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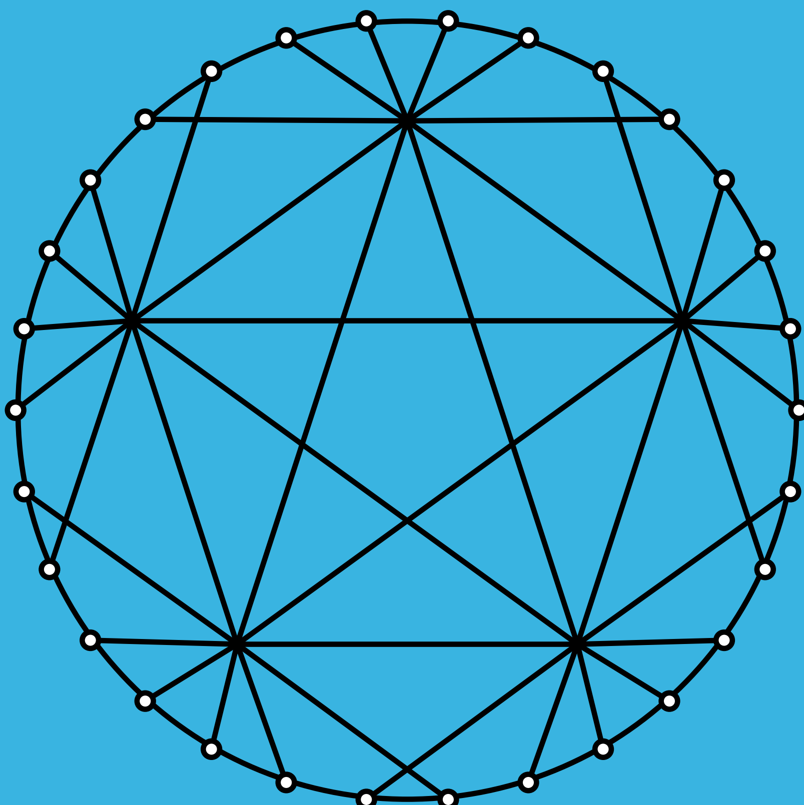

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On the average (edge-)connectivity of minimally k-(edge-)connected graphs

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Abstract. Let G be a graph of order n and let u, v be vertices of G. Let $\kappa_G(u,v)$ denote the maximum number of internally disjoint u–v paths in G. Then the *average connectivity* $\overline{\kappa}(G)$ of G, is defined as $\overline{\kappa}(G) = \sum_{\{u,v\}\subseteq V(G)} \kappa_G(u,v)/\binom{n}{2}$. If $k \geq 1$ is an integer, then G is *minimally k-connected* if $\kappa(G) = k$ and $\kappa(G-e) < k$ for every edge e of G. We say that G is an *optimal* minimally k-connected graph if G has maximum average connectivity among all minimally k-connected graphs of order n. Based on a recent structure result for minimally 2-connected graphs we conjecture that, for every integer $k \geq 3$, if G is an optimal minimally k-connected graph of order $n \geq 2k+1$, then G is bipartite, with the set of vertices of degree k and the set of vertices of degree exceeding k as its partite sets. We show that if this conjecture is true, then $\overline{\kappa}(G) < \frac{9}{8}k$ for every minimally k-connected graph G. For every $k \geq 3$, we describe an infinite family of minimally k-connected graphs whose average connectivity is asymptotically $\frac{9}{8}k$. Analogous results are established for the average edge-connectivity of minimally k-edge-connected graphs.

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1 Introduction

Let G be a nontrivial graph. The *connectivity* of G, denoted by $\kappa(G)$, is the smallest number of vertices whose removal disconnects G or produces a trivial graph. The *edge-connectivity* of G, denoted by $\lambda(G)$, is the smallest number of edges whose removal disconnects G or produces a trivial graph.

Following Beineke, Oellermann, and Pippert [2], for a pair u, v of distinct vertices of G, we define the *connectivity* between u and v in G, denoted by $\kappa_G(u, v)$, to be the maximum number of internally disjoint u–v paths. The *edge-connectivity* between u and v, denoted by $\lambda_G(u, v)$, is the maximum number of edge-disjoint u–v paths. Menger's well-known theorem [11] states that if u and v are non-adjacent, then $\kappa_G(u, v)$ equals the smallest number of vertices whose removal from G separates u and v. The edge-connectivity version of Menger's theorem states that $\lambda_G(u, v)$ equals the minimum number of edges whose removal from G separates u and v. When G is clear from context we omit the subscript G from $\kappa_G(u, v)$ and $\lambda_G(u, v)$.

Whitney [14] showed that $\kappa(G) = \min\{\kappa(u, v) \mid u, v \in V(G)\}$. In a similar manner it follows that $\lambda(G) = \min\{\lambda(u, v) \mid u, v \in V(G)\}$. These results show that both the connectivity and the edge-connectivity of a graph are worst-case measures. A more refined measure of the overall level of connectedness of a graph was introduced by Beineke, Oellermann, and Pippert [2], and is based on the average values of the 'local connectivities' between all pairs of vertices. The *average connectivity* of a graph G of order n, denoted by $\overline{\kappa}(G)$, is the average of the connectivities over all pairs of distinct vertices of G. That is,

$$\overline{\kappa}(G) = \sum_{\{u,v\} \subseteq V(G)} \kappa(u, v) / \tbinom{n}{2}.$$

Analogously, the *average edge-connectivity* of G, studied by Dankelmann and Oellermann [6], and denoted by $\overline{\lambda}(G)$, is the average of the edge-connectivities over all pairs of distinct vertices of G. That is,

$$\overline{\lambda}(G) = \sum_{\{u,v\} \subseteq V(G)} \lambda(u, v) / \tbinom{n}{2}.$$

Several bounds for the average connectivity in terms of various graph parameters, such as for example, the order and size [2], the average degree [5], and the matching number [7] have been determined. Bounds on the average connectivity of graphs belonging to particular families have also been established, including bounds for planar and outerplanar graphs [5], Cartesian product graphs [5], strong product graphs [1], and regular graphs [7]. The average connectivity also plays a role in the assessment of the reliability of real-world networks, including street networks [3] and communication networks [13].

In this paper we study by how much the average (edge-)connectivity can vary in a class of graphs, whose members are in some sense just barely k-(edge-)connected for some integer $k \geq 1$. A graph G is called *minimally k-connected* if $\kappa(G) = k$ and $\kappa(G - e) < k$ for every edge e of G. *Minimally k-edge-connected* graphs are defined similarly. It is natural to ask by how much the average (edge-)connectivity of a minimally k-(edge-)connected graph can differ from k. Trivially the smallest average (edge-)connectivity among all minimally k-(edge-)connected graph is k. For the remainder of the paper we thus focus on an upper bound for the average connectivity for all minimally k-(edge-)connected graphs. We say that G is an *optimal* minimally k-connected graph if G has maximum average connectivity among all minimally k-connected graphs. Since minimally 1-(edge-)connected graphs are precisely the trees, they have average connectivity 1. However, for $k \geq 2$, the average (edge-)connectivity of a minimally k-(edge-)connected graph need not be k. The structure of optimal minimally 2-(edge-)connected graphs, and an upper bound on their average (edge-) connectivity is determined by Casablanca, Mol, and Oellermann [4]. In order to state these results we say that a minimally k-(edge-)connected graph is *degree-partitioned* if it is bipartite, with partite sets the set of vertices of degree k and the set of vertices of degree exceeding k. (Note that every degree-partitioned minimally k-(edge-)connected graph has order at least $2k + 1$.)

Theorem 1.1 (Casablanca, Mol, and Oellermann [4]).

(a) *If G is an optimal minimally 2-connected graph of order $n \geq 5$, then G is degree-partitioned. Moreover, we have $\overline{\kappa}(G) < \frac{9}{4}$, and this bound is asymptotically sharp.*

(b) *If G is an optimal minimally 2-edge-connected graph of order $n \geq 5$, then G is degree-partitioned. Moreover, we have $\overline{\lambda}(G) < \frac{9}{4}$, and this bound is asymptotically sharp.*

In this paper, we continue the study of the average (edge-)connectivity of minimally k-(edge-)connected graphs, which was initiated by Casablanca, Mol, and Oellermann [4]. Mader [10] showed that the vertices of degree exceeding k in a minimally k-connected graph induce a forest. Based on Theorem 1.1, and some computational evidence, we believe that something similar can be said about the structure of optimal minimally k-(edge-)connected graphs for every $k \geq 3$.

Conjecture 1.2. Let $k \geq 3$. If G is an optimal minimally k-(edge-)connected graph of order $n \geq 2k + 1$, then G is degree-partitioned.

In Section 2, we show that if $k \geq 2$ and G is a degree-partitioned minimally k-connected graph of order n, then the average connectivity of G satisfies

$$\overline{\kappa}(G) \leq k + \frac{k(n-2)^2}{8n(n-1)} < \frac{9}{8}k. \tag{1}$$

By a similar argument, it follows that if $k \geq 2$ and G is a degree-partitioned minimally k-edge-connected graph of order n, then the average edge-connectivity of G satisfies

$$\overline{\lambda}(G) \leq k + \frac{k(n-2)^2}{8n(n-1)} < \frac{9}{8}k. \tag{2}$$

We note that, if Conjecture 1.2 holds, then every minimally k-connected graph G satisfies $\overline{\kappa}(G) < \frac{9}{8}k$, and every minimally k-edge-connected graph G satisfies $\overline{\lambda}(G) < \frac{9}{8}k$. The inequalities given in (1) and (2) were established in [4] for the case $k = 2$ and it was remarked that these proofs could be extended to all $k \geq 3$.

In Section 3.1 we describe, for every $k \geq 3$, an infinite family of degree-partitioned minimally k-edge-connected graphs whose average edge-connectivity is asymptotically $\frac{9}{8}k$. In Section 3.2 we describe, for every $k \geq 3$, an infinite family of degree-partitioned minimally k-connected graphs whose average connectivity is asymptotically $\frac{9}{8}k$. Thus, the upper bounds given by (1) and (2) are asymptotically sharp.

2 Upper bounds

In order to establish the upper bounds given by (1) and (2), we generalize the argument given by Casablanca, Mol, and Oellermann [4, Section 2.2] for $k = 2$. We first recall some terminology (c.f. [4]).

Let G be a graph of order n. The *total connectivity* of G, denoted by $K(G)$, is the sum of the connectivities over all pairs of distinct vertices of G, i.e., we have $K(G) = \binom{n}{2}\overline{\kappa}(G)$. The *potential* of a sequence of positive integers $d_1, d_2, \ldots, d_n$ is defined by

$$P(d_1, d_2, \ldots, d_n) = \sum_{1 \le i < j \le n} \min\{d_i, d_j\}.$$

If G has vertices $v_1, v_2, \ldots, v_n$, then the *potential* of G, denoted by $P(G)$, is the potential of the degree sequence of G; that is,

$$P(G) = P(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)) = \sum_{1 \le i < j \le n} \min\{\deg(v_i), \deg(v_j)\}.$$

Since $\kappa(u, v) \le \min\{\deg(u), \deg(v)\}$ for all pairs of distinct vertices u, v of G, we have $K(G) \le P(G)$.

We require the following lemma, which describes the maximum potential among all sequences of n positive integers whose sum is a fixed number D.

Lemma 2.1 (Beineke, Oellermann, and Pippert [2])**.** *Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a graph, and let $D = \sum_{i=1}^{n} d_i$. Let $D = dn + r$, where $d \ge 0$ and $0 \le r < n$. Then*

$$P(d_1, d_2, \ldots, d_n) \le P(\underbrace{d, \ldots, d}_{n-r \text{ terms}}, \underbrace{d+1, \ldots, d+1}_{r \text{ terms}}).$$

We are now ready to prove the upper bound given by (1). Recall that a minimally k-connected graph is called degree-partitioned if it is bipartite, with partite sets the set of vertices of degree k and the set of vertices of degree exceeding k.

Theorem 2.2. *Let $k \ge 2$, and let G be a degree-partitioned minimally k-connected graph of order $n \ge 2k+1$. Then*

$$\overline{\kappa}(G) \le k + \frac{k(n-2)^2}{8n(n-1)} < \frac{9}{8}k.$$

Proof. Suppose that G has s vertices of degree exceeding k, and hence $n - s$ vertices of degree k. Let $d_1, d_2, \ldots, d_s$ be the degrees of the vertices of degree exceeding k. Since G is degree-partitioned, the sum $d_1 + d_2 + \cdots + d_s$ must be equal to $k(n-s)$, the sum of the degrees of the vertices having degree k.

Let $k(n-s) = ds + r$ for $d, r \in \mathbb{Z}$ and $0 \le r < s$. Then by Lemma 2.1, we have

$$\begin{aligned} K(G) \le P(G) &\le k\left[\binom{n}{2} - \binom{s}{2}\right] + P(d_1, d_2, \ldots, d_s) \\ &\le k\left[\binom{n}{2} - \binom{s}{2}\right] + P(\underbrace{d, \ldots, d}_{s-r \text{ terms}}, \underbrace{d+1, \ldots, d+1}_{r \text{ terms}}) \\ &\le k\binom{n}{2} - k\binom{s}{2} + d\binom{s}{2} + \binom{r}{2} \\ &= k\binom{n}{2} + \tfrac{k(n-2s)(s-1)}{2} - \tfrac{r(s-r)}{2} \\ &\le k\binom{n}{2} + \tfrac{k}{2}(n-2s)(s-1) \end{aligned}$$

Using elementary calculus, we find that the quantity $(n-2s)(s-1)$ achieves a maximum of $\frac{(n-2)^2}{8}$ at $s = \frac{n+2}{4}$. Thus we have

$$K(G) \le k\binom{n}{2} + \frac{k}{2}\frac{(n-2)^2}{8}.$$

Now dividing through by $\binom{n}{2}$ gives the desired upper bound on $\overline{\kappa}(G)$. $\square$

The upper bound given by (2) can be established in a strictly analogous manner, so we omit the proof.

Theorem 2.3. *Let $k \ge 2$, and let G be a degree-partitioned minimally k-edge-connected graph of order $n \ge 2k+1$. Then*

$$\overline{\lambda}(G) \le k + \frac{k(n-2)^2}{8n(n-1)} < \frac{9}{8}k.$$

3 Constructions

In this section, we provide constructions of degree-partitioned minimally k-connected graphs and degree-partitioned minimally k-edge-connected graphs for which the upper bounds of Theorem 2.2 and Theorem 2.3, respectively, are attained asymptotically. This has already been done for the case $k = 2$ [4], so we consider only $k \ge 3$. We begin by defining a k-regular graph $G_{k,p}$, which is used as a "building block" in the constructions that follow.

Definition 3.1. Let k, p be integers such that $3 \le k \le p$. Let $W = \{w_0, w_1, \ldots, w_{p-1}\}$ and $X = \{x_0, x_1, \ldots, x_{p-1}\}$. Let $G_{k,p}$ be the graph with vertex set $W \cup X$ and edge set

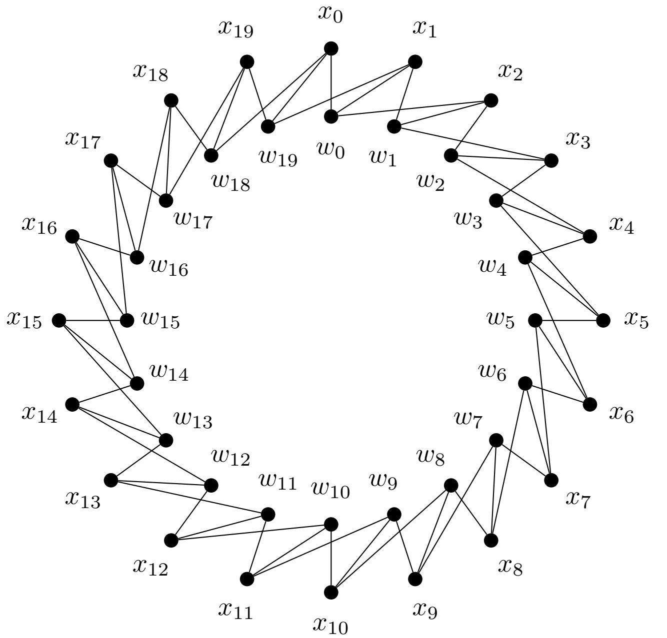

Figure 1: The graph $G_{3,20}$.

$$E = \{w_i x_{i+j} \mid 0 \leq i \leq p-1, 0 \leq j \leq k-1\},$$

where subscripts are expressed modulo p.

For example, the graph $G_{3,20}$ is illustrated in Figure 1. In the sequel, the notation $G_{k,p}$ will always denote the graph of Definition 3.1.

Remark 3.2. It can be shown in a straightforward manner that $G_{k,p}$ is vertex transitive.

3.1 Minimally k-edge connected graphs

We show in this subsection that for all $k \geq 3$, there is an infinite family of degree-partitioned minimally k-edge-connected graphs whose average edge-connectivity asymptotically achieves the $\frac{9}{8}k$ upper bound established in Section 2. The following result due to Mader [9] will be used.

Theorem 3.3. *If G is a connected, k-regular, vertex transitive graph, then G is k-edge-connected.*

As an immediate consequence of this result and Remark 3.2 we have the following:

Corollary 3.4. *The graphs $G_{k,p}$ described in Definition 3.1 are minimally k-edge-connected.*

Theorem 3.5. *There is an infinite family of degree-partitioned minimally k-edge-connected graphs whose average edge-connectivity is asymptotically $\frac{9}{8}k$.*

Proof. Let k, p be integers such that $3 \leq k \leq p$. Let $W=\{w_0, w_1, \ldots, w_{p-1}\}$, and for $m \in \{1, 2, 3\}$, let $X_m = \{x_0^m, x_1^m, \ldots, x_{p-1}^m\}$. Let $\Gamma_{k,p}$ be the graph of order $4p$ with vertex set $W \cup X_1 \cup X_2 \cup X_3$ and edge set $E_1 \cup E_2 \cup E_3$, where

$$E_m = \{w_i x_{i+j}^m \mid 0 \leq i \leq p-1, 0 \leq j \leq k-1\}$$

for $m \in \{1, 2, 3\}$, and where subscripts are expressed modulo p. For $m \in \{1, 2, 3\}$, let $H_m = \Gamma_{k,p}[W \cup X_m]$. Note that $H_m \cong G_{k,p}$ for all $m \in \{1, 2, 3\}$, and that H_1, H_2, and H_3 are pairwise edge-disjoint. The graph $\Gamma_{k,p}$ is bipartite with partite sets W and $X = X_1 \cup X_2 \cup X_3$, and every vertex in W has degree $3k$, while every vertex in X has degree k. (Essentially, the graph $\Gamma_{k,p}$ consists of three copies of $G_{k,p}$, where the three copies of the vertex w_i are identified for all $0 \leq i < p$.) Since $\Gamma_{k,p}$ is obtained from three distinct copies of the minimally k-edge-connected graph $G_{k,p}$, by identifying corresponding vertices of W, it is k-edge-connected. Further, since every edge of $\Gamma_{k,p}$ is incident with a vertex of degree k, we see that $\Gamma_{k,p}$ is minimally k-edge-connected.

We now compute the average connectivity of $\Gamma_{k,p}$. First of all, if $x \in X$ and $v \in V(\Gamma_{k,p}) - \{x\}$, then $\lambda(x, v) = k$, since $\Gamma_{k,p}$ is k-edge-connected and $\deg(x) = k$. If $w_i, w_j \in W$ for $i \neq j$, then $\lambda(w_i, w_j) = 3k$, since there are k edge-disjoint w_i–w_j paths in each of the edge-disjoint subgraphs H_1, H_2 and H_3. Thus the average edge-connectivity of $\Gamma_{k,p}$ is given by

$$\frac{3k\binom{p}{2} + k[\binom{4p}{2} - \binom{p}{2}]}{\binom{4p}{2}} = \left(\frac{9p-3}{8p-2}\right) k,$$

which is asymptotically $\frac{9}{8}k$. □

3.2 Minimally k-connected graphs

We show in this subsection that for all $k \geq 3$, there is an infinite family of degree-partitioned minimally k-connected graphs whose average connectivity asymptotically achieves the $\frac{9}{8}k$ upper bound established in Section 2.

While the graphs $\Gamma_{k,p}$ described in the proof of Theorem 3.5 are minimally k-connected, it can be shown in a fairly straightforward manner that their average connectivity is asymptotically less than $\frac{9}{8}k$.

So we define different families of degree-partitioned minimally k-connected graphs for which the upper bound given in Theorem 2.2 is attained asymptotically. We require two slightly different constructions; one for $k \in \{3, 4, 5\}$, where we compute the average connectivity by constructing internally disjoint paths, and another for $k \geq 6$, where we compute the average connectivity by considering vertex separators. Our constructions use the graphs $G_{k,p}$ described in Definition 3.1. The proof of the main result of this section hinges on the following technical lemma.

Lemma 3.6. *Let k, p be integers such that $3 \leq k \leq p$, and let u and v be nonadjacent vertices of $G_{k,p}$. Let S be a minimal vertex separator of u and v in $G_{k,p}$. Then $|S| = k$ or $|S| = 2k - 2$.*

Proof. First of all, if either u or v is isolated in $G_{k,p} - S$, say u, then S contains the entire neighbourhood $N(u)$ of u, and by the minimality of S, we have $S = N(u)$. We conclude that $|S| = k$ in this case.

So we may assume that neither u nor v is isolated in $G_{k,p} - S$. In this case, we show that $|S| = 2k - 2$. Let C be the component of $G_{k,p} - S$ that contains u, and let D be the union of the remaining components of $G_{k,p} - S$. Colour the vertices of C red, the vertices of D white, and the vertices of S black. Since u is not isolated in $G_{k,p} - S$, the component C has order at least 2, and hence both W and X must contain at least one red vertex. Similarly, since v is not isolated in $G_{k,p} - S$, we see that both W and X must contain at least one white vertex.

By symmetry, we may assume that w_0 is red, and that w_{p-1} is not red; otherwise, we can relabel the vertices of $G_{k,p}$ so that this happens. Since S is a minimal vertex separator of u and v, there are no edges between red and white vertices, and every black vertex must be adjacent with at least one red vertex and at least one white vertex. We illustrate the relevant portion of the graph $G_{k,p}$ in Figure 2.

Let $t \geq 0$ be the largest integer such that all of the vertices in the set $C_W = \{w_0, \ldots, w_t\}$ are coloured red. Thus all of the vertices in the set $N(\{w_0, \ldots, w_t\}) = \{x_0, \ldots, x_{t+k-1}\}$ are coloured either red or black. Let x_ℓ be the first red vertex and x_r be the last red vertex in the sequence $x_0, \ldots, x_{t+k-1}$. (We use ℓ and r for "left" and "right", respectively.) Since C is connected and contains w_0, some neighbour of w_0 must be coloured red, meaning that $\ell \leq k-1$. Similarly, some neighbour of w_t must be coloured red, meaning that $r \geq t$. So we have $0 \leq \ell \leq k-1$ and $t \leq r \leq t+k-1$.

We show first that x_j is coloured red for every $\ell < j < r$. Suppose otherwise that this is not the case, and let j be the smallest integer such that $\ell < j < r$ and x_j is coloured black. Note that the black vertex x_j must have a white neighbour. By the minimality of j, the vertices $x_\ell, \ldots, x_{j-1}$ are coloured red, and hence none of the vertices $w_0, \ldots, w_{j-1}$ are coloured white. Thus we either have $j > t$ and w_j is white, or $j < k-1$ and x_j has a white neighbour in the set $\{w_{p-(k-1-j)}, \ldots, w_{p-1}\}$. In the first case, the white vertex w_j is also adjacent to the red vertex x_r, a contradiction. In the second case, the white neighbour of x_j is also adjacent to the red vertex x_ℓ, a contradiction. We have shown that C contains the vertices in the set $C_W = \{w_0, \ldots, w_t\}$ and the vertices in the set $C_X = \{x_\ell, \ldots, x_r\}$. In fact, we will see that $V(C) = C_W \cup C_X$.

We now show that S has at least $2k-2$ vertices, i.e., that at least $2k-2$ vertices are coloured black. First of all, by the definition of ℓ and r, and the fact that each of the vertices $x_0, \ldots, x_{t+k-1}$ is either red or black, we see that the vertices in the sets

$$L_X = \{x_0, \ldots, x_{\ell-1}\} \quad \text{and} \quad R_X = \{x_{r+1}, \ldots, x_{t+k-1}\}$$

are coloured black. (Note that the set L_X is empty if $\ell = 0$, and that the set R_X is empty if $r = t+k-1$.)

We claim that the vertices in the sets

$$L_W = \{w_{p-(k-1-\ell)}, \ldots, w_{p-1}\} \quad \text{and} \quad R_W = \{w_{t+1}, \ldots, w_r\}$$

are also coloured black. First consider the set L_W. If $\ell = k-1$, then the set L_W is empty, and there is nothing to prove. So suppose $\ell < k-1$. Then the vertex w_{p-1} is adjacent to the red vertex x_ℓ, and since we have assumed that w_{p-1} is not red, it must be black. Since every black vertex must have a white neighbour, and the neighbours $x_0, \ldots, x_{k-2}$ of w_{p-1} are all black or red, the vertex x_{p-1} must be coloured white. So all of the vertices in L_W are adjacent to the white vertex x_{p-1} and the red vertex x_ℓ, and must therefore be black. The argument for R_W is similar. If $r = t$, then R_W

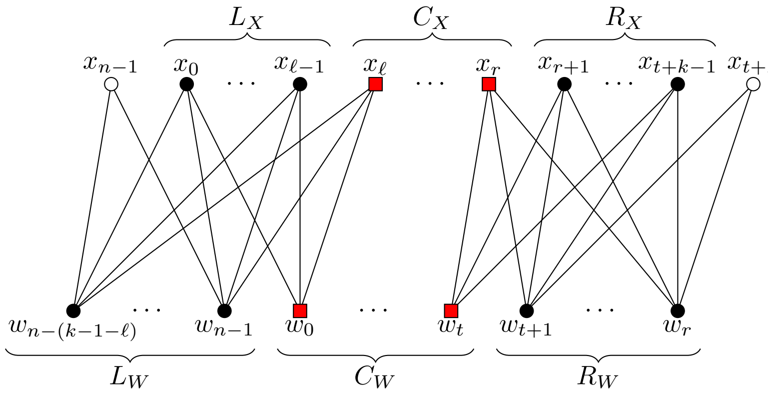

Figure 2: The minimal vertex separator S for u and v. Note that the black and white vertices are represented by black and white circles, respectively, and the red vertices are represented by red squares.

is empty, so suppose that $r > t$. By the maximality of t, the vertex w_{t+1} must be black, and hence must have a white neighbour. It follows that the vertex x_{t+k} must be white. So all of the vertices in R_W are adjacent to the white vertex x_{t+k} and the red vertex x_r, and must therefore be black.

Let $T = L_W \cup L_X \cup R_W \cup R_X$. We have shown that $T \subseteq S$. Since both W and X contain white vertices, the sets L_X, R_X, L_W, and R_W are pairwise disjoint. Note also that $|L_W \cup L_X| = k - 1$ and $|R_W \cup R_X| = k - 1$, so $|S| \geq |T| = 2k - 2$. Moreover, since $N(C_W \cup C_X) = T$, we see that no vertex in $C_W \cup C_X$ has a red neighbour outside of $C_W \cup C_X$. It follows that $V(C) = C_W \cup C_X$, and that $G_{k,p} - T$ is disconnected, hence $S = T$. □

We note that in the terminology of [8], we have shown that $G_{p,k}$ has connectivity k and is *essentially* $(2k-2)$*-connected.*

The proof of Lemma 3.6 reveals more about the minimal vertex separators of $G_{k,p}$ than just their cardinality. We can describe the structure of the minimal vertex separators in $G_{p,k}$ as in the following remark and this is important in the sequel.

Remark 3.7. Let S be a minimal vertex separator of nonadjacent vertices u and v in $G_{k,p}$. Then one of the following holds:

- $|S| = k$, and $S = N(u)$ or $S = N(v)$.

- $|S| = 2k - 2$, and with notation as in the proof of Lemma 3.6, we have $S \cap W = L_W \cup R_W$, where both L_W and R_W consist of at most $k-1$ consecutive vertices from the cyclic arrangement of vertices of W and $S \cap X = L_X \cup R_X$, where L_X and R_X consist of at most $k-1$ consecutive vertices from the cyclic arrangement of vertices of X. Moreover, $|L_W \cup L_X| = k-1$ and $|R_W \cup R_X| = k-1$.

It is also straightforward to prove that $G_{k,p}$ is minimally k-connected using Lemma 3.6.

Corollary 3.8. *Let k, p be integers such that $3 \leq k \leq p$. Then $G_{k,p}$ is minimally k-connected.*

Proof. Since $G_{k,p}$ is k-regular, we must have $\kappa(G_{k,p}) \leq k$. Now let S be a minimal vertex separator of $G_{k,p}$. By Lemma 3.6, we have $|S| = k$ or $|S| = 2k-2$. Since $k \geq 3$, we have $2k-2 > k$, hence $|S| \geq k$. So $\kappa(G_{k,p}) \geq k$, and we conclude that $\kappa(G_{k,p}) = \lambda(G_{k,p}) = k$. Finally, since $G_{k,p}$ is k-regular, we see that for every edge e of $G_{k,p}$, we have $\kappa(G_{k,p} - e) < k$. Thus, we conclude that $G_{k,p}$ is minimally k-connected. □

Theorem 3.9. *If $k \geq 3$, then there is an infinite family of degree-partitioned minimally k-connected graphs whose average connectivity is asymptotically $\frac{9}{8}k$.*

Proof. For $k \in \{3, 4, 5\}$, the proof is completed using a computer algebra system. The details are omitted here but are included in Appendix A of the arXiv version [12], as is a justification why we consider two different constructions: one for $k \in \{3, 4, 5\}$ and another for $k \geq 6$.

Assume now that $k \geq 6$ is fixed and let $p \in \{rk^2 - 1 \mid r \in \{k+1, k+2, \ldots\}\}$. Since k is relatively prime to p, the functions π_1 and π_2 from the set $\mathbb{Z}_p = \{0, 1, 2, \ldots, p-1\}$ to itself defined by

$$\pi_1(i) = ki$$

and

$$\pi_2(i) = k^2 i$$

for $0 \leq i \leq p-1$ (where the output in either case is expressed modulo p) are permutations of $\mathbb{Z}_p$.

Let

$$\begin{aligned} W &= \{w_0, w_1, \ldots, w_{p-1}\}, \\ X &= \{x_0, x_1, \ldots, x_{p-1}\}, \\ Y &= \{y_0, y_1, \ldots, y_{p-1}\}, \text{ and} \\ Z &= \{z_0, z_1, \ldots, z_{p-1}\}. \end{aligned}$$

Let $\Phi_{k,p}$ be the graph of order $4p$ with vertex set $W \cup X \cup Y \cup Z$ and edge set $E_X \cup E_Y \cup E_Z$, where

$$\begin{aligned} E_X &= \{w_i x_{i+j} \mid 0 \leq i \leq p-1, 0 \leq j \leq k-1\}, \\ E_Y &= \{w_{\pi_1(i)} y_{i+j} \mid 0 \leq i \leq p-1, 0 \leq j \leq k-1\}, \text{ and} \\ E_Z &= \{w_{\pi_2(i)} z_{i+j} \mid 0 \leq i \leq p-1, 0 \leq j \leq k-1\}, \end{aligned}$$

where subscripts are expressed modulo p.

Let

$$\begin{aligned} H_X &= \Phi_{k,p}[W \cup X], \\ H_Y &= \Phi_{k,p}[W \cup Y], \text{ and} \\ H_Z &= \Phi_{k,p}[W \cup Z]. \end{aligned}$$

Note that H_X, H_Y, and H_Z are isomorphic to $G_{k,p}$.

We first show that $\Phi_{k,p}$ is degree-partitioned minimally k-connected. Let S be any subset of at most $k-1$ vertices of $\Phi_{k,p}$. We will show that $\Phi_{k,p} - S$ is connected. Since H_X is isomorphic to $G_{k,p}$, it is k-connected by Corollary 3.8. Therefore, the graph $H_X - S$ is connected. Let v be any vertex in $Y \cup Z$ that is not in S. Then v has k neighbours in $\Phi_{k,p}$, all of which belong to $W \subseteq V(H_X)$. At most $k-1$ of these neighbours belong to S, so v is joined to some vertex of $H_X - S$. It follows that $\Phi_{k,p} - S$ is connected, and hence $\Phi_{k,p}$ is k-connected. Note that $\Phi_{k,p}$ is bipartite with partite sets W and $X \cup Y \cup Z$, and that every vertex in W has degree $3k$, while every vertex in $X \cup Y \cup Z$ has degree k. We conclude that $\Phi_{k,p}$ is degree-partitioned minimally k-connected.

It now suffices to show that $\kappa(u,v) = 3k$ for every pair of distinct vertices $u, v \in W$. Since u and v both have degree $3k$, we certainly have $\kappa(u,v) \leq 3k$. So it suffices to show that $|S| \geq 3k$ for every vertex separator S of u and v. Let S be a vertex separator of u and v, and let S_X, S_Y, and S_Z denote the sets $S \cap V(H_X)$, $S \cap V(H_Y)$, and $S \cap V(H_Z)$, respectively. Note that S_X, S_Y and S_Z separate u and v in H_X, H_Y, and H_Z, respectively.

Let $T_X \subseteq S_X$, $T_Y \subseteq S_Y$ and $T_Z \subseteq S_Z$ be minimal separators of u and v in H_X, H_Y and H_Z, respectively. Note that we have

$$|S| = |S_X \cup S_Y \cup S_Z| \geq |T_X \cup T_Y \cup T_Z|.$$

We will use the principle of inclusion and exclusion to show that $|T_X \cup T_Y \cup T_Z| \geq 3k$.

Let $\mathcal{T} = \{T_X, T_Y, T_Z\}$, and let $T \in \mathcal{T}$. First of all, by Lemma 3.6, we have $|T| = k$ or $|T| = 2k - 2$. Since $k \geq 6$, we have $2k - 2 > k$, so $|T| \geq k$. Further, by Remark 3.7, if $|T| = k$, then T is the neighbourhood of u or v in the subgraph H_X, H_Y, or H_Z corresponding to T, and since $u, v \in W$, we see that $T \cap W = \emptyset$ in this case.

We show now that if two distinct sets in $\mathcal{T}$ have nonempty intersection, then they both have cardinality $2k-2$, and their intersection has cardinality at most four. Suppose first that $T_X \cap T_Y \neq \emptyset$. Since $T_X \subseteq W \cup X$ and $T_Y \subseteq W \cup Y$, we see that $T_X \cap T_Y \subseteq W$. Thus, from the previous paragraph, we must have $|T_X| = |T_Y| = 2k - 2$. Further, by Remark 3.7, we have

$$T_X \cap W \subseteq \{w_a, w_{a+1}, \ldots, w_{a+k-2}\} \cup \{w_b, w_{b+1}, \ldots, w_{b+k-2}\}$$

for some $a, b \in \{0, 1, \ldots, p-1\}$, and

$$\begin{aligned} T_Y \cap W \subseteq\ & \{w_{\pi_1(c)}, w_{\pi_1(c+1)}, \ldots, w_{\pi_1(c+k-2)}\} \\ & \cup \{w_{\pi_1(d)}, w_{\pi_1(d+1)}, \ldots, w_{\pi_1(d+k-2)}\} \\ =\ & \{w_{kc}, w_{kc+k}, \ldots, w_{kc+k(k-2)}\} \cup \{w_{kd}, w_{kd+k}, \ldots, w_{kd+k(k-2)}\} \end{aligned}$$

for some $c, d \in \{0, 1, \ldots, p-1\}$. Since each of the sets

$$\{w_a, w_{a+1}, \ldots, w_{a+k-2}\} \text{ and } \{w_b, w_{b+1}, \ldots, w_{b+k-2}\}$$

overlaps with each of the sets

$$\{w_{kc}, w_{kc+k}, \ldots, w_{kc+k(k-2)}\} \text{ and } \{w_{kd}, w_{kd+k}, \ldots, w_{kd+k(k-2)}\}$$

in at most one vertex, we have $|T_X \cap T_Y| \leq 4$. The arguments for $T_X \cap T_Z$ and $T_Y \cap T_Z$ are similar, and are omitted.

We now show that $|T_X \cup T_Y \cup T_Z| \geq 3k$ by considering several cases.

- If the sets in $\mathcal{T}$ are pairwise disjoint, then they each have cardinality at least k, and it follows immediately that $|T_X \cup T_Y \cup T_Z| \geq 3k$.

- If exactly one pair of sets from $\mathcal{T}$ has nonempty intersection, then both of these sets have cardinality $2k-2$, and they overlap in at most four vertices. Further, they are disjoint from the third set, which has cardinality at least k. Thus, by the principle of inclusion and exclusion, we have

  $$|T_X \cup T_Y \cup T_Z| \geq 2(2k-2) - 4 + k = 5k - 8 > 3k,$$

  where we used the fact that $k \geq 6$ at the end.

- If all pairs of sets in $\mathcal{T}$ have nonempty intersection, then all of the sets in $\mathcal{T}$ have cardinality $2k-2$, and each pair overlaps in at most four vertices. Thus, by the principle of inclusion and exclusion, we have

  $$|T_X \cup T_Y \cup T_Z| \geq 3(2k-2) - 3(4) = 6k - 18 \geq 3k,$$

  where we used the fact that $k \geq 6$ at the end.

We conclude in all cases that $|S| \geq |T_X \cup T_Y \cup T_Z| \geq 3k$. Therefore, we have $\kappa(u,v) = 3k$, which completes the proof. □

4 Conclusion

The obvious open problem is to resolve Conjecture 1.2, which states that if G is an optimal minimally k-(edge-)connected graph of order $n \geq 2k+1$ for some $k \geq 3$, then G is degree-partitioned. We showed that if this conjecture is true, then the average (edge-)connectivity of a minimally k-(edge-)connected graph is at most $\frac{9}{8}k$, and we constructed degree-partitioned minimally k-(edge-)connected graphs which attain this upper bound asymptotically.

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