

On Hermitian varieties in $\text{PG}(6, q^2)$

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Abstract

In this paper we characterize the non-singular Hermitian variety $\mathcal{H}(6, q^2)$ of $\text{PG}(6, q^2)$, $q \neq 2$ among the irreducible hypersurfaces of degree $q + 1$ in $\text{PG}(6, q^2)$ not containing solids by the number of its points and the existence of a solid S meeting it in $q^4 + q^2 + 1$ points.

Keywords: unital, Hermitian variety, algebraic hypersurface.

1 Introduction

The set of all absolute points of a non-degenerate unitary polarity in $\text{PG}(r, q^2)$ determines the Hermitian variety $\mathcal{H}(r, q^2)$. This is a non-singular algebraic hypersurface of degree $q + 1$ in $\text{PG}(r, q^2)$ with a number of remarkable properties, both from the geometrical and the combinatorial point of view; see [5, 16]. In particular, $\mathcal{H}(r, q^2)$ is a 2-character set with respect to the hyperplanes of $\text{PG}(r, q^2)$ and 3-character blocking set with respect to the lines of $\text{PG}(r, q^2)$ for $r > 2$. An interesting and widely investigated problem is to provide combinatorial descriptions of $\mathcal{H}(r, q^2)$ among all hypersurfaces of the same degree.

First, we observe that a condition on the number of points and the intersection numbers with hyperplanes is not in general sufficient to characterize Hermitian varieties; see [1],[2]. On the other hand, it is enough to consider

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in addition the intersection numbers with codimension 2 subspaces in order to get a complete description; see [7].

In the present paper, we shall investigate a combinatorial characterization of the Hermitian hypersurface $\mathcal{H}(6, q^2)$ in $\text{PG}(6, q^2)$ among all hypersurfaces of the same degree having also the same number of $\text{GF}(q^2)$ -rational points.

More in detail, in [12, 13] it has been proved that if \mathcal{X} is a hypersurface of degree $q + 1$ in $\text{PG}(r, q^2)$, $r \geq 3$ odd, with $|\mathcal{X}| = |\mathcal{H}(r, q^2)| = (q^{r+1} + (-1)^r)(q^r - (-1)^r)/(q^2 - 1)$ $\text{GF}(q^2)$ -rational points, not containing linear subspaces of dimension greater than $\frac{r-1}{2}$, then \mathcal{X} is a non-singular Hermitian variety of $\text{PG}(r, q^2)$. This result generalizes the characterization of [8] for the Hermitian curve of $\text{PG}(2, q^2)$, $q \neq 2$.

The case where $r > 4$ is even is, in general, currently open. A starting point for a characterization in arbitrary even dimension can be found in [3] where the case of a hypersurface \mathcal{X} of degree $q + 1$ in $\text{PG}(4, q^2)$, $q > 3$ is considered. There, it is shown that when \mathcal{X} has the same number of points as $\mathcal{H}(4, q^2)$, does not contain any subspaces of dimension greater than 1 and meets at least one plane π in $q^2 + 1$ $\text{GF}(q^2)$ -rational points, then \mathcal{X} is a Hermitian variety.

In this article we deal with hypersurfaces of degree $q + 1$ in $\text{PG}(6, q^2)$ and we prove that a characterization similar to that of [3] holds also in dimension 6. We conjecture that this can be extended to arbitrary even dimension.

Theorem 1.1. *Let S be a hypersurface of $\text{PG}(6, q^2)$, $q > 2$, defined over $\text{GF}(q^2)$, not containing solids. If the degree of S is $q + 1$ and the number of its rational points is $q^{11} + q^9 + q^7 + q^4 + q^2 + 1$, then every solid of $\text{PG}(6, q^2)$ meets S in at least $q^4 + q^2 + 1$ rational points. If there is at least a solid Σ_3 such that $|\Sigma_3 \cap S| = q^4 + q^2 + 1$, then S is a non-singular Hermitian variety of $\text{PG}(6, q^2)$.*

Furthermore, we also extend the result obtained in [3] to the case $q = 3$.

2 Preliminaries and notation

In this section we collect some useful information and results that will be crucial to obtain our result.

A Hermitian variety in $\text{PG}(r, q^2)$ is the algebraic variety of $\text{PG}(r, q^2)$ whose points $\langle v \rangle$ satisfy the equation $\eta(v, v) = 0$ where η is a unitary form $\text{GF}(q^2)^{r+1} \times \text{GF}(q^2)^{r+1} \rightarrow \text{GF}(q^2)$. The *radical* of the form η is the vector

subspace of $\text{GF}(q^2)^{r+1}$ given by

$$\text{Rad}(\eta) := \{w \in \text{GF}(q^2)^{r+1} : \forall v \in \text{GF}(q^2)^{r+1}, \eta(v, w) = 0\}.$$

The form η is non-degenerate if $\text{Rad}(\eta) = \{\mathbf{0}\}$. If the form η is non-degenerate, then the corresponding Hermitian variety is denoted by $\mathcal{H}(r, q^2)$ and it is a non-singular, of degree $q + 1$ and contains

$$(q^{r+1} + (-1)^r)(q^r - (-1)^r)/(q^2 - 1)$$

$\text{GF}(q^2)$ -rational points. When η is degenerate we shall call *vertex* R_t of the degenerate Hermitian variety associated to η the projective subspace $R_t := \text{PG}(\text{Rad}(\eta))$ of $\text{PG}(r, q^2)$. A degenerate Hermitian variety can always be described as a cone of vertex R_t and basis a non-degenerate Hermitian variety $\mathcal{H}(r - t, q^2)$ disjoint from R_t where $t = \dim(\text{Rad}(\eta))$ is the vector dimension of the radical of η . In this case we shall write the corresponding variety as $R_t\mathcal{H}(r - t, q^2)$. Indeed,

$$R_t\mathcal{H}(r - t, q^2) := \{P \in \langle P, Q \rangle : P \in R_t, Q \in \mathcal{H}(r - t, q^2)\}.$$

Any line of $\text{PG}(r, q^2)$ meets a Hermitian variety (either degenerate or not) in either 1, $q + 1$ or $q^2 + 1$ points (the latter value only for $r > 2$). The maximal dimension of projective subspaces contained in the non-degenerate Hermitian variety $\mathcal{H}(r, q^2)$ is $(r - 2)/2$, if r is even, or $(r - 1)/2$, if r is odd. These subspaces of maximal dimension are called *generators* of $\mathcal{H}(r, q^2)$ and the generators of $\mathcal{H}(r, q^2)$ through a point P of $\mathcal{H}(r, q^2)$ span a hyperplane P^\perp of $\text{PG}(r, q^2)$, the *tangent hyperplane* at P .

It is well known that this hyperplane meets $\mathcal{H}(r, q^2)$ in a degenerate Hermitian variety $P\mathcal{H}(r - 2, q^2)$, that is in a Hermitian cone having as vertex the point P and as base a non-singular Hermitian variety of $\Theta \cong \text{PG}(r - 2, q^2)$ contained in P^\perp with $P \notin \Theta$.

Every hyperplane of $\text{PG}(r, q^2)$, which is not tangent, meets $\mathcal{H}(r, q^2)$ in a non-singular Hermitian variety $\mathcal{H}(r - 1, q^2)$, and is called a *secant hyperplane* of $\mathcal{H}(r, q^2)$. In particular, a tangent hyperplane contains

$$1 + q^2(q^{r-1} + (-1)^r)(q^{r-2} - (-1)^r)/(q^2 - 1)$$

$\text{GF}(q^2)$ -rational points of $\mathcal{H}(r, q^2)$, whereas a secant hyperplane contains

$$(q^r + (-1)^{r-1})(q^{r-1} - (-1)^{r-1})/(q^2 - 1)$$

$\text{GF}(q^2)$ -rational points of $\mathcal{H}(r, q^2)$.

Throughout this paper, the number of $\text{GF}(q^i)$ -points of an algebraic set \mathcal{X} will be denoted by $N_{q^i}(\mathcal{X})$. For simplicity, we shall also use the convention $|\mathcal{X}| := N_{q^2}(\mathcal{X})$.

We now recall several results which we will make use of in the course of this paper.

Lemma 2.1 ([15]). *Let d be an integer with $1 \leq d \leq q+1$ and \mathcal{C} be a curve of degree d in $\text{PG}(2, q)$ defined over $\text{GF}(q)$, which may have $\text{GF}(q)$ -linear components. Then the number of its rational points is at most $dq+1$ and $N_q(\mathcal{C}) = dq+1$ if and only if \mathcal{C} is a pencil of d lines of $\text{PG}(2, q)$.*

Lemma 2.2 ([10]). *Let d be an integer with $2 \leq d \leq q+2$, and \mathcal{C} a curve of degree d in $\text{PG}(2, q)$ defined over $\text{GF}(q)$ without any $\text{GF}(q)$ -linear components. Then $N_q(\mathcal{C}) \leq (d-1)q+1$, except for a class of plane curves of degree 4 over $\text{GF}(4)$ having 14 rational points.*

Lemma 2.3 ([11]). *Let \mathcal{S} be a surface of degree d in $\text{PG}(3, q)$ over $\text{GF}(q)$. Then*

$$N_q(\mathcal{S}) \leq dq^2 + q + 1$$

Lemma 2.4 ([8]). *Suppose $q \neq 2$. Let \mathcal{C} be a plane curve over $\text{GF}(q^2)$ of degree $q+1$ without $\text{GF}(q^2)$ -linear components. If \mathcal{C} has q^3+1 rational points, then \mathcal{C} is a Hermitian curve.*

Lemma 2.5 ([7]). *A subset of points of $\text{PG}(r, q^2)$ having the same intersection numbers with respect to hyperplanes and spaces of codimension 2 as non-singular Hermitian varieties, is a non-singular Hermitian variety of $\text{PG}(r, q^2)$.*

From [9, Th 23.5.1, Th 23.5.3] we have the following.

Lemma 2.6. *If \mathcal{W} is a set of $q^7 + q^4 + q^2 + 1$ points of $\text{PG}(4, q^2)$, $q > 2$, such that every line of $\text{PG}(4, q^2)$ meets \mathcal{W} in $1, q+1$ or q^2+1 points, then \mathcal{W} is a Hermitian cone with vertex a line and base a unital.*

Finally, we recall that a *blocking set with respect to lines* of $\text{PG}(r, q)$ is a point set which blocks the lines, i.e., intersects each line of $\text{PG}(r, q)$ in at least one point.

3 Proof of Theorem 1.1

We first provide an estimate on the number of points of a curve of degree $q+1$ in $\text{PG}(2, q^2)$, where q is any prime power.

Lemma 3.1. *Let \mathcal{C} be a plane curve over $\text{GF}(q^2)$, without $\text{GF}(q^2)$ -lines as components and of degree $q + 1$. If the number of $\text{GF}(q^2)$ -rational points of \mathcal{C} is $N < q^3 + 1$, then*

$$N \leq \begin{cases} q^3 - (q^2 - 2) & \text{if } q > 3 \\ 24 & \text{if } q = 3 \\ 8 & \text{if } q = 2. \end{cases} \quad (3.1)$$

Proof. We distinguish the following three cases:

- (a) \mathcal{C} has two or more $\text{GF}(q^2)$ -components;
- (b) \mathcal{C} is irreducible over $\text{GF}(q^2)$, but not absolutely irreducible;
- (c) \mathcal{C} is absolutely irreducible.

Suppose first $q \neq 2$.

Case (a) Suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Let d_i be the degree of \mathcal{C}_i , for each $i = 1, 2$. Hence $d_1 + d_2 = q + 1$. By Lemma 2.2,

$$N \leq N_{q^2}(\mathcal{C}_1) + N_{q^2}(\mathcal{C}_2) \leq [(q + 1) - 2]q^2 + 2 = q^3 - (q^2 - 2)$$

Case (b) Let \mathcal{C}' be an irreducible component of \mathcal{C} over the algebraic closure of $\text{GF}(q^2)$. Let $\text{GF}(q^{2t})$ be the minimum defining field of \mathcal{C}' and σ be the Frobenius morphism of $\text{GF}(q^{2t})$ over $\text{GF}(q^2)$. Then

$$\mathcal{C} = \mathcal{C}' \cup \mathcal{C}'^\sigma \cup \mathcal{C}'^{\sigma^2} \cup \dots \cup \mathcal{C}'^{\sigma^{t-1}},$$

and the degree of \mathcal{C}' , say e , satisfies $q + 1 = te$ with $e > 1$. Hence any $\text{GF}(q^2)$ -rational point of \mathcal{C} is contained in $\cap_{i=0}^{t-1} \mathcal{C}'^{\sigma^i}$. In particular, $N \leq e^2 \leq (\frac{q+1}{2})^2$ by Bezout's Theorem and $(\frac{q+1}{2})^2 < q^3 - (q^2 - 2)$.

Case (c) Let \mathcal{C} be an absolutely irreducible curve over $\text{GF}(q^2)$ of degree $q + 1$. Either \mathcal{C} has a singular point or not. Since $N < q^3 + 1$, the curve \mathcal{C} is Frobenius classical from [6, Corollary 1.4]. Therefore, we can apply the Stöhr and Voloch's estimate for the number of rational points of a Frobenius classical curve [17, Theorem 1.1] obtaining

$$N \leq \frac{1}{2}q(q + 1)^2.$$

This bound is lower than the estimation for N in case (a) for $q > 4$ and it is the same for $q = 4$. When $q = 3$ the bound in case (a) is lower than the Stöhr and Voloch's bound.

Finally, we consider the case $q = 2$. Under this assumption, \mathcal{C} is a cubic

curve and neither case (a) nor case (b) might occur. For a degree 3 curve over $\text{GF}(q^2)$ the Stöhr and Voloch's bound is loose, thus we need to change our argument. If \mathcal{C} has a singular point, then \mathcal{C} is a rational curve with a unique singular point. Since the degree of \mathcal{C} is 3, singular points are either cusps or ordinary double points. Hence $N \in \{4, 5, 6\}$. If \mathcal{C} is nonsingular, then it is an elliptic curve and, by the Hasse-Weil bound, see [18], $N \in I$ where $I = \{1, 2, \dots, 9\}$ and for each number N belonging to I there is an elliptic curve over $\text{GF}(4)$ with N points, from [14, Theorem 4.2]. This completes the proof. \square

From now henceforth, we shall always suppose $q > 2$ and we denote by \mathcal{S} a hypersurface of $\text{PG}(6, q^2)$ satisfying the following hypotheses of Theorem 1.1:

- (S1) \mathcal{S} is an algebraic hypersurface of degree $q + 1$ defined over $\text{GF}(q^2)$;
- (S2) $|\mathcal{S}| = q^{11} + q^9 + q^7 + q^4 + q^2 + 1$;
- (S3) \mathcal{S} does not contain projective 3-spaces (solids);
- (S4) there exists a solid Σ_3 such that $|\mathcal{S} \cap \Sigma_3| = q^4 + q^2 + 1$.

We are first going to prove that \mathcal{S} is a blocking set of lines.

Lemma 3.2. *An algebraic hypersurface \mathcal{T} of degree $q+1$ in $\text{PG}(r, q^2)$, $q \neq 2$, with $|\mathcal{T}| = |\mathcal{H}(r, q^2)|$ is a blocking set with respect to lines of $\text{PG}(r, q^2)$*

Proof. Suppose on the contrary that there is a line ℓ of $\text{PG}(r, q^2)$ which is disjoint from \mathcal{T} . Let α be a plane containing ℓ . The algebraic plane curve $\mathcal{C} = \alpha \cap \mathcal{T}$ of degree $q + 1$ cannot have $\text{GF}(q^2)$ -linear components and hence it has at most $q^3 + 1$ points because of Lemma 2.2. If \mathcal{C} had $q^3 + 1$ rational points, then from Lemma 2.4, \mathcal{C} would be a Hermitian curve with an external line, a contradiction since Hermitian curves are blocking sets. Thus $N_{q^2}(\mathcal{C}) \leq q^3$. Since $q > 2$, by Lemma 3.1, $N_{q^2}(\mathcal{C}) < q^3 - 1$ and hence every plane through r meets \mathcal{T} in at most $q^3 - 1$ rational points. Consequently, by considering all planes through r , we can bound the number of rational points of \mathcal{T} by $N_{q^2}(\mathcal{T}) \leq (q^3 - 1) \frac{q^{2r-4}-1}{q^2-1} = q^{2r-3} + \dots < |\mathcal{H}(r, q^2)|$, which is a contradiction. Therefore there are no external lines to \mathcal{T} and so \mathcal{T} is a blocking set w.r.t. lines of $\text{PG}(r, q^2)$. \square

Remark 3.3. The proof of [3, Lemma 3.1] would work perfectly well here under the hypothesis $q > 3$. The alternative argument of Lemma 3.2 is simpler and also holds for $q = 3$.

By the previous Lemma and assumptions (S1) and (S2), \mathcal{S} is a blocking set for the lines of $\text{PG}(6, q^2)$. In particular, the intersection of \mathcal{S} with any 3-dimensional subspace Σ of $\text{PG}(6, q^2)$ is also a blocking set with respect to lines of Σ and hence it contains at least $q^4 + q^2 + 1$ $\text{GF}(q^2)$ -rational points; see [4].

Lemma 3.4. *Let Σ_3 be the solid of $\text{PG}(6, q^2)$ satisfying condition (S4), that is Σ_3 meets \mathcal{S} in exactly $q^4 + q^2 + 1$ points. Then, $\Pi := \mathcal{S} \cap \Sigma_3$ is a plane.*

Proof. $\mathcal{S} \cap \Sigma_3$ must be a blocking set for the lines of $\text{PG}(3, q^2)$; also it has size $q^4 + q^2 + 1$. It follows from [4] that $\Pi := \mathcal{S} \cap \Sigma_3$ is a plane. \square

Lemma 3.5. *Let Σ_3 be the solid of condition (S4). Then, any 4-dimensional projective space Σ_4 through Σ_3 meets \mathcal{S} in a Hermitian cone with vertex a line ℓ_1 and basis a Hermitian curve.*

Proof. Consider all of the $q^6 + q^4 + q^2 + 1$ subspaces Σ'_3 of dimension 3 in $\text{PG}(6, q^2)$ containing Π .

From Lemma 2.3 and condition (S3) we have $|\Sigma'_3 \cap \mathcal{S}| \leq q^5 + q^4 + q^2 + 1$. Hence,

$$|\mathcal{S}| = (q^7 + 1)(q^4 + q^2 + 1) \leq (q^6 + q^4 + q^2)q^5 + q^4 + q^2 + 1 = |\mathcal{S}|$$

Consequently, $|\Sigma'_3 \cap \mathcal{S}| = q^5 + q^4 + q^2 + 1$ for all $\Sigma'_3 \neq \Sigma_3$ such that $\Pi \subset \Sigma'_3$.

Let $C_1 := \Sigma_4 \cap \mathcal{S}$. Counting the number of rational points of C_1 by considering the intersections with the $q^2 + 1$ subspaces of dimension 3 in Σ_4 containing the plane Π we get

$$|C_1| = q^2 \cdot q^5 + q^4 + q^2 + 1 = q^7 + q^4 + q^2 + 1.$$

In particular, $C_1 \cap \Sigma'_3$ is a maximal surface of degree $q + 1$; so it must split in $q + 1$ distinct planes through a line. So C_1 consists of $q^3 + 1$ distinct planes; denote by \mathcal{L} the family of these planes. Also for each $\Sigma'_3 \neq \Sigma_3$, there is a line ℓ' such that all the planes of \mathcal{L} in Σ'_3 pass through ℓ' . It is now straightforward to see that any line contained in C_1 must necessarily belong to one of the planes of \mathcal{L} and no plane not in \mathcal{L} is contained in C_1 .

In order to get the result it is now enough to show that a line of Σ_4 meets C_1 in either 1, $q + 1$ or $q^2 + 1$ points. To this purpose, let ℓ be a line of Σ_4 and suppose $\ell \not\subseteq C_1$. Then, by Bezout's theorem,

$$1 \leq |\ell \cap C_1| \leq q + 1.$$

Assume $|\ell \cap C_1| > 1$. Then we can distinguish two cases:

1. $\ell \cap \Pi \neq \emptyset$. If ℓ and Π are incident, then we can consider the 3-dimensional subspace $\Sigma'_3 := \langle \ell, \Pi \rangle$. Then ℓ must meet each plane of \mathcal{L} in Σ'_3 in different points (otherwise ℓ passes through the intersection of these planes and then $|\ell \cap C_1| = 1$). As there are $q + 1$ planes of \mathcal{L} in Σ'_3 , we have $|\ell \cap C_1| = q + 1$.
2. $\ell \cap \Pi = \emptyset$. Consider the plane Λ generated by a point $P \in \Pi$ and ℓ . Clearly $\Lambda \notin \mathcal{L}$. The curve $\Lambda \cap S$ has degree $q + 1$ by construction, does not contain lines (for otherwise $\Lambda \in \mathcal{L}$) and has $q^3 + 1$ $\text{GF}(q^2)$ -rational points (by a counting argument). So it is a Hermitian curve. It follows that ℓ is a $q + 1$ secant.

We can now apply Lemma 2.6 to see that C_1 is a Hermitian cone. \square

Lemma 3.6. *Let Σ_3 be the space of condition (S_4) and take Σ_5 to be a 5-dimensional projective space with $\Sigma_3 \subseteq \Sigma_5$. Then $S \cap \Sigma_5$ is a Hermitian cone with vertex a point and basis a Hermitian hypersurface $\mathcal{H}(4, q^2)$.*

Proof. Let

$$\Sigma_4 := \Sigma_4^1, \Sigma_4^2, \dots, \Sigma_4^{q^2+1}$$

be the 4-spaces through Σ_3 contained in Σ_5 . Put $C_1 := \Sigma_4^1 \cap S$ and $\Pi = \Sigma_3 \cap C$; clearly $\Pi \subseteq \Sigma_3 \subseteq \Sigma_4^1$ and Π is a plane. Choose a plane $\Pi' \subseteq \Sigma_4^1$ such that $m := \Pi' \cap C_1$ is a line m incident with Π but not contained in it. Let $P_1 := m \cap \Pi$. It is straightforward to see that in Σ_4^1 there is exactly 1 plane through m which is $(q^4 + q^2 + 1)$ -secant, q^4 planes which are $(q^3 + q^2 + 1)$ -secant and q^2 planes which are $(q^2 + 1)$ -secant. Also P_1 belongs to the line ℓ_1 , the vertex of C_1 . There are now two cases to consider:

- (a) There is a plane $\Pi'' \neq \Pi'$ not contained in Σ_4^i for all $i = 1, \dots, q^2 + 1$ with $m \subseteq \Pi'' \subseteq S \cap \Sigma_5$.

We first show that the vertices of the cones $C_i := \Sigma_4^i \cap S$ are all collinear. Consider $m_i := \Pi'' \cap \Sigma_4^i$. Then $\{m_i : i = 1, \dots, q^2 + 1\}$ consists of $q^2 + 1$ lines (including m) all through P_1 . Observe that for all i , the line m_i meets the vertex ℓ_i of the cone C_i in $P_i \in \Pi$. This forces $P_1 = P_2 = \dots = P_{q^2+1}$. So $P_1 \in \ell_1, \dots, \ell_{q^2+1}$.

Let now $\bar{\Sigma}_4$ be a 4-dimensional space in Σ_5 with $P_1 \notin \bar{\Sigma}_4$; in particular $\Pi \not\subseteq \bar{\Sigma}_4$. Put also $\bar{\Sigma}_3 := \Sigma_4^1 \cap \bar{\Sigma}_4$. Clearly, $r := \bar{\Sigma}_3 \cap \Pi$ is a line and $P_1 \notin r$. So $\bar{\Sigma}_3 \cap S$ cannot be the union of $q + 1$ planes, since if this were to be the case, these planes would have to pass through the vertex ℓ_1 . It follows that $\bar{\Sigma}_3 \cap S$ must be a Hermitian cone with vertex a point

and basis a Hermitian curve. Counting the points of $\mathcal{W} := \overline{\Sigma}_4 \cap \mathcal{S}$ by considering $\mathcal{W} \cap \Sigma_4^i$ as i varies, we get

$$|\mathcal{W}| = (q^2 + 1)q^5 + q^2 + 1 = (q^2 + 1)(q^5 + 1);$$

in particular, \mathcal{W} is a hypersurface of $\overline{\Sigma}_4$ of degree $q + 1$, not containing any plane and such that there exists a plane of $\overline{\Sigma}_4$ meeting \mathcal{W} in just one line (such planes exist in $\overline{\Sigma}_3$). So by the characterization of $\mathcal{H}(4, q^2)$ of [3] we have that \mathcal{W} is a Hermitian variety $\mathcal{H}(4, q^2)$.

We also have that $|\mathcal{S} \cap \Sigma_5| = |P_1 \mathcal{H}(4, q^2)|$. Let now r be any line of $\mathcal{H}(4, q^2) = \mathcal{S} \cap \overline{\Sigma}_4$ and let Θ be the plane $\langle r, P_1 \rangle$. The plane Θ meets Σ_4^i in a line $q_i \subseteq \mathcal{S}$ for each $i = 1, \dots, q^2 + 1$ and these lines are concurrent in P_1 . It follows that all the points of Θ are in \mathcal{S} . This completes the proof for the current case and shows that $\mathcal{S} \cap \Sigma_5$ is a Hermitian cone $P_1 \mathcal{H}(4, q^2)$.

- (b) All planes Π'' with $m \subseteq \Pi'' \subseteq \mathcal{S} \cap \Sigma_5$ are contained in Σ_4^i for some $i = 1, \dots, q^2 + 1$. We claim that this case cannot happen. We can suppose without loss of generality $m_1 \cap \ell_1 = P_1$ and $P_1 \notin \ell_i$ for all $i = 2, \dots, q^2 + 1$. So there is exactly one plane through m_1 which is $(q^4 + q^2 + 1)$ -secant, namely the plane $\langle \ell_i, m_1 \rangle$. Furthermore, in Σ_4^1 there are q^4 planes through m_1 which are $(q^3 + q^2 + 1)$ -secant and q^2 planes which are $(q^2 + 1)$ -secant. We can provide an upper bound to the points of $\mathcal{S} \cap \Sigma_5$ by counting the number of points of $\mathcal{S} \cap \Sigma_5$ on planes in Σ_5 through m_1 and observing that a plane through m_1 not in Σ_5 and not contained in \mathcal{S} has at most $q^3 + q^2 + 1$ points in common with $\mathcal{S} \cap \Sigma_5$. So

$$|\mathcal{S} \cap \Sigma_5| \leq q^6 \cdot q^3 + q^7 + q^4 + q^2 + 1.$$

As $|\mathcal{S} \cap \Sigma_5| = q^9 + q^7 + q^4 + q^2 + 1$, all planes through m_1 which are neither $(q^4 + q^2 + 1)$ -secant nor $(q^2 + 1)$ -secant are $(q^3 + q^2 + 1)$ -secant. That is to say that all of these planes meet \mathcal{S} in a curve of degree $q + 1$ which must split into $q + 1$ lines through a point because of Lemma 2.1.

Take now $P_2 \in \Sigma_4^2 \cap \mathcal{S}$ and consider the plane $\Xi := \langle m_1, P_2 \rangle$. The line $\langle P_1, P_2 \rangle$ is contained in Σ_4^2 ; so it must be a $(q + 1)$ -secant, as it does not meet the vertex line ℓ_2 of C_2 in Σ_4^2 . Now, Ξ meets every of Σ_4^i for $i = 2, \dots, q^2 + 1$ in a line through P_1 which is either a 1-secant or a $q + 1$ -secant; so

$$|\mathcal{S} \cap \Xi| \leq q^2(q) + q^2 + 1 = q^3 + q^2 + 1.$$

It follows $|\mathcal{S} \cap \Xi| = q^3 + q^2 + 1$ and $\mathcal{S} \cap \Xi$ is a set of $q + 1$ lines all through the point P_1 . This contradicts our previous construction. □

Lemma 3.7. *Let Σ_3 be the space of condition (S_4) . Then, every hyperplane of $\text{PG}(6, q^2)$ meets \mathcal{S} either in a non-singular Hermitian variety $\mathcal{H}(5, q^2)$ or in a cone over a Hermitian hypersurface $\mathcal{H}(4, q^2)$.*

Proof. Let us denote by Λ a hyperplane of $\text{PG}(6, q^2)$. If Λ contains Σ_3 then, from Lemma 3.6 it follows that $\Lambda \cap \mathcal{S}$ is a Hermitian cone $P\mathcal{H}(4, q^2)$.

Now assume that Λ does not contain Σ_3 . Denote by S_5^j , with $j = 1, \dots, q^2 + 1$ the $q^2 + 1$ hyperplanes through Σ_4^1 , where as before, Σ_4^1 is a 4-space containing Σ_3 . By Lemma 3.6 again we get that $S_5^j \cap \mathcal{S} = P^j\mathcal{H}(4, q^2)$. We count the number of rational points of $\Lambda \cap \mathcal{S}$ by studying the intersections of $S_5^j \cap \mathcal{S}$ with Λ for all $j \in \{1, \dots, q^2 + 1\}$. Setting $\mathcal{W}_j := S_5^j \cap \mathcal{S} \cap \Lambda$, $\Omega := \Sigma_4^1 \cap \mathcal{S} \cap \Lambda$ then

$$|\mathcal{S} \cap \Lambda| = \sum_j |\mathcal{W}_j \setminus \Omega| + |\Omega|.$$

If Π is a plane of Λ then Ω consists of $q + 1$ collinear planes. Otherwise let m be the line in which Λ meets the plane Π . Then Ω is either a Hermitian cone $P_0\mathcal{H}(2, q^2)$, or $q + 1$ collinear planes according as the vertex $P^j \in \Pi$ is an external point with respect to m or not.

In the former case \mathcal{W}_j is a non singular Hermitian variety $\mathcal{H}(4, q^2)$ and thus $|\mathcal{S} \cap \Lambda| = (q^2 + 1)(q^7) + q^5 + q^2 + 1 = q^9 + q^7 + q^5 + q^2 + 1$.

In the case in which Ω consists of $q + 1$ collinear planes then \mathcal{W}_j is either a $P_0\mathcal{H}(3, q^2)$ or a Hermitian cone with vertex a line and basis a Hermitian curve $\mathcal{H}(2, q^2)$.

If there is at least one index j such that $\mathcal{W}_j = \ell_1\mathcal{H}(2, q^2)$ then, there must be a 3-dimensional space Σ_3^j of $S_5^j \cap \Lambda$ meeting \mathcal{S} in a generator. Hence, from Lemma 3.6 we get that $\mathcal{S} \cap \Lambda$ is a Hermitian cone $P'\mathcal{H}(4, q^2)$.

Assume that for all $j \in \{1, \dots, q^2 + 1\}$, \mathcal{W}_j is a $P_0\mathcal{H}(3, q^2)$. In this case $|\mathcal{S} \cap \Lambda| = (q^2 + 1)q^7 + (q + 1)q^4 + q^2 + 1 = q^9 + q^7 + q^5 + q^4 + q^2 + 1 = |\mathcal{H}(5, q^2)|$.

We are going to prove that the intersection numbers of \mathcal{S} with hyperplanes are only two that is $q^9 + q^7 + q^5 + q^4 + q^2 + 1$ or $q^9 + q^7 + q^4 + q^2 + 1$.

Denote by x_i the number of hyperplanes meeting \mathcal{S} in i rational points with $i \in \{q^9 + q^7 + q^4 + q^2 + 1, q^9 + q^7 + q^5 + q^2 + 1, q^9 + q^7 + q^5 + q^4 + q^2 + 1\}$.

Double counting arguments give the following equations for the integers x_i :

$$\begin{cases} \sum_i x_i = q^{12} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1 \\ \sum_i ix_i = |\mathcal{S}|(q^{10} + q^8 + q^6 + q^4 + q^2 + 1) \\ \sum_{i=1} i(i-1)x_i = |\mathcal{S}|(|\mathcal{S}| - 1)(q^8 + q^6 + q^4 + q^2 + 1). \end{cases} \quad (3.2)$$

Solving (3.2) we obtain $x_{q^9+q^7+q^5+q^2+1} = 0$. In the case in which $|\mathcal{S} \cap I| = |\mathcal{H}(5, q^2)|$, since $\mathcal{S} \cap \Lambda$ is an algebraic hypersurface of degree $q + 1$ not containing 3-spaces, from [18, Theorem 4.1] we get that $\mathcal{S} \cap \Lambda$ is a Hermitian variety $\mathcal{H}(5, q^2)$ and this completes the proof. \square

Proof of Theorem 1.1. The first part of Theorem 1.1 follows from Lemma 3.4. From Lemma 3.7, \mathcal{S} has the same intersection numbers with respect to hyperplanes and 4-spaces as a non-singular Hermitian variety of $\text{PG}(6, q^2)$, hence Lemma 2.5 applies and \mathcal{S} turns out to be a $\mathcal{H}(6, q^2)$. \square

Remark 3.8. The characterization of the non-singular Hermitian variety $\mathcal{H}(4, q^2)$ provides in [3] is based on the property that a given hypersurface is a blocking set with respect to lines of $\text{PG}(4, q^2)$, see [3, Lemma 3.1]. This lemma holds when $q > 3$. Since Lemma 3.2 extends the same property to the case $q = 3$ it follows that the result stated in [3] is also valid in $\text{PG}(4, 3^2)$.

4 Conjecture

We propose a conjecture for the general $2n$ -dimensional case.

Let \mathcal{S} be a hypersurface of $\text{PG}(2d, q^2)$, $q > 2$, defined over $\text{GF}(q^2)$, not containing d -dimensional projective subspaces. If the degree of \mathcal{S} is $q + 1$ and the number of its rational points is $|\mathcal{H}(2d, q^2)|$, then every d -dimensional subspace of $\text{PG}(2d, q^2)$ meets \mathcal{S} in at least $\theta_{q^2}(d - 1) := (q^{2d-2} - 1)/(q^2 - 1)$ rational points. If there is at least a d -dimensional subspace Σ_d such that $|\Sigma_d \cap \mathcal{S}| = |\text{PG}(d - 1, q^2)|$, then \mathcal{S} is a non-singular Hermitian variety of $\text{PG}(2d, q^2)$.

Lemma 3.1 and Lemma 3.2 can be a starting point for the proof of this conjecture since from them we get that \mathcal{S} is a blocking set with respect to lines of $\text{PG}(2d, q^2)$.

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