Result.Math. 49 (2006), 81–88
1422-6383/010081-8, DOI 10.1007/s00025-006-0210-8

Results in Mathematics

A Geometric Construction for Some Ovoids of the Hermitian Surface

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Abstract. Multiple derivation of the classical ovoid of the Hermitian surface $\mathcal{H}(3, q^2)$ of $PG(3, q^2)$ is a well known, powerful method for constructing large families of non classical ovoids of $\mathcal{H}(3, q^2)$. In this paper, we shall provide a geometric costruction of a family of ovoids amenable to multiple derivation.

Mathematics Subject Classification (2000). Primary 51E20; Secondary 51A50.

Keywords. Ovoid, Hermitian surface, derivation, polar space, automorphism group.

1. Introduction

A generator of the non–degenerate Hermitian surface $\mathcal{H}(3, q^2)$ of $PG(3, q)$ is a line of $PG(3, q^2)$ fully contained in $\mathcal{H}(3, q^2)$. An *ovoid* $\mathcal{O}$ of $\mathcal{H}(3, q^2)$ is a set of q^3+1 points of $\mathcal{H}(3, q^2)$ meeting each generator of the surface in exactly one point. The intersection of $\mathcal{H}(3, q^2)$ with a non–tangent plane is an ovoid, the so-called *classical* ovoid of $\mathcal{H}(3, q^2)$. Existence of non–classical ovoids of $\mathcal{H}(3, q^2)$ has been known since 1994, see [10]. However, a thorough work on the subject has begun only recently, prompted by the discovery of new large families [2, 4].

The non–classical ovoids in [10] have been constructed using a classical idea, originally introduced in the context of finite translation planes, namely that of deriving a new incidence structure from an old one by partial replacement.

The procedure, *derivation*, is as follows. Consider the classical ovoid $\mathcal{O}$ of $\mathcal{H}(3, q^2)$, cut out on $\mathcal{H}(3, q^2)$ by a non–tangent plane π. Given any $(q + 1)$–secant ℓ of $\mathcal{O}$ in π, that is a line meeting $\mathcal{O}$ in $q+1$ points, denote by ℓ' its polar line with respect to the unitary polarity associated to $\mathcal{H}(3, q^2)$. It is now possible to replace the points $\mathcal{O}$ and ℓ have in common by the points of $\mathcal{H}(3, q^2) \cap \ell'$. The resulting

The present research was performed within the activity of G.N.S.A.G.A. of the Italian INDAM with the financial support of the Italian Ministry M.I.U.R., project "Strutture geometriche, combinatorica e loro applicazioni", 2005–06.

set $\mathcal{O}_\ell$ is still an ovoid of $\mathcal{H}(3,q^2)$. A straightforward generalisation of this idea is to replace more than one $(q+1)$–secant of $\mathcal{O}$, each by its own polar line. This procedure, called *multiple derivation*, provides an ovoid of $\mathcal{H}(3,q^2)$ as long as no two of the chosen $(q+1)$–secants meet in a point of $\mathcal{O}$.

It is not essential for the procedure of (multiple) derivation to work to assume the starting ovoid $\mathcal{O}$ to be classical, as far as $\mathcal{O}$ has some $(q+1)$–secants with good properties. In fact, the non–classical ovoids of [2, 4] are multiply derivable. When the replacement of $\mathcal{O}\cap\ell$ by $\mathcal{H}\cap\ell'$, as described above, is an ovoid $\mathcal{O}_\ell$, we say that $\mathcal{O}_\ell$ is the *derived ovoid* of $\mathcal{O}$, by its *replaceable* $(q+1)$*–secant* ℓ. More generally, given a set $L=\{\ell_1,\dots,\ell_k\}$ of $(q+1)$–secants of $\mathcal{O}$, write

$$\mathcal{O}_L=\left(\mathcal{O}\setminus\left(\bigcup_{\ell_i\in L}\ell_i\right)\right)\cup\bigcup_{\ell_i\in L}(\mathcal{H}\cap\ell_i').$$

If $\mathcal{O}_L$ is still an ovoid, then the set L is *replaceable.* Clearly, the existence and nature of replaceable sets depends heavily on the nature of $\mathcal{O}$. The ovoids found in [4] are multiply derivable.

In this paper, we shall provide a geometric construction of a family of non–classical ovoids which are multiply derivable and determine the corresponding collineation groups.

2. Permutable Polarities

A Hermtian variety and a quadric are said to be in *permutable position* if and only if they are both preserved by the same Baer involution. The properties of varieties in such a position have been investigated by several authors, notably by B. Segre, see [11, 6]. We need now to state some properties of the linear collineation group simultaneously preserving a Hermitian curve and a conic in the Desarguesian plane $PG(2,q^2)$, over the Galois field $\mathrm{GF}(q^2)$ of odd order q^2. These properties shall be used in Section 3 to construct derivable ovoids of the Hermitian surface of $PG(3,q^2)$.

Lemma 2.1. *Any two pairs $(\mathcal{H},\mathcal{C})$ consisting of a non–degenerate Hermitian curve and a conic of $PG(2,q^2)$ in permutable position are projectively equivalent.*

Proof. Recall that any two non–degenerate Hermitian curves $\mathcal{H}$, $\mathcal{H}'$ of $PG(2,q^2)$ are projectively equivalent. Furthermore, the full collineation group $\mathrm{P\Gamma U}(3,q)$ of a non–degenerate Hermitian curve $\mathcal{H}$ contains just one conjugacy class of Baer involutions. The result now follows by observing that, since $\mathcal{H}$ and $\mathcal{C}$ are in permutable position, there exists a Baer involution preserving them both. □

Let s be any non–zero element of $\mathrm{GF}(q)$ and assume $\mathcal{H}(2,q^2)$ as the non–degenerate Hermitian curve of equation

$$X^{q+1}-sY^{q+1}+Z^{q+1}=0; \tag{1}$$

denote also by $\mathcal{C}$ the non–degenerate conic of equation

$$X^2 - sY^2 + Z^2 = 0. \tag{2}$$

For fixed s, the mutual position of $\mathcal{H}(2,q^2)$ and $\mathcal{C}$ is permutable, as the canonical Baer involution $\beta : (X,Y,Z) \mapsto (X^q, Y^q, Z^q)$ preserves both of them. The common points of $\mathcal{H}(2,q^2)$ and $\mathcal{C}$ lie in the Baer subplane $PG(2,q)$ associated with β. Such points are precisely those of the conic $\mathcal{C}_0$ of $PG(2,q)$ with equation $X^2 - sY^2 + Z^2 = 0$.

Definition 2.2. The Hermitian curve $\mathcal{H}(2,q^2)$ of equation (1) and the conic $\mathcal{C}$ of equation (2) are in *canonical permutable position* in $PG(2,q^2)$ with respect to s.

Lemma 2.3. *The linear collineation group G of $PG(2,q^2)$ preserving simultaneously both $\mathcal{H}(2,q^2)$ and $\mathcal{C}$ preserves also the subplane $PG(2,q)$. Furthermore, $G \cong \mathrm{PGL}(2,q)$ and G acts on $\mathcal{C}_0$ as $\mathrm{PGL}(2,q)$ in its 3–transitive permutation representation.*

Proof. The conic $\mathcal{C}_0$ is preserved by G, since $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{H}$. Denote by T the linear collineation group of $PG(2,q)$ preserving $\mathcal{C}_0$; such group is isomorphic to $\mathrm{PGL}(2,q)$ and acts on $\mathcal{C}_0$ as $\mathrm{PGL}(2,q)$ in its 3–transitive permutation representation. We now write explicitely the elements of T. Consider the collineations

$$\gamma_{a,b} : (X,Y,Z) \mapsto (aX + sbY, bX + aY, Z),$$

with $a^2 - sb^2 = 1$, $a,b \in \mathrm{GF}(q)$ and $\delta : (X,Y,Z) \mapsto (-X,Y,Z)$. Clearly, each of these collineations preserves $\mathcal{C}_0$. Furthermore, they generate a dihedral group Γ of order $2(q+1)$. In particular, Γ is a maximal subgroup of T, see [12]. The collineation $\sigma : (X,Y,Z) \mapsto (Z,Y,X)$ preserves $\mathcal{C}_0$, but $\sigma \notin \Gamma$; hence, $T = \langle \Gamma, \sigma \rangle$. As $\mathcal{C}$ is the extension of $\mathcal{C}_0$ to $PG(2,q^2)$, the group T preserves also $\mathcal{C}$. On the other hand, each of the above mentioned collineations preserves also $\mathcal{H}(2,q^2)$. This assertion is obvious for δ and σ. In order to verify that it also holds for $\gamma_{a,b}$, a further computation is required. Indeed, $\gamma_{a,b}$ takes $\mathcal{H}$ to the Hermitian curve $\mathcal{H}^\gamma$ of equation

$$(a^{q+1} - s^q b^{q+1})(X^{q+1} - sY^{q+1}) + Z^{q+1} + (s^q a^q b - sab^q)(X^q Y - Y^q X).$$

Since $a^q = a$, $b^q = b$ and $s^q = s$, it follows that $\mathcal{H}^\gamma = \mathcal{H}$. This proves $G = T$. $\square$

Lemma 2.4. *Let ℓ be a line of $PG(2,q)$ external to $\mathcal{C}_0$. Then, the stabiliser in G of any point $P \in \mathcal{H}(2,q^2) \cap \ell$ has order 2.*

Proof. It suffices to show that for any point $P \in \ell$ not in $PG(2,q)$, the order of G_P is either $q+1$ or 2, according as P lies on $\mathcal{C}$ or not. Following Lemma 2.1, we may take s in (2) to be a non–square in $\mathrm{GF}(q)$. All the lines external to $\mathcal{C}_0$ lie in the same orbit under the action of G. Hence, we may assume without loss of generality that the equation of ℓ is $Z = 0$. The stabiliser G_ℓ of ℓ in G is the dihedral group D_{q+1} of order $2(q+1)$, consisting of the $q+1$ rotations $\gamma_{a,b}$ together with the $q+1$ involutorial symmetries

$$\xi_{a,b} : (X,Y,Z) \mapsto (aX - sbY, bX - aY, Z), \quad a^2 - sb^2 = 1.$$

The action of $\gamma_{a,b}$ on ℓ is given by the rational map $m \mapsto (b+am)/(a+sbm)$. For $(a,b) \notin \{(1,0),(0,1)\}$, the only fixed points of $\gamma_{a,b}$ are $(\sqrt{s},1,0)$ and $(-\sqrt{s},1,0)$, both of them on $\mathcal{C}$ but none on $\mathcal{H}(2,q^2)$. Furthermore, $\gamma_{0,1}$ and the identity $\gamma_{1,0}$ form a subgroup of order 2 which fixes ℓ pointwise. The action of $\xi_{a,b}$ on ℓ can be described in a similar way, using the rational map $m \mapsto (b-am)/(a-sbm)$. In fact, any $\xi_{a,b}$ has exactly two fixed points, namely $(1,a+1,0)$ and $(1,a-1,0)$, both of them lying in $PG(2,q)$ but not on $\mathcal{H}(2,q^2)$. This completes the proof. □

Let Δ_1 denote the set of points of $\mathcal{H}(2,q^2)\setminus\mathcal{C}_0$ which are covered by secants to $\mathcal{C}_0$ and let Δ_2 be the set of points of $\mathcal{H}(2,q^2)$ covered by external lines to $\mathcal{C}_0$.

Lemma 2.5. *The sets Δ_1, Δ_2 and the conic $\mathcal{C}_0$ partition $\mathcal{H}(2,q^2)$.*

Proof. Any point $P \in \mathcal{H}(2,q^2)$ outside $\mathcal{C}_0$ lies on a unique line of $PG(2,q)$. Since $\mathcal{C}$ and $\mathcal{H}$ are in permutable position, this line cannot be tangent to $\mathcal{C}_0$, as it contains two points of $\mathcal{H}(2,q^2)$, namely P and its image under the canonical Baer involution P^β. □

Lemma 2.6. *The group G has three orbits on $\mathcal{H}(2,q^2)$; one of size $q+1$, and two of size $(1/2)q(q+1)(q-1)$. These orbits, with the notation of Lemma 2.5, are precisely $\mathcal{C}_0$, Δ_1 and Δ_2*

Proof. By definition the group G preserves $\mathcal{H}(2,q^2)$. The set $\mathcal{C}_0$ is an orbit of G on $\mathcal{H}$ with size $q+1$. The size of the orbit of any $P \in \mathcal{H}\setminus\mathcal{C}_0$ under the action of G is $|G|/|G_P|$. Hence, by Lemma 2.4, any orbit on $\mathcal{H}$ different from $\mathcal{C}_0$ has size $|G|/2$, that is, $(1/2)q(q+1)(q-1)$. Let now $P \in \Delta_1$ and $Q \in \Delta_2$. Denote respectively by r and s the unique line of $PG(2,q)$ through P and Q. If P and Q were in the same orbit under the action of G, then there would be $\theta \in G$ such that $\theta(r)=s$. On the other hand, r is secant to $\mathcal{C}$, while s is an external line and G preserves $\mathcal{C}$. From this contradiction the result follows. □

Lemma 2.7. *Assume L to be a point of $PG(2,q)$ not on $\mathcal{C}_0$. Consider a a tangent line t to $\mathcal{H}(2,q^2)$ through L such that its tangency point is not on $\mathcal{C}_0$. Then, t is external or secant to $\mathcal{C}$ according as L is external or internal to $\mathcal{C}_0$.*

Proof. Let $\mathcal{H}$ and $\mathcal{C}_0$ be in canonical permutable position, as described in Definition 2.2, with respect to a non–zero element $s \in \mathrm{GF}(q)$. The tangents to $\mathcal{H}$ through the origin $O=(0,0,1)$ are the lines t_m of equation $Y=mX$ with

$$sm^{q+1}=1. \tag{3}$$

Furthermore,

$$sm^2-1=\begin{cases} \text{non–square in } \mathrm{GF}(q^2) & \text{if } t_m \text{ is an external line to } \mathcal{C}, \\ \text{non–zero square in } \mathrm{GF}(q^2) & \text{if } t_m \text{ is secant to } \mathcal{C}. \end{cases}$$

Let $d=sm^2-1$ and assume $d\neq 0$. Then, $(d+1)^{(q+1)/2}=(sm^2)^{(q+1)/2}=s^{(q+1)/2}m^{q+1}$. By (3),

$$(d+1)^{(q+1)/2}=s^{(q-1)/2}. \tag{4}$$

Hence, $(d+1)^{q+1}=1$. Therefore, $d^q+d^{q-1}+1=0$, that is,

$$d^{q-1}=-(d+1)^q. \tag{5}$$

Denote now by $\eta(x)=x^{(q^2-1)/2}$ the quadratic character of $x \in \mathrm{GF}(q^2)^\star$. By (5), $\eta(d)=(d^{q-1})^{(q+1)/2}=(-1)^{(q+1)/2}(d+1)^{q(q+1)/2}$. Taking (4) into account, this may be written as $(-1)^{(q+1)/2}s^{q(q-1)/2}=(-1)^{(q+1)/2}s^{(q-1)/2}$. Hence,

$$\eta(d)=(-1)^{(q+1)/2}s^{(q-1)/2}. \tag{6}$$

To study the case $q \equiv 1 \pmod 4$ and L external to $\mathcal{C}_0$, choose a non–zero square element s in $\mathrm{GF}(q)$. Then, the origin O is an external point to $\mathcal{C}_0$. Up to a linear collineation in $\mathrm{PGL}(2,q)$, as given in Lemma 2.3, L may be taken to be O. In this case, (6) reads $\eta(sm^2-1)=-1$, and t_m is an external line to $\mathcal{C}$. For the case when L is an internal point to $\mathcal{C}_0$, take s as a non–square element in $\mathrm{GF}(q)$; appplying the preceding argument we get $\eta(sm^2-1)=1$, showing that t_m is a secant to $\mathcal{C}$. The same method applies to the case $q \equiv 3 \mod 4$. □

3. Multiply Derivable Ovoids

Let P be the pole of a non–tangent plane π to $\mathcal{H}(3,q^2)$ with respect to the unitary polarity associated with the Hermitian surface. Denote by $\mathcal{H}(2,q^2)$ the Hermitian curve cut out on $\mathcal{H}(3,q^2)$ by π and choose a conic $\mathcal{C}$ of π in permutable position with $\mathcal{H}(2,q^2)$. As before, we write $\mathcal{C}_0=\mathcal{C}\cap\mathcal{H}(2,q^2)$. We now show how Lemmas 2.3, 2.5 and 2.6 can be used to geometrically construct multiply derivable ovoids of $\mathcal{H}(3,q^2)$ containing either $\Delta_1\cup\mathcal{C}_0$ or $\Delta_2\cup\mathcal{C}_0$.

We observe that there are $N=q^2-q$ lines, say $r_1,\ldots,r_N$, joining P to a point of $\mathcal{C}\setminus\mathcal{C}_0$, and each of these lines meets $\mathcal{H}(3,q^2)$ in $q+1$ points. For every $i=1,\ldots N$, take half of the $q+1$ points in common between r_i and $\mathcal{H}(3,q^2)$. The set Θ of all these points has size $(1/2)(q^3-q)$. Add now Θ to either $\Delta_1\cup\mathcal{C}_0$ or $\Delta_2\cup\mathcal{C}_0$. The resulting set $\mathcal{O}'$ contains as many points as an ovoid does. When $\mathcal{O}'$ happens to be an ovoid, it will be called an ovoid of type (1) or (2) according as $\mathcal{O}'$ contains Δ_1 or Δ_2. Examples of ovoids of type (1) were constructed in [4].

We shall now investigate derivability of ovoids of type (1).

Theorem 3.1. *Any ovoid $\mathcal{O}$ of type (1) is derivable.*

Proof. Since $\mathcal{H}(2,q^2)$ and $\mathcal{C}$ are in permutable position, the orthogonal polarity of $PG(2,q)$ arising from $\mathcal{C}_0$ may be extended to the unitary polarity of $PG(2,q^2)$ associated with $\mathcal{H}(2,q^2)$. Assume that $\mathcal{O}$ is an ovoid of type (1). Then, any $(q+1)$–secant ℓ of $\mathcal{O}$ lying in π has $q-1$ points in Δ_1 and 2 points in $\mathcal{C}_0$. In particular, ℓ is a secant to $\mathcal{C}_0$. We claim that ℓ is a replaceable $(q+1)$–secant. To prove it, consider the polar line ℓ' of ℓ with respect to the unitary polarity associated to $\mathcal{H}(3,q^2)$ and assume, by contradiction, that there is a point $R\in\ell'\cap\mathcal{H}(3,q^2)$ conjugate to a point $U\in\mathcal{O}$. Then, $U\notin PG(2,q)$, and the generator g through U and R meets π in a point $V\in\mathcal{H}(2,q^2)$. Since V is conjugate to P, the plane ϕ through P,R and U is tangent to $\mathcal{H}(3,q^2)$ with tangency point V. Therefore, ϕ meets π in the tangent t to $\mathcal{H}(2,q^2)$ at V. Both points $L=\ell'\cap\pi$ and $S=PU\cap\pi$ lie on t. Note that $V\notin\mathcal{C}_0$, otherwise t would be a tangent to $\mathcal{C}_0$ and, hence, to $\mathcal{C}$ contradicting $S\in t$. Now, Lemma 2.7 applies to t; this implies that L must be an internal point to $\mathcal{C}_0$. On the other hand, L is the pole of ℓ with respect to the polarity arising from $\mathcal{C}_0$. Since ℓ is secant to $\mathcal{C}_0$, the point L must be external to $\mathcal{C}_0$. This final contradiction completes the proof. □

To prove that any ovoid of type (1) is indeed multiply derivable, the following result is needed.

Lemma 3.2. *Let $\mathcal{L}=\{\ell_1,\dots\ell_k\}$ be a set of secants to $\mathcal{C}_0$, and let $L_1,\dots,L_k$ denote their poles. If the common point of any two lines ℓ_i,ℓ_j lies outside $\mathcal{C}_0$, or, equivalently, if no line joining two points L_i,L_j is tangent to $\mathcal{C}_0$, then $\mathcal{L}$ is a replaceable set of any ovoid of type* (1) *containing $\mathcal{C}_0$.*

Proof. By Theorem 3.1, it is enough to show that no point on ℓ'_i is conjugate to a point on ℓ'_j. Assume by contradiction that $R_i\in\ell'_i\cap\mathcal{H}(3,q^2)$ and $R_j\in\ell'_j\cap\mathcal{H}(3,q^2)$ are two conjugate points, and let V be the common point of the line R_iR_j with the plane π. Arguing as in the proof of the Theorem 3.1, it turns out that the tangent line t to $\mathcal{H}(2,q^2)$ at V must contain both L_i and L_j. Therefore, t is a line of $PG(2,q)$ and $V\in\mathcal{C}_0$. In particular, t is the tangent to $\mathcal{C}_0$ at V. Then V would be the common point of ℓ_i and ℓ_j — a contradiction. □

Note that examples of replaceable sets $\mathcal{L}$ of size $k\le(1/2)(q+1)$ for an ovoid of type (1) are provided by any k external lines to $\mathcal{C}_0$ through an internal point of $\mathcal{C}$. Such examples are called *linear*. Hence, using Lemma 3.2 we get the following result.

Theorem 3.3. *Any ovoid of type* (1) *is k–fold derivable, for every $k\le(1/2)(q+1)$.*

We remark that any replaceable set has size at most $(1/2)(q+1)$. We now exhibit another infinite family of replaceable sets. Assume that $q^2\equiv 1 \pmod{10}$. Then, $\mathrm{PSL}(2,q)$ contains a subgroup M isomorphic to A_5. Since A_5 has 15 involutions, M contains 15 involutory homologies. The axes of these are pairwise distinct secants to $\mathcal{C}_0$, see [7, 8, 9]. We show that such secants form a replaceable set $\mathcal{L}$. Assume, on the contrary, that there are two involutory homologies $\varphi_1,\varphi_2\in M$ such that their axes meet in a point T of $\mathcal{C}_0$. Then, $\varphi_1\varphi_2$ fixes T but no any other point of $\mathcal{C}_0$. Therefore, the order of $\varphi_1\varphi_2$ is divisible by p. But this is impossible, as p does not divide the order of A_5.

The smallest case is $q=29$ and the size of $\mathcal{L}$ is $15=(1/2)(q+1)$. This shows that replaceable sets of maximum size are not necessarily linear.

It is possible that more infinite families of non–linear replaceable sets may arise from Lemma 3.2. However, if the common point of any two lines in $\mathcal{L}$ is internal to $\mathcal{C}$, then only sporadic examples seem to exist, namely for $q\equiv 3 \pmod 4$ and $q\le 31$. This follows from the main conjecture in [1].

In the above construction, the group M preserves the set $\mathcal{L}$. From Section 4 of [4], the linear collineation group Γ preserving the ovoid of type (1) contains a normal subgroup H such that $\Gamma/H \cong PGL(2,q)$ and H is a homology group of order $(1/2)(q+1)$ with axis π. In particular, Γ/H acts on π as $PGL(2,q)$, preserving both $\mathcal{H}(2,q^2)$ and $\mathcal{C}$. It follows that Γ has a subgroup Φ containing H such that $M = \Phi/H \cong A_5$. In particular, the linear collineation group of every multiply derived ovoid arising from $\mathcal{L}$ is non–solvable.

It is natural to ask whether any non–trivial linear collineation group H of the replaceable set $\mathcal{L}$ may be lifted to a linear collineation group of the derived ovoid $\mathcal{O}'$. Clearly, the answer depends on the geometry of the original ovoid $\mathcal{O}$ from which $\mathcal{O}'$ arose. However, when $\mathcal{O}$ is the ovoid of type (1) constructed in [4], the answer is affirmative, as it is stated in the following theorem.

Theorem 3.4. *Let $\mathcal{O}'$ be an ovoid arising from the ovoid of type* (1) *given in* [4]*, by multiple derivation with respect to a replaceable set $\mathcal{L}$. If $\mathcal{L}$ consists of* $(1/2)(q+1)$ *lines through an internal point to $\mathcal{C}_0$, then the linear collineation group Γ preserving $\mathcal{O}'$ contains a homology group Φ of order* $(1/2)(q+1)$.

Acknowledgements

The author wishes to thank Professor G. Korchmáros for several useful discussions on the topics here–within investigated.

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