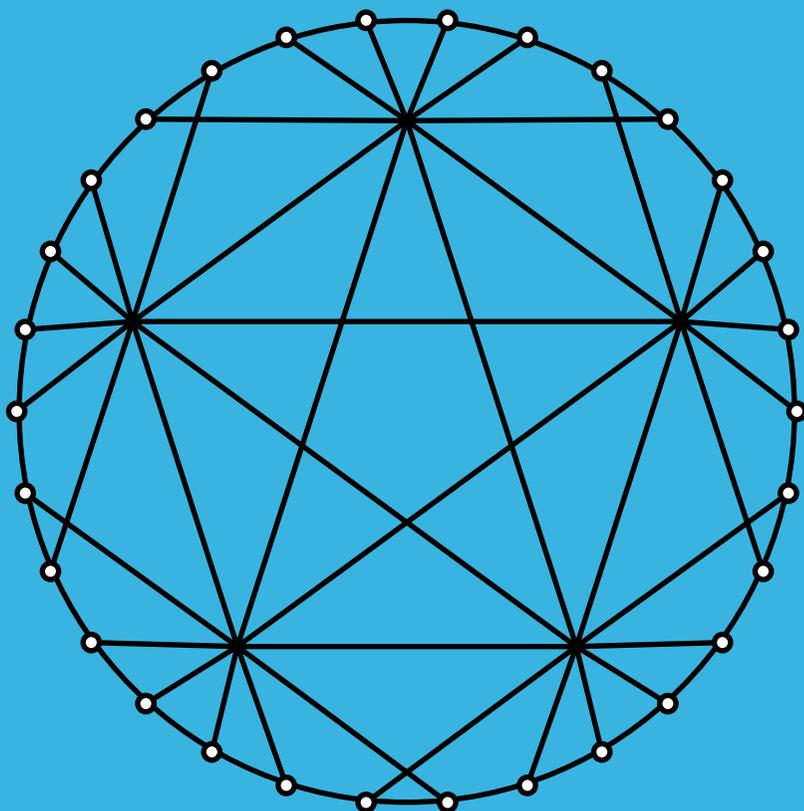


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# Global offensive $k$ -alliances in digraphs

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**Abstract.** In this paper, we initiate the study of global offensive  $k$ -alliances in digraphs. Given a digraph  $D = (V(D), A(D))$ , a global offensive  $k$ -alliance in a digraph  $D$  is a subset  $S \subseteq V(D)$  such that every vertex outside of  $S$  has at least one in-neighbor from  $S$  and also at least  $k$  more in-neighbors from  $S$  than from outside of  $S$ , by assuming  $k$  is an integer lying between two minus the maximum in-degree of  $D$  and the maximum in-degree of  $D$ . The global offensive  $k$ -alliance number  $\gamma_k^o(D)$  is the minimum cardinality among all global offensive  $k$ -alliances in  $D$ . In this article we begin the study of the global offensive  $k$ -alliance number of digraphs. We prove that finding the global offensive  $k$ -alliance number of digraphs  $D$  is an NP-hard problem for any value  $k \in \{2 - \Delta^-(D), \dots, \Delta^-(D)\}$  and that it remains NP-complete even when restricted to bipartite digraphs when we consider the non-negative values of  $k$  given in the interval above. Lower bounds on  $\gamma_k^o(D)$  with characterizations of all digraphs attaining the bounds are given in this work. We also bound this parameter for bipartite digraphs from above. For the particular case  $k = 1$ , an immediate result from the definition shows that  $\gamma(D) \leq \gamma_1^o(D)$  for all digraphs  $D$ , in which  $\gamma(D)$  stands for the domination number of  $D$ . We show that these two digraph parameters are the same for some infinite families of digraphs like rooted trees and contrafunctional digraphs. Moreover, we show that the difference between  $\gamma_1^o(D)$  and  $\gamma(D)$  can be arbitrarily large for directed trees and connected functional digraphs.

## 1 Introduction and preliminaries

Throughout this paper, we consider  $D = (V(D), A(D))$  as a finite digraph with vertex set  $V(D)$  and arc set  $A(D)$  with neither loops nor multiple arcs (although pairs of opposite arcs are allowed). Also,  $G = (V(G), E(G))$

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stands for a simple finite graph with vertex set  $V(G)$  and edge set  $E(G)$ . We use [1] and [17] as references for some very basic terminology and notation in digraphs and graphs, respectively, which are not explicitly defined here.

For any two vertices  $u, v \in V(D)$ , we write  $(u, v)$  as the *arc* with direction from  $u$  to  $v$ , and say  $u$  is *adjacent to*  $v$ , or  $v$  is *adjacent from*  $u$ . Given a subset  $S$  of vertices of  $D$  and a vertex  $v \in V(D)$ , the *in-neighborhood* of  $v$  from  $S$  (*out-neighborhood* of  $v$  to  $S$ ) is  $N_S^-(v) = \{u \in S \mid (u, v) \in A(D)\}$  ( $N_S^+(v) = \{u \in S \mid (v, u) \in A(D)\}$ ). The *in-degree* of  $v$  from  $S$  is  $\deg_S^-(v) = |N_S^-(v)|$  and the *out-degree* of  $v$  to  $S$  is  $\deg_S^+(v) = |N_S^+(v)|$ . Moreover,  $N_S^-[v] = N_S^-(v) \cup \{v\}$  ( $N_S^+[v] = N_S^+(v) \cup \{v\}$ ) is the *closed in-neighborhood* (*closed out-neighborhood*) of  $v$  from (to)  $S$ . In particular, if  $S = V(D)$ , then we simply say (closed) (in or out)-neighborhood and (in or out)-degree of  $v$ , and write  $N_D^-(v)$ ,  $N_D^+(v)$ ,  $N_D^-[v]$ ,  $N_D^+[v]$ ,  $\deg_D^-(v)$  and  $\deg_D^+(v)$  instead of  $N_{V(D)}^-(v)$ ,  $N_{V(D)}^+(v)$ ,  $N_{V(D)}^-[v]$ ,  $N_{V(D)}^+[v]$ ,  $\deg_{V(D)}^-(v)$  and  $\deg_{V(D)}^+(v)$ , respectively (we moreover remove the subscripts  $D, V(D)$ ) if there is no ambiguity with respect to the digraph  $D$ . Given two sets  $A$  and  $B$  of vertices of  $D$ , by  $(A, B)_D$  we mean the sets of arcs of  $D$  going from  $A$  to  $B$ . For a graph  $G$ ,  $\Delta = \Delta(G)$  and  $\delta = \delta(G)$  represent the maximum and minimum degrees of  $G$ . In addition,  $(\Delta^+ = \Delta^+(D))$  and  $(\delta^+ = \delta^+(D))$   $(\Delta^- = \Delta^-(D))$  and  $(\delta^- = \delta^-(D))$  represent the maximum and minimum (out-degrees) in-degrees of the digraph  $D$ .

We denote the *converse* of a digraph  $D$  by  $D^{-1}$ , obtained by reversing the direction of every arc of  $D$ . A *biorientation* of a graph  $G$  is a digraph  $D$  which is obtained from  $G$  by replacing each edge  $xy$  by either  $(x, y)$  or  $(y, x)$  or the pair  $(x, y)$  and  $(y, x)$ . While a *complete biorientation*  $D$  of  $G$  is obtained by replacing each edge  $xy$  by the pair of arcs  $(x, y)$  and  $(y, x)$ . A digraph  $D$  is *connected* if its underlying graph is connected. A *component* of a digraph  $D$  is the digraph induced by a component of the underlying graph of  $D$ . A *directed tree* is a digraph in which its underlying graph is a tree. A *rooted tree* is a connected digraph with a vertex of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1. In general, we call a vertex with in-degree 0 (out-degree 0) in a digraph  $D$  a *source* (*sink*). A digraph is *functional* (*contrafunctional*) if every vertex has out-degree (in-degree) 1.

Given a graph  $G$ , a set  $S \subseteq V(G)$  is a *dominating set* in  $G$  if each vertex in  $V(G) \setminus S$  is adjacent to a vertex in  $S$ . The *domination number*  $\gamma(G)$  of a graph  $G$  is the cardinality of a smallest dominating set of  $G$ . For more information about this concept the reader can consult [9]. The concept of domination in directed graphs was introduced by Fu [5]. A subset  $S$  of

the vertices of a digraph  $D$  is called a *dominating set* if every vertex in  $V(D) \setminus S$  is adjacent from a vertex in  $S$ . The *domination number*  $\gamma(D)$  is the minimum cardinality of a dominating set in  $D$ .

Hedetniemi et al. [10] introduced the concept of global offensive alliances in graphs. A subset  $S \subseteq V(G)$  is said to be a *global offensive alliance* in  $G$  if  $|N[v] \cap S| \geq |N[v] \cap \bar{S}|$  for each  $v \in \bar{S}$ , where  $\bar{S}$  is the complement of the set  $S$  in  $V(G)$ . The *global offensive alliance number*  $\gamma_o(G)$  is the minimum cardinality taken over all global offensive alliances in the graph  $G$ . As a generalization of such alliances, Shafique and Dutton [14, 15] defined the global offensive  $k$ -alliances in graphs. A set  $S$  of vertices of a graph  $G$  is called a *global offensive  $k$ -alliance* (GOkA for short) if  $N[S] = V(G)$  and  $|N(v) \cap S| \geq |N(v) \cap \bar{S}| + k$ , for each  $v \in \bar{S}$ . The *global offensive  $k$ -alliance number*  $\gamma_k^o(G)$  is the minimum cardinality of a GOkA in the graph  $G$ . For more information on global offensive ( $k$ -)alliances in graphs we suggest the surveys [3, 13].

Alliances in graphs have been a relatively popular research topic in graph theory in the last two decades, and a significant number of works dealing with them can be found through the literature. However, although the alliances are arising in a more natural way in a digraph than in a graph, the case of alliances in digraphs has not attracted the attention of any research till the recent work [12], where global defensive alliances in digraphs have been introduced. Consider now a social network (Twitter for instance) and an external entity which wants to spread some information in a positive sense, but that can be taken as false or as true by any user based on the number of opinions received from the other users (if one receives more true opinions, it will take it as true, otherwise it will take it as false). Suppose that the entity gives the information to a set of users  $S$  of the network. Hence, in order that the information arrives in a true way to every user of the network, it is necessary that any other user  $x \notin S$ , that can hear the news from the elements in  $S$ , will have a larger number of connections inside the set  $S$  than outside, otherwise, the information will be taken as false by  $x$ . Thinking in this way, it is readily observed that such a set  $S$  must be a global offensive ( $k$ -)alliance in such network, that can be seen as such set of elements which are more influential among every one. For the sake of efficiency, the search of a minimum number of elements that can be used to spread such kind of information is then connected with precisely finding the global offensive ( $k$ -)alliance number of graphs. If such network uses directions in the connections (like in the case of the Twitter platform), then the definition of global offensive alliances in digraphs is clearly of interest for the study of these kinds of problems, and thus the following definition, and results concerning it are worthy.

**Definition 1.1.** Let  $D$  be a digraph and let  $k \in \{2 - \Delta^-(D), \dots, \Delta^-(D)\}$  be an integer. A set of vertices  $S \subseteq V(D)$  is called a global offensive  $k$ -alliance (GOkA) in  $D$  if  $N^+[S] = V(D)$  and  $\deg_{\bar{S}}^-(v) \geq \deg_{\bar{S}}^-(v) + k$ , for each  $v \in \bar{S}$ . The global offensive  $k$ -alliance number, denoted  $\gamma_k^o(D)$ , is defined as the minimum cardinality of a GOkA in  $D$ . We call the global offensive 1-alliance (number) just global offensive alliance (number), for short.

In this paper, we first dedicate a section to the computational complexity of the problem of computing the global offensive  $k$ -alliance number of digraphs, by proving the NP-completeness of the respectively related decision problem. We next give several bounds on  $\gamma_k^o(D)$  with some emphasis on the case  $k = 1$ . For instance, we prove that  $\gamma_k^o(D)$  can be bounded from below by  $(k + \delta^-)n / (2\Delta^+ + \delta^- + k)$  and characterize all digraphs  $D$  attaining the lower bound for the specific case  $k = 1$ . As a consequence of this result we improve a lower bound on  $\gamma_1^o(G) = \gamma_o(G)$  (for graphs) given in [16]. Moreover, we show that  $(n + n_{<k})/2$  is a sharp upper bound on  $\gamma_k^o(D)$  for a bipartite digraph  $D$ , where  $n_{<k}$  is the number of vertices of in-degree less than  $k$ . Also, we discuss some relationships between  $\gamma_1^o(D)$  and  $\gamma(D)$  with emphasis on (contra)functional digraphs and rooted trees.

From now on, given any parameter  $\eta$  in a digraph  $D$ , a set of vertices of cardinality  $\eta(D)$  is called an  $\eta(D)$ -set. Also, unless specifically stated, in the whole article we shall assume  $k \in \{2 - \Delta^-(D), \dots, \Delta^-(D)\}$ .

## 2 Complexity issues

One first basic observation with respect to  $\gamma_k^o$  is the existent relationship between global offensive  $k$ -alliances of graphs and that of digraphs. Let  $G$  be a graph and  $D$  be a digraph obtained as a complete biorientation of  $G$ . We can immediately observe that a set of vertices  $S$  is a global offensive  $k$ -alliance in  $G$  if and only if  $S$  is a global offensive  $k$ -alliance in  $D$ . This leads to the next result for which we omit its straightforward proof.

**Proposition 2.1.** For any graph  $G$  and any integer  $k$ ,  $\gamma_k^o(G) = \gamma_k^o(D)$ , where  $D$  is the complete biorientation of  $G$ .

Such a relationship is very useful for giving a complexity result for the problem of computing the global offensive alliance number of digraphs. On

the other hand, the result is less useful while studying general digraphs, since only digraphs for which an arc  $(u, v)$  exists if and only if the arc  $(v, u)$  also exists can be considered.

We now consider the problem of deciding whether the global offensive  $k$ -alliance number of a digraph is less than a given integer. That is stated in the following decision problem.

GLOBAL OFFENSIVE  $k$ -ALLIANCE PROBLEM (GOK-A problem)  
 INSTANCE: A digraph  $D$ , an integer  $k$ , and a positive integer  $r$ .  
 QUESTION: Is  $\gamma_k^o(D) \leq r$ ?

The problem clearly belongs to NP since checking that a given subset of  $V(D)$  is indeed a global offensive  $k$ -alliance of cardinality at most  $r$  can be done in polynomial time. Moreover, proving the NP-completeness of the GOK-A problem above can be easily done (and therefore omitted) by making use of Proposition 2.1, and the fact that the decision problem concerning computing the global offensive  $k$ -alliance number of graphs is NP-complete (see [4]).

**Corollary 2.2.** *For a digraph  $D$  and an integer  $k$ , the GOK-A PROBLEM is NP-complete.*

We now focus on bipartite digraphs, and prove that the GOK-A PROBLEM remains NP-complete, even when restricted to such class of digraphs if we consider  $k \in \{0, \dots, \Delta^+(D)\}$ . By a *bipartite digraph* we mean a biorientation of a bipartite graph (see [1]). In order to deal with this, we make a reduction from the well-known exact cover by 3-sets problem (EC3S problem). That is, we have a set  $A$  of exactly  $n$  different elements, where  $n$  is a multiple of three, and exactly  $n$  subsets of  $A$  such that every subset contains exactly 3 elements of  $A$  and every element occurs in exactly 3 sets. It can be readily seen that at least  $\frac{n}{3}$  sets are needed to cover all the  $n$  elements. Further, it is well-known that deciding whether there are  $\frac{n}{3}$  such (pairwise disjoint) sets is in fact NP-complete (see [6]).

**Theorem 2.3.** *For a digraph  $D$  and an integer  $k \in \{0, \dots, \Delta^-(D)\}$ , the GOK-A PROBLEM is NP-complete for bipartite digraphs.*

*Proof.* As already mentioned, the problem is in NP. We now describe a polynomial transformation of the EC3S problem to the GOK-A PROBLEM.

Consider a set  $A$  of exactly  $n$  different elements, where  $n$  is a multiple of three, and exactly  $n$  subsets of  $A$ , such that every subset contains exactly 3 elements of  $A$  and every element occurs in exactly 3 sets. Let  $A = \{v_1, \dots, v_n\}$  and  $U = \{U_1, \dots, U_n\}$  be the set of elements and the collection of subsets of elements of  $A$ , respectively. Let us construct a digraph  $D$  as follows. For any element  $v_i \in A$  we create a vertex  $v_i$  of  $D$ , and for any set  $U_i \in U$ , we create a vertex  $u_i$  of  $D$ . If an element  $v_i$  occurs in a set  $U_j$ , then we add the arcs  $(v_i, u_j)$  and  $(u_j, v_i)$  (two opposite arcs). Now, for any vertex  $v_i \in A$ , we add  $k + 2$  vertices  $v_{i,1}, \dots, v_{i,k+2}$  and the arcs  $(v_{i,1}, v_i), \dots, (v_{i,k+2}, v_i)$ , and for any vertex  $u_i$ , we add  $k + 3$  vertices  $u_{i,1}, \dots, u_{i,k+3}$  and the arcs  $(u_{i,1}, u_i), \dots, (u_{i,k+3}, u_i)$ . We can easily note that the digraph constructed in this way is bipartite (an example for the case  $k = 1$  is depicted in Figure 1, for which  $A = \{v_1, v_2, v_3, v_4, v_5, v_6\}$  and  $U = \{U_1, U_2, U_3, U_4, U_5, U_6\}$  where  $U_i = \{v_i, v_{i+1}, v_{i+2}\}$ , in which  $v_7 = v_1$  and  $v_8 = v_2$ . Note that an edge with two sided arrows stands for a pair of opposite arcs). Furthermore, this is a polynomial time reduction.

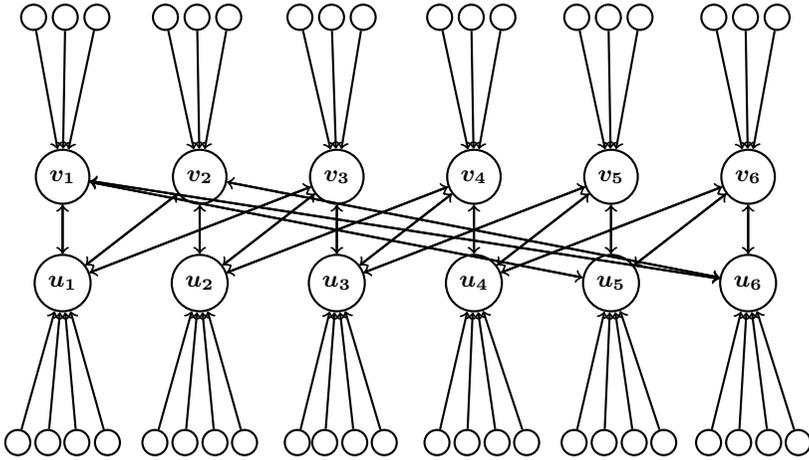


Figure 1: The depiction of the example for the case  $k = 1$ .

We shall now prove that deciding whether there are  $\frac{n}{3}$  subsets in  $U$  which cover the set  $A$  is equivalent to prove that  $D$  has global offensive  $k$ -alliance number equals  $\frac{n}{3} + n(2k + 5)$ .

We first assume that there are  $\frac{n}{3}$  sets, w.l.g. say  $U_1, \dots, U_{n/3}$ , which cover the set  $A$ . Let  $S$  be the set of vertices of  $D$  given by the union of the sets  $S_1$  and  $S_2$  such that

$$S_1 = \{u_1, \dots, u_{n/3}\}$$

and

$$S_2 = \{v_{i,j}, u_{l,q} : i, l \in \{1, \dots, n\}, j \in \{1, \dots, k+2\}, q \in \{1, \dots, k+3\}\}.$$

Note that any vertex  $u_j$  with  $j > n/3$  satisfies that  $\deg_S^-(u_j) = k+3 = \deg_S^-(u_j) + k$ . Moreover, since any vertex  $v_i$  with  $i \in \{1, \dots, n\}$  occurs in exactly one set  $U_l$  with  $l \in \{1, \dots, n/3\}$ , it is satisfied that  $\deg_S^-(v_i) = k+3 > k+2 = \deg_S^-(v_i) + k$ . Thus,  $S$  is a GOkA in  $D$ , and so,  $\gamma_k^o(D) \leq \frac{n}{3} + n(2k+5)$ .

On the other hand, let  $S'$  be a  $\gamma_k^o(D)$ -set. Since any vertex  $v_{i,j}$  and any vertex  $u_{l,q}$  with  $i, l \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, k+2\}$  and  $q \in \{1, \dots, k+3\}$  has in-degree zero, we deduce that such vertices must belong to  $S'$ , which means

$$|S' \cap S_2| = n(2k+5). \quad (1)$$

Now, if there is a vertex  $v_i$  for which  $N_{S'}^-(v_i) \cap \{u_1, \dots, u_n\} = \emptyset$ , then  $\deg_{S'}^-(v_i) = k+2 < k+3 = \deg_{S'}^-(v_i) + k$ , which is not possible. Thus, any vertex  $v_i$ , with  $i \in \{1, \dots, n\}$  must have an in-neighbor in  $S' \cap \{u_1, \dots, u_n\}$ . Let  $t = |S' \cap \{u_1, \dots, u_n\}|$ . Since every vertex  $v_i$  has at least one in-neighbor in  $S' \cap \{u_1, \dots, u_n\}$  and every vertex  $u_i$  has three out-neighbors in  $\{v_1, \dots, v_n\}$ , we have

$$3t \geq \sum_{i=1}^n |N_{S' \cap \{u_1, \dots, u_n\}}^-(v_i)| \geq n. \quad (2)$$

Therefore, by using (1) and (2), we deduce that  $\gamma_k^o(D) \geq \frac{n}{3} + n(2k+5)$ , which leads to the desired equality.

We now assume that  $\gamma_k^o(D) = \frac{n}{3} + n(2k+5)$  and let  $Q$  be a  $\gamma_k^o(D)$ -set. As stated while proving the previous implication, it must happen that

$$\{v_{i,j}, u_{l,q} : i, l \in \{1, \dots, n\}, j \in \{1, \dots, k+2\}, q \in \{1, \dots, k+3\}\} \subset Q.$$

Moreover, we can similarly see that  $|Q \cap \{u_1, \dots, u_n\}| \geq n/3$ , and that every vertex in the set  $\{v_1, \dots, v_n\}$  has at least one in-neighbor in  $Q \cap \{u_1, \dots, u_n\}$ . Since  $\gamma_k^o(D) = \frac{n}{3} + n(2k+5)$ , it must happen that  $|Q \cap \{u_1, \dots, u_n\}| = n/3$ , which leads to that every vertex in  $\{v_1, \dots, v_n\}$  has exactly one in-neighbor in  $Q \cap \{u_1, \dots, u_n\}$ . Let  $W = Q \cap \{u_1, \dots, u_n\}$  (note that  $|W| = n/3$ ). If the sets (without loss of generality, say,  $C_1, \dots, C_{n/3}$ ), corresponding

to the vertices of  $W$ , do not form an exact cover of  $U$ , then either there is an element of  $A$  which is not in any set  $C_1, \dots, C_{n/3}$  or there is an element of  $A$  which belongs to two sets of  $C_1, \dots, C_{n/3}$ . Both situations lead to a contradiction with the fact that  $|W| = n/3$  and every vertex  $v_i$ ,  $i \in \{1, \dots, n\}$ , has exactly one in-neighbor in  $W$ . Therefore,  $C_1, \dots, C_{n/3}$  form an exact cover of the elements in  $A$ , and this completes the proof of this implication, and the desired reduction.  $\square$

### 3 Bounding $\gamma_k^o(D)$

Since the problem of computing the global offensive  $k$ -alliance number of digraphs is NP-hard, it is then desirable to bound it for general digraphs. We begin exhibiting a lower bound on  $\gamma_k^o(D)$  for a general digraph  $D$ . In order to characterize all digraphs attaining the bound with  $k = 1$ , we define the family  $\Phi$  of digraphs as follows. Suppose that  $\widehat{D}$  is a digraph with the set of vertices  $\{v_1, \dots, v_{n'}, u_1, \dots, u_p\}$  such that

- (i)  $(r' + 1)n' \equiv 0 \pmod{p}$  and  $(r' + 1)n'/p \geq \deg_{\widehat{D}}^+(v_i)$ , for each  $1 \leq i \leq n'$ ,
- (ii) the in-degrees of all vertices  $v_i$  in  $\widehat{D}(\{v_1, \dots, v_{n'}\})$  equal  $r'$ ,
- (iii)  $\deg_{\widehat{D}}^+(u_i) = 0$  and  $\deg_{\widehat{D}}^-(u_i) \geq 2r' + 1$ , for each  $1 \leq i \leq p$ .

We now add  $r = (r' + 1)n'/p$  arcs from each  $u_i$ ,  $1 \leq i \leq p$ , to the vertices in  $\{v_1, \dots, v_{n'}\}$  such that all vertices  $v_i$  are incident to  $r' + 1$  such arcs. Let  $D$  be the obtained digraph, and let  $\Phi$  be the family of all digraphs  $D$ .

As an example, let  $D$  be obtained from the complete biorientation of the cycle  $C_t$  on vertices  $v_1, \dots, v_t$  with  $t \geq 5$ , by adding three new vertices  $u_1$ ,  $u_2$  and  $u_3$  and the set of new arcs

$$\{(u_i, v_1), \dots, (u_i, v_t)\}_{i=1}^3 \cup \{(v_j, u_1), (v_j, u_2), (v_j, u_3)\}_{j=1}^5.$$

Then,  $D$  is a member of  $\Phi$  with  $(n', p, r', r) = (t, 3, 2, t)$ , in which  $\widehat{D}$  is the graph with  $V(\widehat{D}) = V(D)$  and  $E(\widehat{D}) = E(D) \setminus \{(u_i, v_1), \dots, (u_i, v_t)\}_{i=1}^3$ . Such a digraph  $D$ , for  $t = 7$ , is depicted in Figure 2

**Theorem 3.1.** *If  $D$  is a digraph of order  $n$ , minimum in-degree  $\delta^-$  and maximum in-degree  $\Delta^-$ , then*

$$\gamma_k^o(D) \geq \left( \frac{k + \delta^-}{2\Delta^- + \delta^- + k} \right) n.$$

Moreover, for the case  $k = 1$ , the equality in the bound holds if and only if  $D \in \Phi$ .

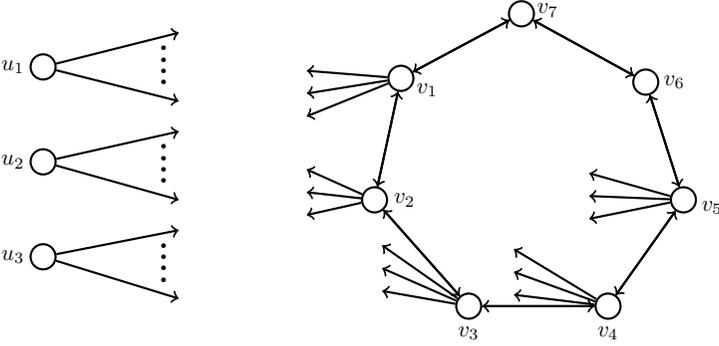


Figure 2: An edge with two sided arrows stands for a pair of opposite arcs. Every vertex  $u_i$  is adjacent to all vertices  $v_j$ , and each vertex  $v_j$  with  $j \in \{1, \dots, 5\}$  is adjacent to  $u_1, u_2$  and  $u_3$ .

*Proof.* Let  $S$  be a  $\gamma_k^o(D)$ -set. We have

$$\begin{aligned}
 \Delta^+|S| &\geq |(S, \bar{S})_D| = \sum_{v \in \bar{S}} \deg_{\bar{S}}^-(v) \geq \sum_{v \in \bar{S}} (\deg_{\bar{S}}^-(v) + k) \\
 &= k|\bar{S}| + \sum_{v \in \bar{S}} \deg^-(v) - \sum_{v \in \bar{S}} \deg_{\bar{S}}^-(v) \\
 &\geq (k + \delta^-)|\bar{S}| - \Delta^+|S|.
 \end{aligned} \tag{3}$$

Therefore, the bound can be deduced from the above. We next consider the case  $k = 1$ , which particularly means  $\gamma_1^o(D) \geq \left(\frac{1 + \delta^-}{2\Delta^+ + \delta^- + 1}\right)n$ , and present the characterization of the digraphs achieving the equality in this situation.

Suppose that the lower bound holds with equality for a digraph  $D$ . Hence, all the inequalities in (3) necessarily hold with equality. In particular, this means  $\sum_{v \in \bar{S}} \deg^-(v) = \delta^-|\bar{S}|$ , which is equivalent to say that the in-degrees  $\deg^-(v) = \deg_{\bar{S}}^-(v) + \deg_{\bar{S}}^-(v)$  of all vertices in  $v \in V(D')$  equal  $\delta^-$ , where  $D'$  is the subdigraph induced by  $\bar{S}$ . Moreover,  $\deg_{\bar{S}}^-(v) = \deg_{\bar{S}}^-(v) + 1$  (note that  $k = 1$ ) for all  $v \in V(D')$ , by the equality in the second inequality in (3). Therefore, all vertices in  $V(D')$  have the same in-degree, say  $r'$ , in the subgraph induced by  $V(D')$ . On the other hand, every vertex in  $S$  is adjacent to precisely  $\Delta^+$  vertices of  $D'$  since  $\Delta^+|S| = |(S, \bar{S})_D|$ . Now, since  $\deg_{\bar{S}}^-(v) = \deg_{\bar{S}}^-(v) + 1 = r' + 1$  for all  $v \in V(D')$ , and  $\Delta^+|S| = |(S, \bar{S})_D|$ , we have that  $|V(D')|(r' + 1) = |S|\Delta^+$ . Thus, the membership of  $D$  in  $\Phi$  easily follows by choosing  $|V(D')|$ ,  $D - (S, \bar{S})_D$ ,  $\Delta^+$  and  $S$  for  $n'$ ,  $\hat{D}$ ,  $r$  and the set  $\{u_1, \dots, u_p\}$ , respectively, in the description of  $\Phi$ . Thus,  $D \in \Phi$ .

Conversely, let  $D \in \Phi$ . It can be observed that  $\{u_1, \dots, u_p\}$  is a GO1A in  $D$ . Moreover,  $(n, \delta^-, \Delta^+) = (n' + p, 2r' + 1, n'(r' + 1)/p)$ . Thus,  $\gamma_1^o(D) \leq p = (1 + \delta^-)n/(2\Delta^+ + \delta^- + 1)$ . This completes the proof.  $\square$

As a result of the lower bound in Theorem 3.1 we have the following.

**Corollary 3.2.** *For any graph  $G$  of order  $n$ , minimum degree  $\delta$  and maximum degree  $\Delta$ ,  $\gamma_k^o(G) \geq \lceil (k + \delta)n/(2\Delta + \delta + k) \rceil$ .*

*Proof.* Let  $D$  be the complete biorientation of  $G$ . It is then straightforward to note that  $|V(D)| = n$ ,  $\delta^+(D) = \delta^-(D) = \delta$ ,  $\Delta^+(D) = \Delta^-(D) = \Delta$ , and that  $\gamma_k^o(D) = \gamma_k^o(G)$  by Proposition 2.1. Now the result follows from Theorem 3.1.  $\square$

For the particular case of  $k = 1$ , Sigarreta and Rodríguez-Velázquez [16] proved that

$$\gamma_1^o(G) \geq \begin{cases} \lceil (1 + \delta)n/(2\Delta + \delta + 1) \rceil, & \text{if } \delta \text{ is odd,} \\ \lceil n\delta/(2\Delta + \delta) \rceil, & \text{otherwise.} \end{cases} \quad (4)$$

Since  $\lceil (1 + \delta)n/(2\Delta + \delta + 1) \rceil \geq \lceil n\delta/(2\Delta + \delta) \rceil$ , Corollary 3.2 is an improvement of the lower bound given in (4) when  $\delta$  is even.

We next continue with an upper bound on  $\gamma_k^o(D)$ .

**Theorem 3.3.** *Let  $D$  be a bipartite digraph of order  $n$  and let  $n_{<k}$  be the number of vertices of in-degree less than  $k$  in  $D$ . Then,*

$$\gamma_k^o(D) \leq \frac{n + n_{<k}}{2},$$

and this bound is sharp.

*Proof.* We consider  $V_{<k}$  as the set of all vertices of in-degree less than  $k$ . Let  $X$  and  $Y$  be the partite sets of  $D$  and  $X' = X \setminus V_{<k}$  and  $Y' = Y \setminus V_{<k}$ . Moreover, we may assume that  $|X'| \geq |Y'|$ . The above argument guarantees that each vertex in  $X'$  has at least  $k$  in-neighbors and all such in-neighbors belong to  $Y$ , necessarily. Therefore,  $V(D) \setminus X'$  is a  $\text{GO}k\text{A}$  in  $D$ . Therefore,

$$\gamma_k^o(D) \leq n - |X'| \leq n - \frac{|X'| + |Y'|}{2} = n - \frac{|X| + |Y| - n_{<k}}{2} = \frac{n + n_{<k}}{2}. \quad (5)$$

The sharpness of the upper bound can be seen as follows. We begin with the complete biorientation  $D'$  of the complete bipartite graph  $K_{p,p}$  with partite sets  $X = \{x_1, \dots, x_p\}$  and  $Y = \{y_1, \dots, y_p\}$  such that  $k \leq p \leq 2k - 1$ . We obtain the digraph  $D$  by removing the set of arcs  $\{(x_i, y_j), (y_j, x_i)\}_{i,j=1}^k$ . This shows that  $n_{<k} = 2k$ . Now, let  $S$  be a  $\gamma_k^o(D)$ -set. Clearly,  $\{x_i, y_i\}_{i=1}^k \subseteq S$ . If a partite set, say  $X$ , is a subset of  $S$ , then  $|S| = |X| + |Y \cap S| \geq |X| + k = (n + n_{<k})/2$  which implies the equality in the upper bound. So, we may assume that both  $X \setminus S$  and  $Y \setminus S$  are nonempty. Let  $t_1 = |X \setminus S|$  and  $t_2 = |Y \setminus S|$ . Suppose that  $y \in Y \setminus S$ . Since  $\deg_{\overline{S}}(y) \geq \deg_{\overline{S}}(y) + k$ , we have  $|X \cap S| \geq t_1 + k$ . Moreover,  $|Y \cap S| \geq t_2 + k$  by a similar fashion. Together the last two inequalities imply  $|S| \geq t_1 + t_2 + 2k = n - |S| + n_{<k}$ . Thus,  $|S| \geq (n + n_{<k})/2$  which results in  $\gamma_k^o(D) = (n + n_{<k})/2$  by (5). This completes the proof.  $\square$

As an immediate consequence of the definitions given in the introduction, we have  $\gamma(D) \leq \gamma_1^o(D)$ , for any digraph  $D$ . On the other hand, one can observe that any directed tree is a bipartite digraph. So, as an immediate consequence of Theorem 3.3 we have the following result.

**Corollary 3.4.** *Let  $T$  be a directed tree of order  $n$ . Then, the following statements hold.*

- (i)  $\gamma(T) \leq \lfloor (n + q)/2 \rfloor$ , where  $q$  is the number of sources.
- (ii) If  $T$  is a rooted tree, then  $\gamma(T) \leq \lceil n/2 \rceil$ . ([11])

We shall show in Section 3.1 that there are some infinite families of directed trees for which  $\gamma_1^o$  and  $\gamma$  differ.

### 3.1 The specific case of (contra)functional digraphs and rooted trees when $k = 1$

In this section, we investigate the global offensive alliance number and the domination number for (contra)functional digraphs and rooted trees.

**Proposition 3.5.** *For any rooted tree or contrafunctional digraph  $D$ ,*

$$\gamma_1^o(D) = \gamma(D).$$

*Proof.* Since every vertex in a contrafunctional digraph has in-degree one, every dominating set is a GO1A. Similarly, every dominating set is a GO1A in a rooted tree (the root of a rooted tree belongs to every dominating set and every GO1A). Therefore,  $\gamma_1^o(D) \leq \gamma(D)$  which implies the equality.  $\square$

Note that the difference between  $\gamma_1^o(D)$  and  $\gamma(D)$  can be arbitrarily large, even for connected functional digraphs and directed trees as we can see in the following example. Let  $b$  be an arbitrary positive integer. Let  $D'$  be obtained from a directed cycle  $C$  on vertices  $v_1, \dots, v_{2b}$  by adding new vertices  $v'_1, \dots, v'_{2b}$  and arcs  $(v'_1, v_1), \dots, (v'_{2b}, v_{2b})$ . Then,  $\{v'_1, \dots, v'_{2b}\}$  is the minimum dominating set in  $D'$  while  $\{v'_1, \dots, v'_{2b}\} \cup \{v_{2i}\}_{i=1}^b$  is a minimum GO1A in  $D'$ . Thus,  $\gamma_1^o(D') - \gamma(D') = b$  (see Figure 3 when  $b = 4$ ). We now let  $T$  be a directed tree by removing one arc from the directed cycle  $C$  of  $D'$ . It is easy to see that  $\gamma_1^o(T) - \gamma(T) = 3b - 2b = b$ .

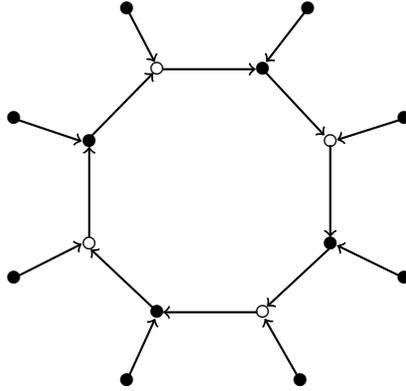


Figure 3: The sources form a (unique) minimum dominating set and the black vertices form a minimum GO1A.

A vertex  $y$  is called *accessible* or *reachable* from  $x$  if there is a path in  $D$  from  $x$  to  $y$ . Let  $R(x)$  be the set of all vertices accessible from  $x$  and let  $R^{-1}(x)$  be the set of all vertices from which  $x$  is accessible. We make use of the following lemma due to Harary.

**Lemma 3.6.** ([7]) *A digraph  $D$  is functional if and only if each of its components consists of exactly one directed cycle  $C$  and for each vertex  $v$  of  $C$ , the converse of subgraph induced by  $R^{-1}(v)$  of the digraph  $D - C$  is a rooted tree with the root  $v$ .*

**Theorem 3.7.** *Let  $D$  be a connected functional digraph of order  $n$  with  $q$  sources. Then,*

$$\gamma_1^o(D) \leq \left\lfloor \frac{n+q+1}{2} \right\rfloor.$$

*Furthermore, this bound is sharp.*

*Proof.* We consider a connected functional digraph  $D$  in view of Lemma 3.6. Let  $C$  be the unique directed cycle of  $D$ . Let  $Q$  be the set of all sources in  $D$ . Then  $D' = D - Q$  is still a connected functional digraph with the unique directed cycle  $C$ . We define the height  $h(D')$  of the connected functional digraph  $D'$  as  $\max\{d_{D'}(v, V(C)) \mid v \in V(D')\}$  where  $d_{D'}(v, V(C))$  represents the length of a shortest path between  $v$  and a vertex of  $V(C)$ . Let  $D'_1 = D'$ . We select a source  $v_1$  with maximum distance from  $C$  and let  $u_1$  be its unique out-neighbor. Let  $D'_2 = D'_1 - N_{D'_1}^-[u_1]$ . Iterate this process for the remaining connected functional digraph  $D'_i$  until  $D'_p$  is the directed cycle  $C$  on vertices  $w_1, \dots, w_{|V(C)|}$  or a connected functional digraph with height one. In fact, we have a partition  $\{Q, N_{D'_1}^-[u_1], \dots, N_{D'_{p-1}}^-[u_{p-1}], V(D'_p)\}$  of  $V(D)$ . If  $D'_p$  is the directed cycle  $C$ , then

$$S_1 = Q \cup \{u_i\}_{i=1}^{p-1} \cup \{w_{2j-1}\}_{j=1}^{\lfloor (|V(C)|+1)/2 \rfloor}$$

is a GO1A in  $D$ . Let  $D'_p$  contain the directed cycle  $C$  and some arcs  $(w'_j, w_j)$  for some  $1 \leq j \leq |V(C)|$ . We observe that  $D'_p - \{w'_j, w_j\}_j$ , with  $j \in \{1, \dots, |V(C)|\}$ , is either a disjoint union of some directed paths  $P_r$  on vertices  $x_1, \dots, x_r$ , or it is empty. If it is empty, then

$$S_2 = Q \cup \{u_i\}_{i=1}^{p-1} \cup \{w_j\}_{j=1}^{|V(C)|}$$

is a GO1A in  $D$ . So, we assume  $D'_p - \{w'_j, w_j\}_j$ , with  $j \in \{1, \dots, |V(C)|\}$ , is not empty. Let  $V_e$  be the set of vertices on the directed paths  $P_r$  with even subscripts. Then,

$$S_3 = Q \cup \{u_i\}_{i=1}^{p-1} \cup \{w_j\}_j \cup V_e$$

is a GO1A in  $D$ .

On the other hand,  $p-1 \leq (n-q-|V(D'_p)|)/2$ . Moreover, the cardinalities of the sets  $\{w_{2j-1}\}_{j=1}^{\lfloor (|V(C)|+1)/2 \rfloor}$ ,  $\{w_j\}_{j=1}^{|V(C)|}$  and  $\{w_j\}_j \cup V_e$  are bounded from above by  $(|V(D'_p)|+1)/2$  for  $S_1$ ,  $S_2$  and  $S_3$ , respectively. Therefore, for any  $i \in \{1, 2, 3\}$  we have,

$$\gamma_1^o(D) \leq |S_i| \leq q + (n-q-|V(D'_p)|)/2 + (|V(D'_p)|+1)/2 = (n+q+1)/2.$$

To see the sharpness of the bound, consider a directed cycle  $C$  on vertices  $y_1, y_2, \dots, y_t$  and add disjoint directed paths  $x_{i,1}, \dots, x_{i,2k_i+1}$  for each  $1 \leq i \leq t$ , for which  $x_{i,2k_i+1}$  is adjacent to  $y_i$ . Let  $D^*$  be the obtained connected functional digraph. It is now easy to see that  $\{x_{i,1}, x_{i,3}, \dots, x_{i,2k_i+1}\}_{i=1}^t \cup \{y_1, y_3, \dots, y_{2\lfloor (t+1)/2 \rfloor - 1}\}$  is a minimum dominating set in  $D^*$  of cardinality  $\lfloor \frac{n+q+1}{2} \rfloor$ . Therefore,  $\gamma_1^o(D^*) = \lfloor \frac{n+q+1}{2} \rfloor$ .  $\square$

## 4 Concluding remarks

We have introduced and begun the study of several combinatorial and computational properties of the global offensive  $k$ -alliances in digraphs. The results presented above have allowed us to generate a new research line on the theory of digraphs which we intend to continue exploring by possibly dealing with some and/or all the following open problems.

- Similarly to the case of graphs, alliances can be analyzed not only from a global way, but also in a local way. That is, for a given digraph  $D = (V(D), A(D))$ , one can consider a set of vertices  $S \subseteq V(D)$  as an offensive  $k$ -alliance in  $D$  if  $\deg_S^-(v) \geq \deg_S^-(v) + k$  for all  $v \in N_D(S) \setminus S$  (which is equivalent to say that  $S$  is not necessarily a dominating set in  $D$ ). The offensive  $k$ -alliance number, which could be denoted  $a_k^o(D)$ , is then defined as the minimum cardinality of an offensive  $k$ -alliance in  $D$ . The study of the non global case for an offensive  $k$ -alliance may be of potential interest to continue this research line, which we have presented in this article.
- Another issue that requires to be dealt with concerns completing the NP-hardness property of computing the global offensive  $k$ -alliance number of bipartite digraphs. That is, finding which is the complexity of the  $GO_k$ -A problem studied above for suitable negative values of  $k$ . It would probably not be surprising if such problem belongs to the so-called NP-hard class (as for the other values for which it is already proved here): however, a proof of it is required. In this sense, it is maybe possible to adapt the idea of the proof [4, Theorem 2] to directed graphs to make the reduction. In addition, finding some classes of digraphs for which such a problem could be polynomially solved will give more insight into the study of global offensive  $k$ -alliances in digraphs.
- Since the global offensive  $k$ -alliances can be used to model the situation of finding the most influential elements of a network, it is worth

finding some algorithms (that could even not be polynomial) together with some heuristics that would allow one to make some implementations and experiments on real social networks in order to detect the “influencers” (according to the social network terminology) of such networks.

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