The extended graphical and bigraphical
generalized Steiner systems

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Abstract. A set system \((X, \mathcal{B})\) consists of a \(v\)-element set \(X\) of points and a collection \(\mathcal{B}\) of subsets of \(X\) called blocks. A set system \((X, \mathcal{B})\) is a (proper) \(t\)-(\(v, \mathcal{K}, \lambda\)) design, provided

1. every \(t\)-element set of points is contained in precisely \(\lambda\) blocks,
2. \(\mathcal{K}\) is a set of integers, each of which is \(> t\) and \(< v\), and
3. if \(B \in \mathcal{B}\), then \(|B| \in \mathcal{K}\).

The parameters \(t\) and \(\lambda\) are called respectively the strength and index of the design. When the index is 1, they are called generalized Steiner systems. With respect to a given point \(x \in X\), the blocks containing \(x\) (with \(x\) discarded) are called the derived design and the blocks that do not contain \(x\) are called the residual design.

If \(X = E(K_n)\), then the blocks are subgraphs of \(K_n\). We say that the set system is graphical, if whenever \(B\) is a block then all subgraphs of \(K_n\) isomorphic to \(B\) are also blocks. A \(t\)-(\(v, \mathcal{K}, \lambda\)) design is an extended graphical design if respect to some fixed point \(x\), both the residual and derived designs are graphical. In this note all extended graphical generalized Steiner systems are determined. In a similar fashion bigraphical designs are defined for \(X = E(K_m,n)\) and all extended bigraphical generalized Steiner systems are also determined. This note concludes with several open problems that arise naturally from this investigation.

1 Exordium

A \(2\)-(\(v, \mathcal{K}, \lambda\)) design is a pairwise balanced design, see [4, Part IV]. Among the first examples of pairwise balanced designs are those constructed by

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Kirkman [10] in 1847, which he used to solve the schoolgirls problem on 
\(v = 5 \cdot 3^m + 1\) points, [4, page 15]. Kramer [12] referred to \(t-(v, \mathcal{K}, \lambda)\) designs as \(t\)-wise balanced designs or \(t\)BDs. However Hanani [8] in 1963 first introduced such designs as \(C\)-designs, but called them pairwise balanced designs when \(t = 2\). For the case \(t = 3\) Brouwer [2] used the term triplewise balanced designs. When the index \(\lambda = 1\), the adjective Steiner is affixed even though they are only loosely related to Steiner’s original problem, see [5]. Indeed \(t-(v, \mathcal{K}, 1)\) designs were called Hanani generalized Steiner systems in [17] and generalized Steiner systems in [1, 18]. Note that generalized Steiner systems, as introduced in [7], are \(H\)-designs (see [16]) with certain coding properties and they should not be confused with the \(t-(v, \mathcal{K}, 1)\) designs that are discussed here. Regardless of the terminology it is not difficult to see that \(t-(v, \mathcal{K}, 1)\) designs have deep connections to geometries and codes. Furthermore they have proven to be invaluable in the construction of desired combinatorial configurations.

A set system \((X, \mathcal{B})\) is graphical if \(X\) is the set of \(v = \binom{n}{2}\) labelled edges of the complete graph \(K_n\) with vertex set \(\{1, 2, \ldots, n\}\) and it has the natural action of the symmetric group \(S_n\) on the edges as an automorphism group. Thus if \(B \in \mathcal{B}\), then all subgraphs isomorphic to \(B\) are also in \(\mathcal{B}\). Hence a graphical set system can be presented as a list of unlabeled graphs. For example

![Graphical set system example](image)

graphically represents a 3-(10, 4, 1) design. The actual blocks are:

\[
\begin{align*}
\mathcal{B}_1 &= \{\{12,13,14,15\},\{21,23,24,25\},\{31,32,34,35\},\{41,42,43,45\},\{51,52,53,54\}\} \\
\mathcal{B}_2 &= \{\{12,23,31,45\},\{12,24,41,35\},\{14,43,31,25\},\{42,23,34,15\},\{12,25,51,34\}\} \\
&\quad \quad \quad \{\{13,35,51,24\},\{32,25,53,14\},\{14,45,51,23\},\{24,45,52,13\},\{34,45,53,12\}\} \\
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{B}_3 &= \{\{12,23,34,41\},\{13,34,42,21\},\{14,42,23,31\},\{12,23,35,51\},\{13,35,52,21\}\} \\
&\quad \quad \quad \{\{15,52,23,31\},\{12,24,45,51\},\{14,45,52,21\},\{15,52,24,41\},\{13,34,45,51\}\} \\
&\quad \quad \quad \{\{14,45,53,31\},\{15,53,34,41\},\{23,34,45,52\},\{24,45,53,32\},\{25,53,34,42\}\} \\
\end{align*}
\]

where \(ab = ba = \{a, b\}\). A complete list of graphical \(t-(v, K, 1)\) designs can be found in [3] and is provided for reference in Table 2. The presentation
of a design by a list of unlabeled graphs is I think quite beautiful. It gives
a visual interpretation for the blocks of the design.

If \((X, \mathcal{B})\) is a set system and \(x \in X\), then the derived set system with
respect to \(x\) is \((X \setminus \{x\}, \mathcal{B}_x)\), where
\[
\mathcal{B}_x = \{B \setminus \{x\} : x \in B \in \mathcal{B}\}
\]
and the residual set system with respect to \(x\) is \((X \setminus \{x\}, \mathcal{B}^x)\), where
\[
\mathcal{B}^x = \{B \in \mathcal{B} : x \notin B\}.
\]

A \(t-(v, \mathcal{K}, \lambda)\) design \((X, \mathcal{B})\) is an extended graphical design, if there is a
point \(x\) such that both \(\mathcal{B}_x\) and \(\mathcal{B}^x\) are graphical. This article was motivated
by my discovery that the \(3-(16, \{4,6\}, 1)\) design studied by Assmus and
Sardi \cite{1} is the extended graphical design listed as \(D_4\) in Table 1. There
is, by the way, a unique \(3-(16, \{4,6\}, 1)\) design. It consists of the best
biplane on 16 points and its 60 ovals. Having made this discovery (see \cite{6})
I became interested in finding all extended graphical generalized Steiner
systems. This article is the results of my efforts.

## 2 Proventus

It is not difficult to determine the extended graphical \(t-(v, \mathcal{K}, 1)\) designs
\((X \cup \{\infty\}, \mathcal{B})\), when the derived set system \((X, \mathcal{B}_\infty)\) is a known graphical
\((t-1)-(v-1, \mathcal{K}', 1)\) design. Consider \(\mathcal{B}_\infty\) as a list of unlabeled graphs on
\(v-1\) points and let \(\mathcal{R}\) be the \(t\)-edge unlabeled graphs on \(v-1\) points that
are not a subgraph of any graph in \(\mathcal{B}_\infty\). Let \(\mathcal{C}\) be the set of unlabeled
graphs on \(v-1\) points that have more than \(t\) edges. (Incidentally \cite[Corollary 1.3]{13}
shows that subgraphs with more than \(\frac{1}{2}(v-1)\) edges need not be considered.) Define the matrix \(A: \mathcal{R} \times \mathcal{C} \to \mathbb{Z}_{\geq 0}\) by
\[
A[R, C] = |\{c \in C : r \text{ is a subgraph of } c\}|,
\]
where \(r\) is any fixed labeled subgraph in \(R\). Then there will be a solution
\(U : \mathcal{C} \to \{0, 1\}\) to the matrix equation \(AU = J\), where \(J[r] = 1\), for
all \(r \in \mathcal{R}\), if and only if \(\mathcal{B}_\infty = \{C \in \mathcal{C} : U[C] = 1\}\) is such that
\((X, \mathcal{B}_\infty \cup \mathcal{B}^\infty)\) is an extended graphical \(t-(v, \mathcal{K}, 1)\) design. Columns \(C \in \mathcal{C}\)
of \(A\) that contain an entry exceeding 1 can of course be removed. Although the matrix equations \(AU = J\) that need to be considered in this
note are perhaps small enough that they could possibly be constructed and
solved by hand, I employed computer algorithms. I used the algorithms
Table 1: Derived and residual systems

Extended graphical generalized Steiner systems

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$X$</th>
<th>$\mathcal{B}_\infty$</th>
<th>$\mathcal{B}_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>2-(7, 3, 1)</td>
<td>$E(K_4) \cup {\infty}$</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>$D_2$</td>
<td>2-(7, 3, 1)</td>
<td>$E(K_4) \cup {\infty}$</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>$D_3$</td>
<td>3-(16, 4, 1)</td>
<td>$E(K_6) \cup {\infty}$</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>$D_4$</td>
<td>3-(16, {4, 6}, 1)</td>
<td>$E(K_6) \cup {\infty}$</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>$D_5$</td>
<td>5-(16, {6, 8}, 1)</td>
<td>$E(K_6) \cup {\infty}$</td>
<td><img src="image5" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Extended BW-bigraphical generalized Steiner systems

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$X$</th>
<th>$\mathcal{B}_\infty$</th>
<th>$\mathcal{B}_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_6$</td>
<td>2-(10, {3, 4}, 1)</td>
<td>$E(K_{33}) \cup {\infty}$</td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
<tr>
<td>$D_7$</td>
<td>2-(10, {3, 4}, 1)</td>
<td>$E(K_{33}) \cup {\infty}$</td>
<td><img src="image7" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Extended bigraphical generalized Steiner systems

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$X$</th>
<th>$\mathcal{B}_\infty$</th>
<th>$\mathcal{B}_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_8$</td>
<td>3-(10, 4, 1)</td>
<td>$E(K_{33}) \cup {\infty}$</td>
<td><img src="image8" alt="Diagram" /></td>
</tr>
</tbody>
</table>
described in [14, Chapter 6] to construct the matrices and I used the dancing links data structure described in [11] to solve them. The dancing link implementation I employed can be downloaded from:

http://pottonen.kapsi.fi/libexact.html

**Theorem 1.** The only extended graphical $t$-$(v,\mathcal{K},1)$ designs are $D_1, D_2, \ldots, D_5$, which are listed in Table 1.

**Proof.** Let $(X\cup\{\infty\},\mathcal{B})$ be an extended graphical $t$-$(v,\mathcal{K},1)$ design. Then $(X,\mathcal{B}_\infty)$ is a graphical $(t-1)-(v-1,\mathcal{K}',1)$ design. There is only one on $K_4$, one on $K_5$ and three on $K_6$. It is computationally easy, by the method described above, to determine how a $(t-1)-(v-1,\mathcal{K}',1)$ design $(X,\mathcal{B}_\infty)$ can be completed to an extended graphical $t$-$(v,\mathcal{K},1)$ design. \hfill $\Box$

The automorphism group $\text{Aut}(K_{m,n})$ of the undirected complete bipartite graph $K_{m,n}$ with vertex set $\{1',2',\ldots,m',1,2,\ldots,n\}$ is the cross product $S_m \times S_n$, if $m \neq n$ and is the wreath product $S_n \rtimes S_2$, if $m = n$. A set system $(X,\mathcal{B})$ is bi graphical if $X$ is the set of $v = mn$ labelled edges of $K_{m,n}$ and the natural action of $\text{Aut}(K_{m,n})$ on the edges of $K_{m,n}$ is an automorphism group. Thus if $B \in \mathcal{B}$, then all subgraphs of $K_{m,n}$ isomorphic to $B$ are also in $\mathcal{B}$.

If the vertices of $K_{n,n}$ are 2-colored, so that one independent set is colored black and the other is colored white, then the automorphism group $G$ is $S_n \times S_n$. ($S_n \times S_n$ is subgroup of index 2 in $\text{Aut}(K_{n,n})$.) A set system $(X,\mathcal{B})$ is BW-bigraphical if $X$ is the set of $v = n^2$ labelled edges of the 2-colored $K_{n,n}$ and the natural action of $G$ is an automorphism group. Thus if $B \in \mathcal{B}$, then all 2-colored subgraphs of $K_{m,n}$ isomorphic to $B$ are also in $\mathcal{B}$.

A complete list of bi graphical $t$-$(v,\mathcal{K},1)$ designs can be found in [9] and is provided in Table 3.

The proofs of Theorems 2 and 3 are similar to the proof of Theorem 1.

**Theorem 2.** The only extended bi graphical $t$-$(v,\mathcal{K},1)$ design is $D_8$, which is listed in Table 1.

**Theorem 3.** The only extended BW-bigraphical $t$-$(v,\mathcal{K},1)$ designs are $D_6$ and $D_7$, which are listed in Table 1.
Table 2: Graphical generalized Steiner systems (see [3]).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>X</th>
<th>Graphical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-(6, 2, 1)</td>
<td>$E(K_4)$</td>
<td><img src="image1" alt="Graphical representation" /></td>
</tr>
<tr>
<td>2-(15, 3, 1)</td>
<td>$E(K_6)$</td>
<td><img src="image2" alt="Graphical representation" /></td>
</tr>
<tr>
<td>2-(15, {3, 5}, 1)</td>
<td>$E(K_6)$</td>
<td><img src="image3" alt="Graphical representation" /></td>
</tr>
<tr>
<td>3-(10, 4, 1)</td>
<td>$E(K_5)$</td>
<td><img src="image4" alt="Graphical representation" /></td>
</tr>
<tr>
<td>4-(15, {5, 7}, 1)</td>
<td>$E(K_6)$</td>
<td><img src="image5" alt="Graphical representation" /></td>
</tr>
</tbody>
</table>

Table 3: Bigraphical generalized Steiner systems (see [9]).

<table>
<thead>
<tr>
<th>Parameters</th>
<th>X</th>
<th>Graphical representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-(n², n, 1)</td>
<td>$E(K_{n,n})$</td>
<td>$K_{1,n}$</td>
</tr>
<tr>
<td>1-(4, 2, 1)</td>
<td>$E(K_{2,2})$</td>
<td><img src="image6" alt="Graphical representation" /></td>
</tr>
<tr>
<td>2-(9, 3, 1)</td>
<td>$E(K_{3,3})$</td>
<td><img src="image7" alt="Graphical representation" /></td>
</tr>
<tr>
<td>3-(16, 4, 1)</td>
<td>$E(K_{4,4})$</td>
<td><img src="image8" alt="Graphical representation" /></td>
</tr>
<tr>
<td>3-(16, {4, 6}, 1)</td>
<td>$E(K_{4,4})$</td>
<td><img src="image9" alt="Graphical representation" /></td>
</tr>
<tr>
<td>5-(16, {6, 8}, 1)</td>
<td>$E(K_{4,4})$</td>
<td><img src="image10" alt="Graphical representation" /></td>
</tr>
</tbody>
</table>
Here are a few problems to consider.

1. What are the extended graphical and bigraphical designs with index 2? A complete list of graphical designs with index 2 can be found in [3] and the bigraphical designs with index 2 are in [15].

2. Do there exist doubly extended graphical and bigraphical designs? How should multi-extended graphical and bigraphical designs be defined?

3. Are there other actions of the symmetric group that yield interesting designs? Do they have extensions?

I close by thanking the reviewer for the inspiration to add more history and detail to my note. I also thank Ortrud Oellermann and Doug Stinson who read a preprint of this note and provided me with useful comments.

References


[7] T. Etzion Optimal constant weight codes over $\mathbb{Z}_k$ and generalized de-


[9] D.G. Hoffman and D.L. Kreher, The bigraphical $t$-wise balanced de-


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