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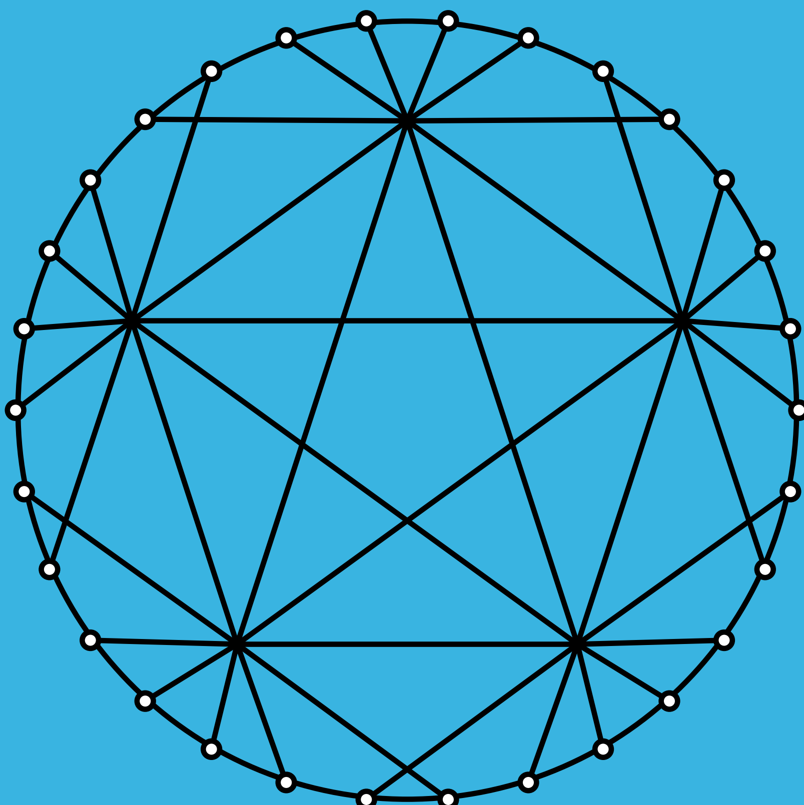

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A note on the Buratti-Horak-Rosa conjecture about hamiltonian paths in complete graphs

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Abstract. The conjecture posed by Buratti, Horak and Rosa states that a (multiset) list L of $v-1$ positive integers not exceeding $\lfloor v/2 \rfloor$ is the list of edge-lengths of a suitable Hamiltonian path of the complete graph with vertex-set $\{0,1,\ldots,v-1\}$ if and only if for every divisor d of v, the number of multiples of d appearing in L is at most $v-d$. A list L is called realizable if there exists such Hamiltonian path P of the complete graph with $|L|+1$ vertices whose edge-lengths is the given list L. If the initial and the final vertices in P are 0 and $v-1$, respectively, then P is called perfect.

In this note, we show some properties of some perfect linear realizations. Also, we present a new operation over well-known linear realizations. For example, we give a linear realizations of the lists $\{1^a, 2^b, 4^c\}$, where $a, c \geq 1$ and $b \geq 3$ integers, $\{1^a, 2^b, 4^{2c}, 8^d\}$, for all $a, d \geq 1$, $b \geq 3$ and $c \geq 2$ integers, and $\{1^a, 2^b, 4^c, 8^d\}$, for all $a, d \geq 1$, $b \geq 3$ and $c \geq 8$ integers.

1 Introduction

Throughout the paper, K_p will denote the complete graph on p vertices, labeled by the integers of the set $\{0,1,\ldots,p-1\}$. For the basic terminology on graphs we refer to [1] and for basic facts about the Buratti-Horak-Rosa

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conjecture we refer to [10]. The *length* of the edge xy, where $x, y \in V(K_p)$ is given by

$$\ell(x,y) = \min\{|y-x|, p-|y-x|\}.$$

Given a path $P = (x_0, x_1, \ldots, x_k)$, the list of edge-lengths of P is the list $\ell(P)$ of the lengths (taken with their respective multiplicities) of all the edges of P. Hence, if a list L consists of a_1 $1s$, a_2 $2s, \ldots, a_t$ ts, then we write $L = \{1^{a_1}, 2^{a_2}, \ldots, t^{a_t}\}$ and $|L| = \sum_{i=1}^{t} a_i$. The set $U_L = \{i : a_i > 0\} \subseteq L$ is called the *underlying set* of L.

The following conjecture was proposed in a private communication by Buratti to Rosa in 2007:

Conjecture 1.1 (M. Buratti). *For any prime $p = 2n+1$ and any list L of $2n$ positive integers not exceeding n, there exists a Hamiltonian path P of K_p with $\ell(P) = L$.*

Talking with Professor Buratti, the origin of this problem comes from the study of dihedral Hamiltonian cycle decompositions of the cocktail party graph (see comments before Corollary 3.19 in [2]).

Buratti's conjecture is almost trivially true in the case when $|U_L| = 1$. On the other hand, the case of exactly two distinct edge-lengths has been solved independently by Dinitz and Janiszewski [4] and Horak and Rosa [5]. Using a computer, Meszka has verified the validity of Buratti's conjecture for all primes ≤ 23. Monopoli [6] showed that the conjecture is true when all the elements of the list L appear exactly twice.

In [5] Horak and Rosa proposed a generalization of Buratti's conjecture, which has been restated in an easier way in [9] as follows:

Conjecture 1.2 (P. Horak and A. Rosa). *Let L be a list of $v-1$ positive integers not exceeding $\lfloor v/2 \rfloor$. Then there exists a Hamiltonian path P of K_v such that $\ell(P) = L$ if and only if the following condition holds:*

for any divisor d of v, the number of multiples of d appearing in L does not exceed $v-d$.

The case of exactly three distinct edge-lengths has been solved when the underlying set is $U_L = \{1,2,3\}$ in [3], when U_L is one of the sets

$$\{1,2,5\}, \{1,3,5\}, \{2,3,5\}$$

in [9], and when $U_L = \{1,3,4\}$ or $U_L = \{2,3,4\}$ in [8]. In [10] the authors give a complete solution when $U_L = \{1,2,t\}$, where $t \in \{4,6,8\}$, and when $L = \{1^a, 2^b, t^c\}$ with $t \geq 4$ an even integer and $a+b \geq t-1$. The case with four distinct edge-lengths for which the conjecture has been shown to be true is when $U_L = \{1,2,3,4\}$ or $U_L = \{1,2,3,5\}$, see [6] and [10]. Recently, Ollis et al. [8] proved some partial results in which $U_L = \{x, y, x+y\}$, $U_L = \{1,2,4,\ldots,2x\}$ and $U_L = \{1,2,4,\ldots,2x,2x+1\}$; many other lists were considered, see [8].

A *cyclic realization* of a list L with $v-1$ elements each from the set $\{1,2,\ldots,\lfloor v/2 \rfloor\}$ is a Hamiltonian path P of K_v such that the multiset of edge-lengths of P equals L. Hence, it is clear that the Conjecture 1.2 can be formulated as follow: every such a list L has a cyclic realization if and only if condition (1,1) is satisfied.

Example 1. *The path* $P = (0,1,2,3,6,4,5,7)$ *is a cyclic realization of the list* $L = \{1^4, 2^2, 3\}$.

A *linear realization* of a list L with $v-1$ positive integers not exceeding $v-1$ is a Hamiltonian path $P = (x_0, x_1, ..., x_{v-1})$ of K_v such that $L = \{|x_i - x_{i+1}| : i = 0, \ldots, v-2\}$. The linear realization is *standard* if $x_0 = 0$ (see [8]). In this note we assume that any realization P of a given list L is standard. On the other hand, if $x_{v-1} = v-1$, the (standard) linear realization is called *perfect* (see [3]).

Example 2. *The path* $P = (0,2,4,6,5,3,1,7)$ *is a perfect linear realization of the list* $L = \{1^1, 2^5, 6\}$.

Remark 1. *From the definitions presented before, it is not hard to see that any linear realization of a list L can be viewed as a cyclic realization of a list $\hat{L}$ (not necessarily of the same list); however if all the elements in the list are less than or equal to $\lfloor \frac{|L|+1}{2} \rfloor$, then every linear realization of L is also a cyclic realization of the same list L. For example, the path* $P = (0,5,7,8,6,4,3,1,2)$ *is a linear realization of the list* $L = \{1^3, 2^4, 5\}$ *and a cyclic realization of the list* $\hat{L} = \{1^3, 2^4, 4\}$.

In this note, we show some properties of some perfect linear realizations. Also, we present a new operation over well-known linear realizations and we give several examples.

2 Some perfect linear realizations

Let $P = (x_0, x_1, \ldots, x_{v-1})$ and $P' = (y_0, y_1, \ldots, y_{w-1})$ be two paths (in general) such that $V(P) \cap V(P') = \emptyset$. If x_{v-1} and y_0 are adjacent, then we can generate the path:

$$P + P' = (x_0, x_1, \ldots, x_{v-1}, y_0, y_1, \ldots, y_{w-1}).$$

The path $P + P'$ is also well-defined if $x_{v-1} = y_0$, in this case

$$P + P' = (x_0, x_1, \ldots, x_{v-1}, y_1, \ldots, y_{w-1}).$$

Theorem 2.1 ([3])**.** *Let P be a perfect linear realization of a list L and P' be a linear realization of the list L'. Then there exists a linear realization P'' of the list $L \cup L'$. Furthermore, if P' is also perfect, then P'' is perfect.*

Remark 2. *Let $P = (x_0 = 0, x_1, \ldots, x_{v-1} = v-1)$ be a perfect linear realization of a list L. Applying the previous theorem to the perfect linear realization $(0, 1, \ldots, A)$ of $\{1^A\}$, $P' = P + (v-1, v, \ldots, v-1+A)$ is a perfect linear realizations of $L \cup \{1^A\}$, for all $A \geq 0$ integer, see [3].*

Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a path. For every $k \in \mathbb{Z}$ integer, let $\pi_k : \mathbb{Z} \to \mathbb{Z}$ given by $\pi_k(x) = x + k$. Hence, if $P = (x_0 = 0, x_1, \ldots, x_{v-1})$ is a linear realization of a list L, then $P' = (0, 1, \ldots, A) + \pi_A(P)$ is a linear realization of the list $L \cup \{1^A\}$.

Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a path. For each $j \in \{1, 2, \ldots, v-1\}$, the path P is called *j-partitionable* if $P = P_j + P_j^c$, where

$$V(P_j) = \{x_0, x_1, \ldots, x_j\} = \{0, 1, \ldots, j\}$$

and $x_j = j$. A path P is called *partitionable* if P is j-partitionable for some $j \in \{1, 2, \ldots, v-1\}$.

Example 3. *The path $P = (0, 1, 2, 5, 3, 4, 6)$ is j-partitionable for $j \in \{1, 2, 6\}$. On the other hand, the path $P' = (0, 1, 2, 5, 3, 4, 6, 7, 8)$ is j-partitionable for $j \in \{1, 2, 6, 7, 8\}$. In particular, both paths are perfect.*

Let P be a j-partitionable, for some $j > 0$. Then P is *weakly j-partitionable* if P is also $(j+1)$-partitionable; otherwise the path is called *strong*.

Lemma 2.2 ([10])**.** *Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list L. If there exists $i \in \{0, 1, \ldots, v-2\}$ such that $\{x_i, x_{i+1}\} = \{v-2, v-1\}$, then $P = (x_0, \ldots, x_i, v, x_{i+1}, \ldots, x_{v-1})$ is a linear realization of $L \cup \{2\}$.*

Corollary 2.3 ([10]). *Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list L. If there exists $i \in \{0, 1, \ldots, v-2\}$ such that $\{x_i, x_{i+1}\} = \{v-2, v-1\}$, then the list $L' = L \cup \{2^b\}$ admits a linear realization, for any positive integer b.*

Lemma 2.4. *If a list L admits a weakly j-partitionable linear realization, for some $j \in \{1, \ldots, |L|-2\}$, then the list $L \setminus \{1\}$ admits a linear realization.*

Proof. Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a weakly j-partitionable linear realization of a list L, for some $j \in \{1, \ldots, |L| - 2\}$. Since the path is weakly j-partitionable, then j and $j+1$ are adjacent in P and $1 \in L$. Therefore, the path

$$P' = (x_0, \ldots, x_j, \pi_{-1}(x_{j+2}), \ldots, \pi_{-1}(x_{v-1}))$$

is a linear realization of $L \setminus \{1\}$. □

Proposition 2.5. *If a list L admits a perfect weakly i-partitionable linear realization, for all $i \in \{i_1, \ldots, i_k\}$, then $L = L_{i_1,} \cup \cdots \cup L_{i_k} \cup L_{v-1}$ where L_i admits a perfect strong linear realization for all $i \in \{i_1, \ldots, i_k\} \cup \{v-1\}$.*

Proof. Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a perfect weakly i-partitionable linear realization of a list L, where $i \in \{i_1, \ldots, i_k\}$ and $i_1 < i_2 < \cdots < i_k$. Hence

$$P = (x_0, \ldots, x_{i_1}) + (x_{i_1+1}, \cdots, x_{i_2}) + \ldots + (x_{i_k+1}, \ldots, x_{v-1}).$$

Setting $i_0 = 0$, $i_{k+1} = v-1$ and $P_{i_j} = (x_{i_{j-1}+1}, \ldots, x_{i_j})$, for all $j \in \{1, \ldots, k\}$, then $P = P_{i_1} + P_{i_2} + \cdots + P_{i_{k+1}}$. Since P is perfect and partitionable,

$$P_{i_1}, \pi_{-(x_{i_1}+1)}(P_{i_2}), \ldots, \pi_{-(x_{i_k}+1)}(P_{i_{k+1}})$$

are perfect strong linear realizations of $L_{i_1}, L_{i_2}, \ldots, L_{i_{k+1}}$, respectively, where $L_{i_j} \subseteq L$, for all $j \in \{1, \ldots, k+1\}$ and $L = L_{i_1} \cup \cdots \cup L_{i_{k+1}}$ (by Theorem 2.1). □

Lemma 2.6 ([10]). *If a list $L = \{1^{a_1}, 2^{a_2}, \ldots, t^{a_t}\}$ admits a linear realization, then $a_i + i - 1 \leq |L|$ for all $i = 1, \ldots, t$.*

Proposition 2.7. *If a list $L = \{1^a, 2^b, t^c\}$ admits a perfect linear realization, then $b + (t-1)c$ is even.*

Proof. The proof is obtained straightforwardly of proof given by Proposition 3.1 in [3]. □

In particular of Lemma 2.6, if a list $L = \{1^a, 2^b, t^c\}$ admits a linear realization, then $a + b \geq t - 1$.

Remark 3. *If $P = (x_0, x_1, \ldots, x_t)$ is a perfect linear realization of $L_t = \{1^a, 2^b, t\}$ with $a + b = t$, then either $x_1 = t$ or $x_{t-1} = 1$.*

Proposition 2.8. *There exist a perfect linear realization of the list $L_t = \{1, 2^{t-1}, t\}$, for all $t \geq 3$ integer.*

Proof. It is very easy to see that the following paths are perfect linear realizations of L.

(*a*) $P_t = (0, 2, 4, \ldots, t, t-1, t-3, \ldots, 1, t+1)$ if $t \geq 4$ is even.

(*b*) $P_t = (0, 2, 4, \ldots, t-1, t, t-2, \ldots, 1, t+1)$ if $t \geq 3$ is odd.

(*c*) $\hat{P}_t = (0, t, t-2, \ldots, 2, 1, 3, \ldots, t-1, t+1)$ if $t \geq 4$ is even.

(*d*) $\hat{P}_t = (0, t, t-2, \ldots, 1, 2, 4, \ldots, t-1, t+1)$ if $t \geq 3$ is odd.

□

Example 4. *The paths $P_4 = (0, 2, 4, 3, 1, 5)$ and $\hat{P}_4 = (0, 4, 2, 1, 3, 5)$ are perfect linear realizations of the list $L_4 = \{1, 2^3, 4\}$.*

Theorem 2.9. *Let $a + b = t \geq 3$ with $a, b \geq 1$ integers. The list $L_t = \{1^a, 2^b, t\}$ admits a perfect linear realization if and only if $(a, b) = (1, t-1)$, in which the paths P_t and $\hat{P}_t$ are the unique perfect linear realization of the list L_t.*

Proof. Suppose that $t \geq 4$ is an even integer (the proof for $t \geq 3$ odd is completely analogous). Let $P = (x_0, x_1, \ldots, x_{t+1})$ be a perfect linear realization of L_t. By Remark 3 either $x_t = 1$ or $x_1 = t$. Without loss of generality assume that $x_t = 1$, which implies that $x_1 = 2$, which implies that $x_{t-1} = 3$, which implies that $x_2 = 4$, which implies that $x_{t-2} = 5$, and so on until $x_{\frac{t}{2}+2} = t - 3$ and $x_{\frac{t}{2}} = t$. Which implies that $x_{\frac{t}{2}+1} = t - 1$. Hence, we have that $P = P_t$. The proof to the case $x_1 = t$ is analogous to the proof presented before. □

3 Even-odd applications over paths

If $P = (x_0, x_1, \ldots, x_{v-1})$ is a standard linear realization of a list L, then this path is called (x_1, x_{v-1})-realization of L. Let P^* be the sub-path of P without initial vertex, that is $P^* = P \setminus \{x_0\}$. Hence, P^* is a (non-standard) linear realization of the list $L \setminus \{x_1\}$. The *reverse* of P, $rev(P) = (x_{v-1}, x_{v-2}, \ldots, x_0)$, is also a liner realization of L, see [8]. The *even-application* of P, $E(P)$, is defined by the path

$$E(P) = (2x_0, 2x_1, \ldots, 2x_{v-1}).$$

This application satisfies that $\ell(E(P)) = 2L$. Finally, the *odd-application* of P, $O(P)$, is defined by the path:

$$O(P) = (2x_1 - 1, 2x_2 - 1, \ldots, 2x_{v-1} - 1)$$

and the *odd reverse-application* of P, $OR(P)$, is defined as the path

$$OR(P) = (2x_{v-1} - 1, 2x_{v-2} - 1, \ldots, 2x_1 - 1).$$

These applications satisfy $\ell(O(P)) = \ell(OR(P)) = 2L \setminus \{2x_1\}$.

We define two operations over a linear realization P of a list L, called *even-odd extension*, $EO(P)$, and *even-odd reverse extension* of P, $EOR(P)$, as follow:

$$EO(P) = E(P) + O(P) \text{ and } EOR(P) = E(P) + OR(P).$$

The even-odd extension of P is a linear realization of the list

$$(2L \cup 2L \setminus \{2x_1\}) \cup \{|2(x_{v-1} - x_1) + 1|\}.$$

On the other hand, the even-odd reverse extension of P is a linear realization of the list

$$(2L \cup 2L \setminus \{2x_1\}) \cup \{1\}.$$

To the next, we are going to construct some linear realization from well-known linear realizations.

Example 5. *As we have already seen, $P = (0, 1, \ldots, k)$ is a (perfect) linear realization of the list $\{1^k\}$. On the other hand, $E(P) = (0, 2, 4, \ldots, 2k)$ and*

$O(P) = (1, 3, \ldots, 2k-1)$. Hence, $\ell(E(P)) = \{2^k\}$ and $\ell(O(P)) = \{2^{k-1}\}$. It follows that the even-odd reverse extension of P:

$$EOR(P) = (0, 2, 4, \ldots, 2k, 2k-1, 2k-3, \ldots, 3, 1)$$

is a linear realization of the list $\{1, 2^{2k-1}\}$. Notice that the new path is a $(2,1)$-realization.

Example 6. *Let $k \geq 1$ be an integer, and take*

$$P_k = (0, 2, \ldots, 2k, 2k-1, 2k-3, \ldots, 1),$$
$$P'_k = (0, 2, \ldots, 2k, 2k+1, 2k-1, \ldots, 1).$$

It is easy to see that P_k is a $(2,1)$-realization of $\{1, 2^{2k-1}\}$ (see Example 5) and P'_k is $(2,1)$-realization of $\{1, 2^{2k}\}$. Hence, the even-application of P_k and P'_k are

$$E(P_k) = (0, 4, \ldots, 4k, 4k-2, 4k-6, \ldots, 2),$$
$$E(P'_k) = (0, 4, \ldots, 4k, 4k+2, 4k-2, \ldots, 2),$$

satisfying $\ell(E(P_k)) = \{2, 4^{2k-1}\}$ and $\ell(E(P'_k)) = \{2, 4^{2k}\}$, respectively. The odd-application of P_k and P'_k are

$$O(P_k) = (3, 7, \ldots, 4k-1, 4k-3, 4k-7, \ldots, 1),$$
$$O(P'_k) = (3, 7, \ldots, 4k-1, 4k+1, 4k-3, \ldots, 1),$$

satisfying $\ell(O(P_k)) = \{2, 4^{2k-2}\}$ and $\ell(O(P'_k)) = \{2, 4^{2k-1}\}$, respectively.

Hence, the even-odd extension of P_k and P'_k, $EO(P_k)$ and $EO(P'_k)$, are $(4,1)$-realization of the lists $\{1, 2^2, 4^{4k-3}\}$ and $\{1, 2^2, 4^{4k-1}\}$, respectively. Also, the even-odd reverse extension of P_k and P'_k, $EOR(P_k)$ and $EOR(P'_k)$, are $(4,3)$-realization of the same lists.

Lemma 3.1 ([10], Lemma 9)**.** *Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a standard linear realization of a list L. If $x_{v-1} = 1$, then the list $L' = L \cup \{2^b\}$ is linear realizable, for any $b \geq 2$ integer.*

By Remark 2, Example 6 and Lemma 3.1, we have the following result, which is a particular case of Proposition 20 in [10]:

Corollary 3.2. *There are linear realizations of the lists $\{1^a, 2^2, 4^{2c-1}\}$ and $\{1^a, 2^b, 4^{2c-1}\}$, for all positive integers a, b, c such that $b \geq 4$.*

Theorem 3.3 ([3])**.** *If $a \geq 2$ and $b \geq 0$ are integers, then the list $\{1^a, 3^b\}$ admits a linear realization. Also, this realization can be assumed to be perfect when $b \not\equiv 1 \pmod 3$.*

Corollary 3.4. *Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list L, where the vertices $v-1, v-2$ are adjacent. If $EO(P)$ is a linear realization of L_{EO} and $EOR(P)$ is a linear realization of L_{EOR}, then the lists $L_{EO} \cup \{4\}$ and $L_{EOR} \cup \{4\}$ are linear realizable.*

Proof. The proof is completely analogous to the proof of Lemma 7 of [10]. Since the vertices $v-1, v-2$ are adjacent in P, the vertices $2v-3, 2v-5$ are adjacent in $O(P)$ (and in $OR(P)$), and the vertices $2v-2, 2v-4$ are adjacent in $E(P)$. Hence, the new vertex $2v-1$ can be added between $2v-3, 2v-5$. So, there is a linear realization of $L_{EO} \cup \{4\}$ or $L_{EOR} \cup \{4\}$. Else, if we also add the vertex $2v$ between $2v-2$ ans $2v-4$, we obtain a linear realization of $L_{EO} \cup \{4^2\}$ and of $L_{EOR} \cup \{4^2\}$. □

Corollary 3.5. *Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list L, where the vertices $v-1, v-2$ are adjacent. If $EO(P)$ is a linear realization of L_{EO} and $EOR(P)$ is a linear realization of L_{EOR}, then the lists $L_{EO} \cup \{4^b\}$ and $L_{EOR} \cup \{4^b\}$ are linear realizable, for any positive integer b.*

By Remark 2, Example 6, Corollary 3.5 and Lemma 3.1, we have the following:

Corollary 3.6. *There are linear realizations of the lists $\{1^a, 2^2, 4^c\}$ and $\{1^a, 2^b, 4^c\}$, for all positive integers a, b, c such that $b \geq 4$.*

Corollary 3.7. *Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a standard linear realization of a list L, where $x_{v-1} = 1$. There exists a linear realization of $2L \cup 2L \cup \{1, 4^{2b-1}\}$.*

Proof. Following the proof of Lemma 9 of [10], there exists a $(2,1)$-realization P' of $L \cup \{2^b\}$. Then $EO(P')$ and $EOR(P')$ are linear realizations of $2L \cup 2L \cup \{1, 4^{2b-1}\}$. □

Proposition 3.8. *There exists a standard linear realization of the list $\{1^a, 2^b, 4^c\}$, for all positive integers a, b, c where $b \geq 3$.*

Proof. Let $k \geq 2$ be an integer. Consider the path P_k of Example 5, obtained by applying the even-odd reverse extension of the perfect linear realization $I_k = \{0, 1, 2, \ldots, k\}$ of the list $\{1^k\}$: $P_k = EOR(I_k)$. Then, we can write $P_k = P^E_{k,0} + rev(P^O_{k,0})$, where

$$P^E_{k,0} = E(I_k) = (0, 2, \ldots, 2k) \text{ and } P^O_{k,0} = O(I_k) = (1, 3, \ldots, 2k-1).$$

So, $\ell(P^E_{k,0}) = \{2^k\}$ and $\ell(P^O_{k,0}) = \{2^{k-1}\}$. Now, let t be a positive integer. For all $j = 1, \dots, t$, we construct a path $P^E_{k,j}$ by adding the vertex $2k + 2j$ between the consecutive vertices $2k + 2(j - 2), 2k + 2(j - 1)$ of the path $P^E_{k,j-1}$. Then, $\ell(P^E_{k,j}) = \{2^k, 4^j\}$. Similarly, for all $j = 1, \dots, t$, we construct a path $P^O_{k,j}$ by adding the vertex $2k+2j-1$ between the consecutive vertices $2k+2j-5, 2k+2j-3$ of the path $P^O_{k,j-1}$. In this case, $\ell(P^O_{k,j}) = \{2^{k-1}, 4^j\}$. Hence, the path $P_{k,t} = P^E_{k,t} + rev(P^O_{k,t})$ is a $(2, 1)$-realization of the list $\{1, 2^{2k-1}, 4^{2t}\}$. Now, the path

$$P^E_{k,0} + rev(P^O_{k,0}) = (0, 2, 4, \dots, 2k, 2k - 1, 2k + 1, 2k - 3, 2k - 5, \dots, 1)$$

is a $(2, 1)$-realization of the list $\{1, 2^{2k-1}, 4\}$. Finally, for any positive integer t, the path $P^E_{k,t} + P^O_{k,t+1}$ is a $(2, 1)$-realization of the list $\{1, 2^{2k-1}, 4^{2t+1}\}$. Hence, there is a $(2, 1)$-realization of $\{1, 2^{2x+1}, 4^y\}$ for all positive integers x, y. Finally, by Remark 2, Lemma 3.1, Corollary 3.6 and Corollary 3.7, there is a linear realization of $\{1^a, 2^b, 4^c\}$, for all positive integers a, b, c where $b \geq 3$. □

Example 7. *For instance, taking $t = 2$ and $k = 3$, we have*

$$P^E_{3,0} = (0, 2, 4, 6), \quad P^E_{3,1} = (0, 2, 4, \mathbf{8}, 6), \quad P^E_{3,2} = (0, 2, 4, 8, \mathbf{10}, 6),$$

$$P^O_{3,0} = (1, 3, 5), \quad P^O_{3,1} = (1, 3, \mathbf{7}, 5), P^O_{3,2} = (1, 3, 7, \mathbf{9}, 5).$$

Hence, $P_{3,2} = (0, 2, 4, 8, 10, 6, 5, 9, 7, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^5, 4^4\}$, $P^E_{3,0}+rev(P^O_{3,1}) = (0, 2, 4, 6, 5, 7, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^5, 4\}$, and $P^E_{3,2} = (0, 2, 4, 8, 10, 6, 5, 9, 11, 7, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^5, 4^5\}$. Furthermore, $P^E_{3,2} + rev(P^O_{4,2}) = (0, 2, 4, 8, 10, 6, 7, 11, 9, 5, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^6, 4^4\}$, $P^E_{3,1} = rev(P^O_{4,0}) = (0, 2, 4, 8, 7, 5, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^6, 4\}$, and $P^E_{3,3} + P^O_{4,2} = (0, 2, 4, 8, 12, 10, 6, 7, 11, 9, 5, 3, 1)$ is a $(2, 1)$-realization of $\{1, 2^6, 4^5\}$.

Proposition 3.9. *There exists a standard linear realization of the list $\{1^a, 2^b, 4^{2c}, 8^d\}$, for all positive integers a, b, c, d where $b \geq 3$ and $c \geq 2$. Moreover, there exists a standard linear realization of the list $\{1^a, 2^b, 4^c, 8^d\}$, for all positive integers a, b, c, d such that $a \geq 2$, $b \geq 3$ and $c \geq 4$.*

Proof. Let $Q = P_{2,4} = (0, 2, 6, 8, 4, 3, 7, 5, 1)$ be a $(2, 1)$-linear realization of $\{1, 2^3, 4^4\}$ (see Proposition 3.8). Let $Q^2 = (2, 6)$, $Q^0 = (4, 8)$, $Q^1 = (3, 7)$ and $Q^3 = (1, 5)$. So,

$$Q = (0) + Q^2 + rev(Q^0) + Q^1 + rev(Q^3).$$

Let $l \geq 3$ be an integer and $i \in \{0,1,2,3\}$, we construct the path $Q_{l+1,i}$ by adding the vertex $4l-i$ to the path $Q_{l,i}$, where $Q_3^i = Q^i$ for $i = 0,1,2,3$, as follow:

- If $i = 3$, we add the vertex $4l-3$ between the vertices $4(l-1)-3$ and $4(l-2)-3$ to the path Q_l^3. Hence,

$$Q_{l+1,3} = (0) + Q_l^2 + rev(Q_l^0) + Q_l^1 + rev(Q_{l+1}^3).$$

- If $i = 2$, then $Q_{l+1}^2 = Q_{l+1}^3 + 1$ (since $Q^2 = Q^3 + 1$), we are adding the vertex $4l-2$ between the vertices $4(l-1)-2$ and $4(l-2)-2$ of the path Q_l^2. Hence,

$$Q_{l+1,2} = (0) + Q_{l+1}^2 + rev(Q_l^0) + Q_l^1 + rev(Q_{l+1}^3).$$

- If $i = 1$, we add the vertex $4l-1$ between the vertices $4(l-1)-1$ and $4(l-2)-1$ to the path Q_l^1. Hence,

$$Q_{l+1,1} = (0) + Q_{l+1}^2 + rev(Q_l^0) + Q_{l+1}^1 + rev(Q_{l+1}^3).$$

- If $i = 0$, then $Q_{l+1}^0 = Q_{l+1}^1 + 1$ (since $Q^0 = Q^0 + 1$), we are adding the vertex $4l$ between the vertices $4(l-1)$ and $4(l-2)$ of the path Q_l^0. Hence,

$$Q_{l+1,0} = (0) + Q_{l+1}^2 + rev(Q_{l+1}^0) + Q_{l+1}^1 + rev(Q_{l+1}^3).$$

So, $\ell(Q_{l+1}^i) = \{4, 8^i\}$, for $i = 0,1,2,3$. Therefore, the path $Q_{l+1,i}$ is a $(2,1)$-realization of $\{1, 2^3, 4^4, 8^{4l-8-i}\}$, proving that there exists a $(2,1)$-realization of $\{1, 2^3, 4^4, 8^t\}$, for all positive integer t. Proceeding as the same way as before taking $Q = P_{2,2k}$ (see Proposition 3.8), for $k \geq 1$ integer, we can prove that there is a $(2,1)$-realization of the list $\{1, 2^3, 4^{4k}, 8^s\}$, for all positive integers k, s.

Now, let $\hat{Q} = (0,2,6,10,8,4,5,9,11,7,3,1)$ be a $(2,1)$-linear realization of $\{1, 2^4, 4^6\}$. If $\hat{Q}^2 = (2,6)$, $\hat{Q}^0 = (4,8)$, $\hat{Q}^1 = (5,9)$ and $\hat{Q}^3 = (3,7)$, we have

$$\hat{Q} = (0) + \hat{Q}^2 + (10) + rev(\hat{Q}^0) + \hat{Q}^1 + (11) + rev(\hat{Q}^3) + (1).$$

As the same way as before, we can construct a $(2,1)$-linear realization of the list $\{1, 2^4, 4^6, 8^s\}$, for $s \geq 1$ integer. Moreover, if we take $\hat{Q} = P'_{2,2k+1}$ (see Proposition 3.8), for $k \geq 1$ integer, we can prove that there is a $(2,1)$-realization of the list $\{1, 2^4, 4^{4k+2}, 8^s\}$, for all positive integers k, s.

On the other hand, let $Q = (0, 2, 6, 8, 4, 5, 9, 7, 3, 1)$ be a $(2,1)$-linear realization of $\{1, 2^4, 4^4\}$. Let $Q^2 = (2, 6)$, $Q^0 = (4, 8)$, $Q^1 = (5, 9)$ and $Q^3 = (3, 7)$, we have

$$Q = (0) + Q^2 + rev(Q^0) + Q^1 + rev(Q^3) + (1).$$

Let $l \geq 3$ be an integer and $i \in \{0, 1, 2, 3\}$, we construct the path $Q_{l+1,i}$ by adding the vertex $(4l - 3) + i$ to the path $Q_{l,i}$, where $Q_3^i = Q^i$ for $i = 0, 1, 2, 3$, as follow:

- If $i = 0$, we add the vertex $(4l - 2)$ between the vertices $4(l - 2) - 2$ and $4(l - 3) - 2$ to the path Q_l^2. Hence,

$$Q_{l+1,0} = (0) + Q_{l+1}^2 + rev(Q_l^0) + Q_l^1 + rev(Q_l^3).$$

- If $i = 1$, then $Q_{l+1}^3 = Q_{l+1}^2 + 1$ (since $Q^2 = Q^3 + 1$). Hence,

$$Q_{l+1,1} = (0) + Q_{l+1}^2 + rev(Q_l^0) + Q_l^1 + rev(Q_{l+1}^3).$$

- If $i = 2$, we add the vertex $4l - 1$ between the vertices $4(l - 1) - 1$ and $4(l - 2) - 1$ to the path Q_l^1. Hence,

$$Q_{l+1,2} = (0) + Q_{l+1}^2 + rev(Q_l^0) + Q_{l+1}^1 + rev(Q_{l+1}^3).$$

- If $i = 3$, then $Q_{l+1}^0 = Q_{l+1}^1 + 1$ (since $Q^0 = Q^0 + 1$), we are adding the vertex $4l$ between the vertices $4(l - 1)$ and $4(l - 2)$ of the path Q_l^0. Hence,

$$Q_{l+1,3} = (0) + Q_{l+1}^2 + rev(Q_{l+1}^0) + Q_{l+1}^1 + rev(Q_{l+1}^3).$$

So, we can construct a $(2,1)$-linear realization of the list $\{1, 2^4, 4^4, 8^t\}$, for $t \geq 1$ integer. Moreover, if we take $\hat{Q} = P'_{2,2k}$ (see Proposition 3.8), for $k \geq 1$ integer, we can prove that there is a $(2,1)$-realization of the list $\{1, 2^4, 4^{4k}, 8^s\}$, for all positive integers k, s. Finally, taking the path $\hat{Q} = P_{2,2k+1}$ (see Proposition 3.8) and all ideas presented before, we can construct a $(2,1)$-linear realization of the list $\{1, 2^3, 4^{4k+2}, 8^s\}$, for all positive integers k, s. By Remark 2 and Lemma 3.1, there is a linear realization of $\{1^a, 2^b, 4^{2c}, 8^d\}$, for all positive integers a, b, c, d such that $b \geq 3$ and $c \geq 2$. Moreover, By Remark 2, Corollary 3.7 and Lemma 3.1 there exists a linear realization of $\{1^a, 2^b, 4^c, 8^d\}$, for all positive integers a, b, c, d such that $a \geq 2$, $b \geq 3$ and $c \geq 4$. □

Proposition 3.10. *There are linear realizations of the lists*

$$\{1^a, 2^4, 4^c, 6^{6d+1}\}, \{1^a, 2^5, 4^c, 6^{6d-2}\} \textit{ and } \{1^a, 2^b, 4^c, 6^{6d-2}\},$$

for all positive integer a, b, c, d such that $b \geq 7$.

Proof. Let $k \geq 1$ be an integer. The path

$$Q_k = (0, 3, \ldots, 3k+3, 3k+2, 3k-1, \ldots, 2, 1, 4, \ldots, 3k+1),$$

is a realization of the list $\{1^2, 3^{3k+1}\}$. Then, the even-odd reverse extension of Q_k, $EOR(Q_k)$, is a linear realization of the list $\{1, 2^4, 6^{6k+1}\}$. By Remark 2 and Corollary 3.5, there exists a linear realization of $\{1^a, 2^4, 4^c, 6^{6k+1}\}$ for all positive integer a, c.

On the other hand, the path

$$\hat{Q}_k = (0, 1, 4, \ldots, 3k+1, 3k+2, 3k-1, \ldots, 2, 3, 6, \ldots, 3k),$$

is a linear realization of the list $\{1^3, 3^{3k-1}\}$. Then, the even-odd reverse extension of $\hat{Q}_k$, $EOR(\hat{Q}_k)$, is a (2, 1)-linear realization of the list $\{1, 2^5, 6^{6k-2}\}$. Using Remark 2, Corollary 3.5 and Lemma 3.1, there are linear realizations of the lists $\{1^a, 2^5, 4^c, 6^{6k-2}\}$ and $\{1^a, 2^b, 4^c, 6^{6k-2}\}$, for all positive integer a, b, c such that $b \geq 7$. □

Let $P = (x_0, x_1, \ldots, x_{v-1})$ be a linear realization of a list L, and let $P' = (y_0, y_1, \ldots, y_{v-1})$ be a standard linear realization of the list L', such that $|L| = |L'|$. The *even-odd extension* of P and P', denoted by $EO(P, P')$, is defined as follow:

$$\begin{aligned} EO(P, P') &= E(P) + O(P') \\ &= (2x_0, 2x_1, \ldots, 2x_{v-1}, 2y_1 - 1, 2y_2 - 1, \ldots, 2y_{v-1} - 1); \end{aligned}$$

the *even-odd reverse extension* of P and P', denoted by $EO(P, P')$, is defined as follow:

$$\begin{aligned} EOR(P, P') &= E(P) + OR(P') \\ &= (2x_0, 2x_1, \ldots, 2x_{v-1}, 2y_{v-1} - 1, 2y_{v-2} - 1, \ldots, 2y_1 - 1). \end{aligned}$$

The even-odd extension of P and P' is a linear realization of the list $(2L \cup 2L' \setminus \{2y_1\}) \cup \{|2(x_{v-1} - y_1) + 1|\}$, while the even-odd reverse extension of P and P' is a linear realization of the list $(2L \cup 2L' \setminus \{2y_1\}) \cup \{|2(x_{v-1} - y_{v-1}) + 1|\}$. In particular, if $P' = P$, then $EO(P, P) = EO(P)$ and $EOR(P, P) = EOR(P)$.

To the next, we are going to construct some linear realization from well-known linear realizations.

Example 8. *Let $a \geq 2$ and $b \geq 1$ integers. Let $P = (x_0, x_1, \ldots, x_{a+b})$ be a linear realization of the list $\{1^a, 3^b\}$, and let $Q = (0, 1, 2, \ldots, a+b)$ be a* *linear realization of the list* $\{1^{a+b}\}$*. If* P *is a perfect linear realization, then the even-odd reverse extension of* P *and* Q*,* $EOR(P,Q)$*, is a (standard) linear realization of the list* $\{1, 2^{2a+b-1}, 6^b\}$*. Also, if* $x_{a+b} = 1$*, then the even-odd extension of* P *and* Q*,* $EO(P,Q)$*, is a linear realization of the same list.*

Example 9. *Let* $k = 2s$ *and* $h = 3s+1$*, where* $s \geq 1$ *is an integer. Let* $P_h = (0, 1, \ldots, h)$ *be a linear realization of the list* $\{1^h\}$*. By Example 5, the even-odd reverse extension of* P_h*,* $EOR(P_h)$*, is a* $(2,1)$*-realization of the list* $\{1, 2^{2h-1}\}$*. If* $\hat{P}_h = EOR(P_h)$*, then the even-odd extension of* $\hat{P}_h$ *and* $\hat{Q}_k$ *(see Proposition 3.10),* $EO(\hat{P}_h, \hat{Q}_k)$*, is* $(2,1)$*- realization of the list* $\{1, 2^2, 4^{2h-1}, 6^{3k}\}$*; that is, the even-odd extension of* $\hat{P}_{3s+1}$ *and* $\hat{Q}_{2s}$ *is a* $(2,1)$*-realization of the list* $\{1, 2^2, 4^{6s+1}, 6^{6s}\}$*. By Remark 2, Corollary 3.5 and Lemma 3.1 there are linear realizations of* $\{1^a, 2^2, 4^c, 6^{6d}\}$ *and* $\{1^a, 2^b, 4^c, 6^{6d}\}$ *for all positive integers* a, b, c, d *such that* $b \geq 4$ *and* $c \geq 7$*.*

4 *k*-extension of linear realizations

In this section, we are going to generalize the even-odd extension given in Section 3 for well-known linear realizations.

Let $P = (x_0 = 0, x_1, \ldots, x_{v-1})$ be a (standard) linear realization of a list L. For each $i \in \{1, 2, \ldots, k-1\}$, the *i-application* of P is defined by the path

$$P_{k,i} = (kx_1 - i, kx_2 - i, \ldots, kx_{v-1} - i).$$

So, $\ell(P_{k,i}) = kL \setminus \{kx_1\}$. We define the *k-extension* of P, denoted by $E_k(P)$, as follow:

$$E_k(P) = P_{k,0} + P_{k,1} + \cdots + P_{k,k-1},$$

where $P_{k,0} = kP = (0, kx_1, \ldots, kx_{v-1})$. Notice that

$$|kx_1 - (i+1) - (kx_{v-1} - i)| = |k(x_1 - x_{v-1}) - 1|.$$

Hence, the k-extension of P is a linear realization of the list

$$(kL \cup kL \setminus \{kx_1\} \cup kL \setminus \{kx_1\} \cup \cdots \cup kL \setminus \{kx_1\}) \cup \{|k(x_1 - x_{v-1}) - 1|^{k-1}\}.$$

Corollary 4.1. *Let* $P = (x_0, x_1, \ldots, x_{v-1})$ *be a standard linear realization of* L*, where the vertices* $v-1$ *and* $v-2$ *are adjacent. If* $E_k(P)$ *is the linear realization of* L_k*, then the list* $L_k \cup \{(2k)^b\}$ *admits a linear realization for any positive integer* b*.*

Proof. Note that the vertices $k(v-1)-i, k(v-2)-i$ are adjacent in $E_k(P)$ for all $i = 0, \dots, k-1$. Then, one can proceed as the proof of Lemma 7 of [10]. □

Example 10. *Let $P = (0, 1, \dots, s)$ be a linear realization of the list $\{1^s\}$. For each $i = \{1, 2\}$ (in this case $k = 3$), the i-application of P is given by the path*

$$P_{3,i} = (3 \cdot 1 - i, 3 \cdot 2 - i, \dots, 3 \cdot s - i),$$

which satisfies that $\ell(P_{3,i}) = \{3^{s-1}\}$. Hence, the 3-extension of P:

$$E_3(P) = (0, 3, 6, \dots, 3s, 2, 5, \dots, 3s-1, 1, 4, \dots, 3s-2)$$

is a linear realization of the list $\{3^{3s-2}, (3s-2)^2\}$. By Remark 2 and Corollary 4.1 there exists a linear realization of the list $\{1^a, 3^{3s-2}, 6^b, (3s-2)^2\}$ for all positive integer a, b, s.

Proposition 4.2. *There are linear realizations of the lists $\{1^a, 2^2, 3^3, 6^c\}$ and $\{1^a, 2^b, 3^3, 6^c\}$, for all positive integers a, b, c such that $b \geq 4$.*

Proof. Let $s \geq 1$ be an integer. Consider the linear realizations P_s and P'_s of the lists $\{1, 2^{2s-1}\}$ and $\{1, 2^{2s}\}$, respectively, given in Example 6:

$$P_s = (0, 2, \dots, 2s, 2s-1, 2s-3, \dots, 1),$$
$$P'_s = (0, 2, \dots, 2s, 2s+1, 2s-1, \dots, 1).$$

For each $i = \{1, 2\}$ (in this case $k = 3$), the i-realization of P_s, and P'_s are:

$$P_{s,i} = (3 \cdot 2 - i, \dots, 3 \cdot 2s - i, 3 \cdot (2s-1) - i, 3 \cdot (2s-3) - i, \dots, 3 \cdot 1 - i),$$
$$P'_{s,i} = (3 \cdot 2 - i, \dots, 3 \cdot 2s - i, 3 \cdot (2s+1) - i, 3 \cdot (2s-1) - i, \dots, 3 \cdot 1 - i),$$

respectively. So, $\ell(P_{s,i}) = \{3, 6^{2s-2}\}$ and $\ell(P'_{s,i}) = \{3, 6^{2s-1}\}$. Therefore, the 3-extensions of P_s, $E_3(P_s)$, and P'_s, $E_3(P'_s)$, are $(6,1)$-realization of the lists $\{2^2, 3^3, 6^{6s-5}\}$ and $\{2^2, 3^3, 6^{6s-2}\}$, respectively. By Remark 2, Corollary 4.1 and Lemma 3.1, there are linear realizations of the lists $\{1^a, 2^2, 3^3, 6^c\}$ and $\{1^a, 2^b, 3^3, 6^c\}$, for all positive integers a, b, c such that $b \geq 4$. □

For each $i \in \{0, 1, 2, \dots, k-1\}$ let $Q_i = (x_{i,0} = 0, x_{i,1}, \dots, x_{i,v-1})$ be k (standard) linear realizations of the list L_i, such that $|L_i| = |L_j|$, for every $0 \leq i < j \leq k-1$. A *k-extension* of $Q_0, Q_1, \dots, Q_{k-1}$, denoted by $E_k(Q_0^{T_0}, Q_1^{T_1}, \dots, Q_{k-1}^{T_k})$, is defined as follow:

$$E_k(Q_0^{T_0}, Q_1^{T_1}, \dots, Q_{k-1}^{T_k}) = Q_{k,0}^{T_0} + Q_{k,1}^{T_1} + \cdots + Q_{k,k-1}^{T_k},$$

where either $Q_{k,i}^{T_i} = Q_{k,i}$ if $Q_i^{T_i} = Q_i$ or $Q_{k,i}^{T_i} = rev(Q_{k,i})$ if $Q_i^{T_i} = Q_i^{rev}$, for all $i = 0, 1, \ldots, k-1$, and where $Q_{k,i} = (kx_{i,1} - i, kx_{i,2} - i, \ldots, kx_{i,v-1} - i)$, for all $i = 1, \ldots, k-1$, and $Q_{k,0} = (0, kx_{0,1}, \ldots, kx_{0,v-1})$. So, $\ell(Q_{k,i}) = kL_i \setminus \{kx_{i,1}\}$, for all $i = 1, \ldots, k-1$, and $\ell(Q_{k,0}) = kL_0$.

A k-extension of $Q_0, Q_1, \ldots, Q_{k-1}$, $E_k(Q_0^{T_0}, Q_1^{T_1}, \ldots, Q_{k-1}^{T_{k-1}})$, is a linear realization of the list

$$\big(kL_0 \cup kL_1 \setminus \{kx_{1,1}\} \cup kL_2 \setminus \{kx_{2,1}\} \cup \cdots \cup kL_{k-1} \setminus \{kx_{(k-1),1}\}\big) \cup R,$$

where $R = \bigcup_{i=0}^{k-2} |k(x_{(i+1),p} - x_{i,q}) - 1|$, where either $p = 1$ if $Q_{i+1}^{T_{i+1}} = Q_{i+1}$ or $p = v-1$ if $Q_{i+1}^{T_{i+1}} = Q_{i+1}^{rev}$, and $q = v-1$ if $Q_i^{T_i} = Q_i$ or $q = 1$ if $Q_i^{T_i} = Q_i^{rev}$, for all $i = 0, 1, \ldots, k-2$.

Proposition 4.3. *For all $k \geq 2$ an even integer and s a positive integer, there exists a linear realization of the list*

$$\{1^{k-1}, k^k, (2k)^k, \ldots, ((s-1)k)^k, (sk)\}.$$

Proof. Let $C = C_s = (x_0, x_1, \ldots, x_s)$ with $x_{2i} = i$ and $x_{2i+1} = s - i$, for all $i \in \{0, 1, \ldots, \lfloor s/2 \rfloor\}$, be the well-known Walecki linear realization of the list $\{1, 2, \ldots, s\}$, see [8] (page 3). The following k-extension of C:

$$E_k(C, C^{rev}, \ldots, C, C^{rev}) = C_{k,0} + rev(C_{k,1}) + \cdots + C_{k,k-2} + rev(C_{k,k-1}),$$

is a linear realization of the list $\{1^{k-1}, k^k, (2k)^k, \ldots, ((s-1)k)^k, (sk)\}$. □

Corollary 4.4. *For $i = 0, 1, \ldots, k$, let $Q_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,v-1})$ be a standard linear realization of L_i, where the vertices $v-1$ and $v-2$ are adjacent for all i. If $E_k(Q_0^{T_0}, Q_1^{T_1}, \ldots, Q_{k-1}^{T_{k-1}})$ is a (standard) linear realization of L, then the list $L \cup \{(2k)^b\}$ is linear realizable, for any positive integer b.*

Proof. See proof of Corollary 4.1. □

Proposition 4.5. *Let $k \geq 2$ be an integer, then there exists a linear realization of the list*

$$\{1^a, 2^b, k^{k(t-1)+1}, (2k)^c\}$$

for all integers a, b, c, t such that $a \geq k-1$ and $t, b \geq 2$.

Proof. Let $I = I_t = \{0, 1, \ldots, t\}$ be the linear realization of the list $\{1^t\}$, where $t \geq 2$ is a positive integer. The following k-extension of I:

$$E_k(I, I^{rev}, \ldots, I, I^{rev}) = I_{k,0} + rev(I_{k,1}) + \cdots + I_{k,k-2} + rev(I_{k,k-1}),$$

is a $(k, 1)$-realization of $\{1^{k-1}, k^{k(t-1)+1}\}$. By Remark 2, Corollary 4.4 and Lemma 3.1, there exists a linear realization of the list

$$\{1^a, 2^b, k^{k(t-1)+1}, (2k)^c\},$$

for all integers a, b, c such that $a \geq k - 1$ and $b \geq 2$. □

Proposition 4.6. *There exists a linear realization of the lists* $\{1^a, 4^4, 8^c\}$, *for all positive integers* a, b, c *such that* $a \geq 3$ *and* $c \geq 1$.

Proof. Let $s \geq 1$ be an integer. Consider the linear realizations P_s of the list $\{1, 2^{2s-1}\}$ (given in Example 6): $P_s = (0, 2, \ldots, 2s, 2s-1, 2s-3, \ldots, 1)$. The following 4-extension $E_4(P_s, P_s^{rev}, P_s, P_s^{rev})$ is a linear extension of $\{1^3, 4^4, 8^{8s-7}\}$. By Remark 2 and Corollary 4.4 there exists a linear realization of the lists $\{1^a, 4^4, 8^c\}$, for all positive integers a, b, c such that $a \geq 3$ and $c \geq 1$. □

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