=1}^t B_{i,F}$ with respect to a given basis of $V_N(\mathbb{K})$ (chosen as in Section 2.4 when $X_n = D_n$).

As S is finite, $C(S)$ is finite as well; in fact $|C(S)| \leq t \cdot N \cdot |S|$. Therefore, and since $\mathbb{K}$ is not finitely generated, $C(S)$ generates a proper sub-division ring $\mathbb{K}_0$ of $\mathbb{K}$. Then $\Gamma(\mathbb{K}_0) := \mathrm{Gr}_J(X_n(\mathbb{K}_0))$ spans a proper subspace of $\Gamma(\mathbb{K})$, by Theorem 1.3. Obviously, S is contained in $\Gamma(\mathbb{K}_0)$. Hence S spans a proper subspace of $\Gamma(\mathbb{K})$. Thus we have proved that no finite subset of $\Gamma(\mathbb{K})$ generates $\Gamma(\mathbb{K})$, as claimed in Lemma 1.4.

4.2 Proof of Theorem 1.5

Put $\Gamma := \mathrm{Gr}_{1,n}(A_n(\overline{\mathbb{F}}_p))$. We have $\mathrm{gr}(\Gamma) = \infty$ by Lemma 1.4, since $\overline{\mathbb{F}}_p$ is not finitely generated. The geometry Γ admits a (full) projective embedding of dimension $(n+1)^2 - 1$, namely the embedding e_{Lie} mentioned in Remark 1.6. Therefore $\mathrm{er}(\Gamma) \geq (n+1)^2 - 1$.

By way of contradiction, suppose that $\mathrm{er}(\Gamma) > (n+1)^2$. Then Γ admits a (full) projective emedding $e : \Gamma \to \mathrm{PG}(V)$ of dimension $\dim(e) \geq (n+1)^2 + 1$. Consequently, there exists a set S of $(n+1)^2 + 1$ points of Γ such that $\cup_{x \in S} e(x) \subset V$ spans a subpace V_S of V of dimension $\dim(V_S) = (n+1)^2 + 1$.

Every point $x \in S$ is a point-hyperplane flag (p_x, H_x) of $A_n(\overline{\mathbb{F}}_p)$. For every $x \in S$ we choose a non-zero vector $v_x \in p_x$ and a basis B_x of H_x. Chosen a basis E of $V_{n+1}(\overline{\mathbb{F}}_p)$, let $C(S)$ be the set of all elements of $\overline{F}_p$ which occur as coordinates (with respect to E) of either v_x or a vector of B_x, for $x \in S$. The set $C(S)$ is finite. Hence it generates a finite subfield $\mathbb{L}$ of $\overline{\mathbb{F}}_p$. Every point $x \in S$ is obviously $\mathbb{L}$-rational. Therefore $S \subset \Gamma_{\mathbb{L}} := \mathrm{Gr}_{1,n}(A_n(\mathbb{L})) \subset \Gamma$.

Let $V_{\mathbb{L}}$ be the subspace of V corresponding to the span of $e(\Gamma_{\mathbb{L}})$. Clearly $V_{\mathbb{L}} \supseteq V_S$. Hence $\dim(V_{\mathbb{L}}) \geq \dim(V_S) = (n+1)^2 + 1$. The restriction $e_{\mathbb{L}}$ of e to $\Gamma_{\mathbb{L}}$ is a lax embedding of $\Gamma_{\mathbb{L}}$ in $\mathrm{PG}(V_{\mathbb{L}})$. As noticed in Remark 1.8, inequality (1) holds for lax embeddings too. Therefore $\Gamma_{\mathbb{L}}$ has generating rank $\mathrm{gr}(\Gamma_{\mathbb{L}}) \geq \dim(e_{\mathbb{L}}) = \dim(V_{\mathbb{L}}) > (n+1)^2$.

On the other hand, the field $\mathbb{L}$ is a simple extension of the prime field $\mathbb{F}_p$ and $\mathrm{Gr}_{1,n}(A_n(\mathbb{F}_p))$ has generating rank equal to $(n+1)^2 - 1$, by Cooperstein [7]. Therefore $\mathrm{gr}(\Gamma_{\mathbb{L}}) \leq (n+1)^2$ by Blok and Pasini [2, Corollary 4.8]. We have reached a contradiction. Consequently, $\mathrm{er}(\Gamma) \leq (n+1)^2$. The proof of Theorem 1.5 is complete.

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