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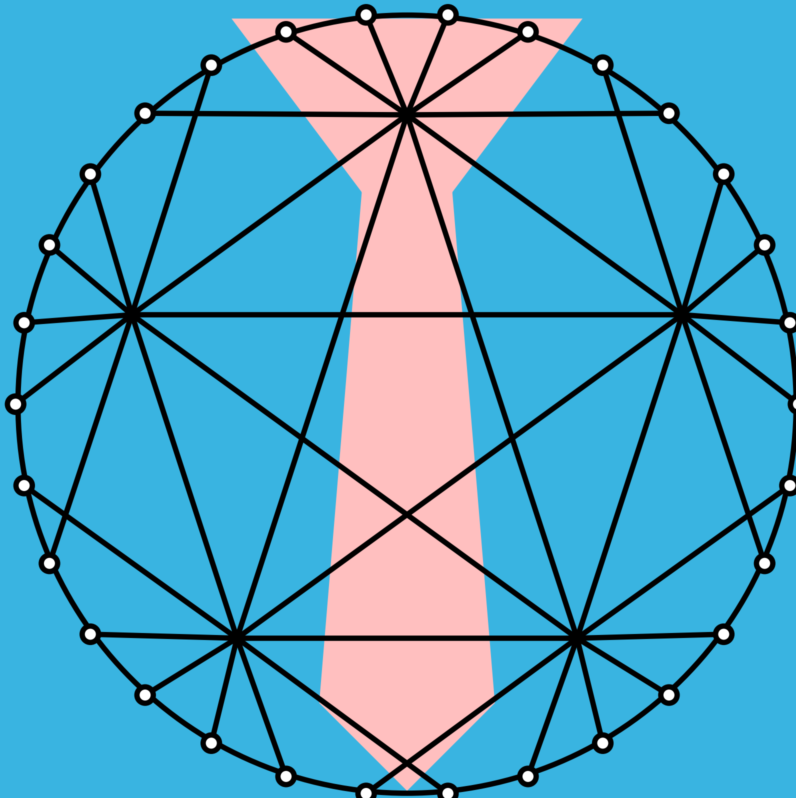

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A note on sets of type $(3, n)$ in $PG(3, q)$

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Abstract. We prove that if K is a set of type $(3, n)$ with $n > 3$ in $PG(3, q)$, then either K is the point-set of a plane in $PG(3, 2)$ or $n = q + 3$ or $(q, n) = (8, 7)$.

1 Introduction

In the three-dimensional projective space $PG(3, q)$ over a finite field $GF(q)$ with $q = p^h$ a prime power, a k-set of type (m, n), with $n > m$, is a set of k points which meets any plane in m or n points and both intersection numbers occur, see [2]. In [6], the Author proved that $n - m = p^s$ with $0 \leq s \leq 2h$.

If K is a set of type $(0, n)$, then K is a point or the complementary set of the point-set of a plane, see [6]. The sets of type $(1, n)$ and $(2, n)$ have been completely determined by Thas in [7] and by Durante and Olanda in [1], respectively. In [4], Napolitano and Olanda studied sets of type $(3, n)$. First they proved that if $q = 2$, then either $n = 7$ and K is the point-set of a plane or $n = 5$ and K is the point-set of three pairwise skew lines. Let us note that if $q = 2$, then $n \geq q+3$. Then, they proved that if $q > 2$, then $n-3$ divides q and, hence, $n \leq q + 3$. Finally, they completely classified the sets of type $(3, q + 3)$. The Authors also searched by computer for admissible values (q, n) with $n < q + 3$. Within a huge range of tested parameters they founded only the case $(q, n) = (8, 7)$. So, they conjectured that if such sets do exist, then they are very rare. Furthermore, in [5] (cf Theorem 3), Napolitano devoted the last part of the paper to sets of type $(3, n)$ with $n < q+3$. Without completing their classification, the Author showed that in this case it is possible to determine the sizes of the intersections of the lines of the projective space with the set K. In this paper we prove the following

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Theorem. *Let K be a k-set of type $(3,n)$, with $n>3$, of $PG(3,q)$. Then one of the following conditions holds*

- *K is the point-set of a plane in $PG(3,2)$;*
- *$n=q+3$;*
- *$q=8$ and $n=7$.*

2 The proof of the Theorem

From now on K is a k-set of type $(3,n)$, with $n>3$, of $PG(3,q)$, $q=p^h$.

Remark 2.1. *If the set K is the point-set of a plane, then $q=2$.*

Proof. The point-set of a plane is a set of type $(q+1, q^2+q+1)$. □

From now on K is not the point-set of a plane in $PG(3,2)$. Hence, as already seen, we have that $n-3$ divides q. So, $n=3+p^s$ with $s\le h$.

Double counting the number of planes of $PG(3,q)$, the number of pairs (A,α), where $A\in K$ and α is a plane of $PG(3,q)$ passing through A, and the number of pairs $((A,B),\alpha)$, where $\{A,B\}\subset K$ and α is a plane of $PG(3,q)$ passing through A and B, we get the following equations:

$$t_3+t_n=q^3+q^2+q+1 \tag{1}$$

$$3t_3+nt_n=k(q^2+q+1) \tag{2}$$

$$6t_3+n(n-1)t_n=k(k-1)(q+1) \tag{3}$$

where t_i is the number of planes meeting K in exactly i points.

By Equations (1), (2) and (3) we get

$$k[(q+1)k-(2q^2+3q+3)]=n[(q^2+q+1)k-3(q^2+1)(q+1)] \tag{4}$$

Given that $n\le q+3$, by Equation (4) we obtain

$$k[(q+1)k-(2q^2+3q+3)]\le (q+3)[(q^2+q+1)k-3(q^2+1)(q+1)] \tag{5}$$

which can be rewritten in the following way

$$[k - 3(q + 1)][(q + 1)k - (q + 1)(q^2 + 2q - 1) - 4] \leq 0 \tag{6}$$

thus we have that

$$3(q + 1) \leq k \leq q^2 + 2q - 1 + \frac{4}{q + 1} \tag{7}$$

where equalities hold if and only if $n = q + 3$. So we have the following

Proposition 2.2. *If K is a k-set of type $(3, q + 3)$ in $PG(3, q)$, then*

- $k = 3(q + 1)$;
- $q = 3$ *and* $k = 15$.

Now, we will prove the following

Proposition 2.3. *Let K be a set of type $(3, n)$ in $PG(3, q)$. If $n < q + 3$, then $q = 8$ and $n = 7$.*

From now on, $n < q + 3$. So, $s < h$ and $k > 3(q + 1)$. Furthermore, $q > 2$.

Step 1. *Any 3-plane contains at least one line l_0 meeting K in no point.*

Proof. Let α be a 3-plane and let Q be a point of $\alpha \setminus K$. Since $q + 1 > 3$, on α there is at least one line l_0 passing through Q and meeting K in no point. □

Step 2. *There is an integer u such that $k = 3(q+1) + up^s$ with $0 < u \leq q$.*

Proof. Let α be a 3-plane and l_0 a line of α meeting K in no point. Now, let $u \leq q$ be the number of n-planes passing through l_0. Counting the points of K via the planes through l_0 we obtain $k = 3(q + 1 - u) + nu = 3(q + 1 - u) + (3 + p^s)u$ from which we get $k = 3(q + 1) + up^s$. Finally, $k > 3(q + 1)$ implies $u > 0$. □

Step 3. *We have that $1 < u < q$. Furthermore, p^{h-s} divides either u or $u - 1$.*

Proof. Putting $n = 3 + p^s$ and $k = 3(q+1) + up^s$ into Equation (4) we get

$$Ap^{h-s} + u(u-1) = 0 \tag{8}$$

where

$$A = 3p^{2h-s} - up^{h+s} + (u-3)p^h + u(u-1)p^s + 3p^{h-s} + 3(2u-1). \tag{9}$$

If $u = 1$, then by Equations (8) and (9) we obtain

$$3p^{2h-s} - p^{h+s} - 2p^h + 3p^{h-s} + 3 = 0,$$

Hence, $p = 3$ and $s = h-1$, since p^{h-s} divides 3. Substituting these values into the previous Equation we obtain $3^{2h-1} - 7(3^h) - 12 = 0$. So, $h = 1$ necessarily which implies $30 = 0$, a contradiction. Thus, $u > 1$.

If $u = q = p^h$, then by Equations (8) and (9) we obtain

$$(p^{2h-s} - 1)(p^s + 3) + 3(p^s + 1)p^{h-s} = 0$$

a contradiction since the first side is positive. So, $1 < u < q$. Finally, by Equation (8), we have that p^{h-s} divides $u(u-1) > 0$ and, hence, p^{h-s} divides either u or $u-1$. □

2.1 The case where p^{h-s} divides u

If p^{h-s} divides u, then there is an integer d such that $u = dp^{h-s}$ and, hence, $k = 3(q+1) + dq$. Furthermore, $1 < u < q$ implies that $0 < d < p^s$. So $s > 0$.

Step 4. *We have that p^{h-s} divides $d+3$ and $h \leq 2s$.*

Proof. Putting $n = 3+p^s$ and $k = 3(q+1)+dq$ into Equation (4) we obtain

$$(dp^s - d^2 - d - 3)q^2 - [(d+3)p^s - (d^2 + 6d + 3)]q + (d+3)p^s = 0 \tag{10}$$

Equation (10) implies that p^{h-s} divides $d+3$. If $p^{h-s} = d+3$, then substituting $d = p^{h-s} - 3$ into Equation (10) we obtain

$$p^{3h-2s} - 5p^{2h-s} + p^{2(h-s)} - p^{2h} + 3p^{h+s} + 8p^h - 7 = 0$$

Given that $1 \leq s < h$, we have that p^2 divides 7, a contradiction.
Finally, $p^{h-s} \leq \frac{d+3}{2} \leq d+1 \leq p^s$ implies $h - s \leq s$, i.e. $h \leq 2s$. □

Step 5. *K is a 35-set of type $(3,7)$ in $PG(3,8)$.*

Proof. Since $q = p^h$, Equation (10) can be rewritten in the following way

$$[(dp^s - d^2 - d - 3)p^{h-s} + 3](p^h + 1) - (5p^{h-s} - 1)d = 0 \tag{11}$$

Hence, $p^h + 1$ divides $(5p^{h-s} + 1)d > 0$. So there is a positive integer x such that

$$x = \frac{(5p^{h-s} - 1)d}{p^h + 1}. \tag{12}$$

Given that $d < p^s$, we have that

$$(5p^{h-s} - 1)d < (5p^{h-s} - 1)p^s = 5p^h - p^s < 5p^h < 5(p^h + 1).$$

So $0 < x < \frac{5(p^h+1)}{p^h+1} = 5$, i.e. $1 \leq x \leq 4$.

Substituting $(5p^{h-s} - 1)d = x(p^h + 1)$ into Equation (11) we obtain

$$(dp^s - d^2 - d - 3)p^{h-s} + 3 - x = 0. \tag{13}$$

Equation (13) implies that p^{h-s} divides $3 - x$. Since $1 \leq x \leq 4$, the possible cases are $x = 3$ and $(x, p, s) = (1, 2, h - 1)$.

First, let us consider the case $x = 3$. Putting $x = 3$ into Equation (12), respectively (13), we obtain

$$3p^h - 5dp^{h-s} + (d + 3) = 0, \tag{14}$$

respectively

$$dp^s - d^2 - d - 3 = dp^s - (d - 2)(d + 3) - 9 = 0. \tag{15}$$

Since p^{h-s} divides $d+3$ and $s \geq h-s$, then, by Equation (15), we have that p^{h-s} divides 9. Hence, $p = 3$ and $s = h-1$ or $s = h-2$. Putting $p = 3$ and $s = h-1$ into Equations (14), respectively (15), we obtain $3^{h+1} = 14d - 3$, respectively $3^h d - 3(d^2 + d + 3) = 0$. Finally, we get $5d^2 - 12d - 27 =$ which has no integer solution, a contradiction. Putting $p = 3$ and $s = h - 2$ into Equations (14), respectively (15), we obtain $3^{h+1} = 44d - 3$, respectively $3^h d - 9(d^2 + d + 3) = 0$. Finally, we get $17d^2 - 30d - 81 =$ which has no integer solution, a contradiction.

Now, let us consider the case $(x, p, s) = (1, 2, h - 1)$. Substituting $p = 2$ and $s = h - 1$ and $x = 2$ into Equation (12) and (13) respectively, we get $2^h = 9d - 1$ and $2^h d - 2d^2 - 2d - 4 = 0$ respectively. Finally, we get $(d - 1)(7d + 4)$. So, $d = 1$ and, hence, $2^h = 9d - 1 = 8$ from which we get $h = 3$. Thus, $s = h - 1 = 2$. Furthermore, $q = p^h = 8$, $n = 3 + p^s = 7$ and $k = 3(q + 1) + dq = 35$. $\square$

2.2 The case where p^{h-s} divides $u - 1$

If p^{h-s} divides $u-1$, then there is an integer d such that $u = 1 + dp^{h-s}$ and, hence, $k = 3(q+1) + dq + p^s$. Furthermore, $1 < u < q$ implies that $0 < d < p^s$.

Step 6. *We have that p^{h-s} divides $d+3$ and $h \leq 2s$.*

Proof. Putting $n = 3 + p^s$ and $k = 3(q+1) + dq + p^s$ into Equation (4) we obtain

$$(dp^s - d^2 - d - 3)q^2 - [(p^s - d + 2)p^s - (d^2 + 6d + 3)]q + (d+3)p^s = 0. \quad (16)$$

Given that $q = p^h$, we have that p^{h-s} divides $d+3$. If $p^{h-s} = d+3$, then substituting $d = p^{h-s} - 3$ into Equation (16) we obtain

$$p^{3h-2s} - 5p^{2h-s} + p^{2(h-s)} - p^{2h} + 3p^{h+s} - p^{2s} + 10p^h - 5(p^s + 1) = 0$$

So, p^2 divides $5(p^s + 1)$, since $1 \leq s < h$. Hence, p^2 divides 5, a contradiction. Finally, $p^{h-s} \leq \frac{d+3}{2} \leq d + 1 \leq p^s$ implies $h - s \leq s$, i.e. $h \leq 2s$. □

Step 7. *K is a 39-set of type $(3, 7)$ in $PG(3, 8)$.*

Proof. Since $q = p^h$, Equation (16) can be rewritten in the following way

$$B(p^h + 1) - [(5p^{h-s} - 1)d + p^s + 5] = 0, \quad (17)$$

where

$$B = dp^h + p^s - (d^2 + d + 3)p^{h-s} - 2d + 2. \quad (18)$$

Hence, $p^h + 1$ divides $[(5p^{h-s} - 1)d + p^s + 5] > 0$. So there is a positive integer x such that

$$x = \frac{(5p^{h-s} - 1)d + p^s + 5}{p^h + 1}. \quad (19)$$

Given that $d < p^s$, we have that

$$(5p^{h-s} - 1)d + p^s + 5 < (5p^{h-s} - 1)p^s + p^s + 5 = 5p^h + 5 = 5(p^h + 1)$$

Hence, $0 < x < \frac{5(p^h+1)}{p^h+1} = 5$, i.e. $1 \leq x \leq 4$.

Substituting $(5p^{h-s} - 1)d + p^s + 5 = x(p^h + 1)$ into Equation (17) we obtain

$$dp^h + p^s - (d^2 + d + 3)p^{h-s} - 2d + 2 - x = 0. \quad (20)$$

Since $s \geq h-s$, p^{h-s} divides $-2d+2-x = -2(d+3)+8-x$. Hence, p^{h-s} divides $8-x$ since p^{h-s} divides $d+3$. Given that $1 \leq x \leq 4$, let us analyze the six possible cases.

1. $(x, p, s) = (1, 7, h-1)$. By Equation (19) and (20) respectively, we get $3(7^{h-1}) = 17d+2$ and $7^{h-1}(7d+1) - 7d^2 - 9d - 20 = 0$ respectively. Finally, we have that $49d^2 + 2d - 29 = 0$. The last equation has no integer solution, a contradiction.

2. $(x, p, s) = (2, 2, h-1)$. By Equation (19) and (20) respectively, we get $2^{h-1} = 3d+1$ and $2^{h-1}(2d+1) - 2(d^2+2d+3) = 0$ respectively. Finally, we have that $(d-1)(4d+5) = 0$. So, $d = 1$. So, $d = 1$ and, hence, $2^{h-1} = 3d+1 = 4$ from which we get $h = 3$. Thus, $s = h-1 = 2$. Furthermore, $q = p^h = 8$, $n = 3 + p^s = 7$ and $k = 3(q+1) + dq + p^s = 39$.

3. $(x, p, s) = (2, 3, h-1)$. By Equation (19) and (20) respectively, we get $5(3^{h-1}) = 14d+3$ and $3^{h-1}(3d+1) - 3d^2 - 5d - 9 = 0$ respectively. Finally, we have that $27d^2 - 2d - 42 = 0$. The last equation has no integer solution, a contradiction.

4. $(x, p, s) = (3, 5, h-1)$. By Equation (19) and (20) respectively, we get $7(5^{h-1}) = 12d+1$ and $5^{h-1}(5d+1) - 5d^2 - 7d - 16 = 0$ respectively. Finally, we have that $25d^2 - 32d - 111 = 0$. The last equation has no integer solution, a contradiction.

5. $(x, p, s) = (4, 2, h-1)$; By Equation (19) and (20) respectively, we get $7(2^{h-1}) = 9d+1$ and $2^{h-1}(2d+1) - 2(d^2+2d+4) = 0$ respectively. Finally, we have that $4d^2 - 17d - 55 = 0$. The last equation has no integer solution, a contradiction.

6. $(x, p, s) = (4, 2, h-2)$; By Equation (19) and (20) respectively, we get $15(2^{h-2}) = 19d+1$ and $2^{h-2}(4d+1) - 2(2d^2+3d+7) = 0$ respectively. Finally, we have that $16d^2 - 67d - 209 = 0$. The last equation has no integer solution, a contradiction. □

Remark 2.4. *In $PG(3, 8)$ there is a 39-set of type $(3, 7)$, see* [3].

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