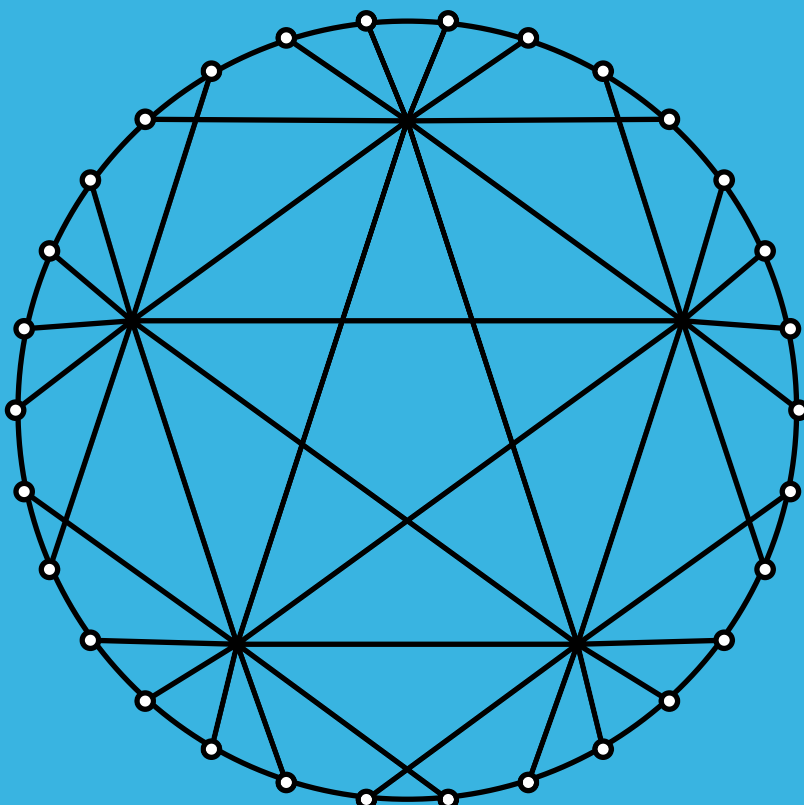
Volume 93
October 2021

BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS



Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **93** (2021), 119–125

The Steiner distance problem for large vertex subsets in the hypercube

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Abstract

We find the asymptotic behavior of the Steiner k-diameter of the n-cube if k is large. Our main contribution is the lower bound, which utilizes the probabilistic method.

1 Introduction

For a connected graph G of order at least 2 and $S \subseteq V(G)$, the *Steiner distance* $d(S)$ among the vertices of S is the minimum size among all connected subgraphs whose vertex sets contain S. Necessarily, such a minimum subgraph must be a tree and such a tree is called a *Steiner tree*.

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AMS (MOS) Subject Classifications: Primary 05C12; secondary 05C05, 05C35, 05C69
Key words and phrases: hypercube, Steiner distance, domination

Received: 19 July 2019
Accepted: 22 June 2021

The Steiner distance was introduced by G. Chartrand, O.R. Oellermann, S. Tian and H.B. Zou [2], and it has turned into a well-studied parameter of graphs. Tao Jiang, Zevi Miller, and Dan Pritikin [6] studied how large the Steiner distance of k vertices can be in the n-dimensional hypercube Q_n as $n \to \infty$, while Zevi Miller and Dan Pritikin [5] gave near tight bounds for the Steiner distance of a layer, i.e. vertices with the same number of 1's, in the n-dimensional hypercube Q_n as $n \to \infty$. For a given $2 \leq k \leq n$, the *Steiner k-diameter* of the n-cube, $sdiam_k(Q_n)$, is the maximum Steiner distance among all k subsets of $V(Q_n)$.

In this note we give natural upper bounds for the Steiner distance of a *large* vertex set in the hypercube. It turns out that even the second order term in this estimate is close to tight. With these bounds, we determine $sdiam_k(Q_n)$ asymptotically for large k.

2 Upper bound

For the upper bound, we utilize connected dominating sets of Q_n. A set $S \subset V(Q_n)$ is a *dominating set* of Q_n if every vertex of Q_n is either an element of S or has a neighbor in S. The minimum size of all dominating sets is called the *domination number* of Q_n and is denoted $\gamma(Q_n)$. The *connected domination number*, denoted by $\gamma_c(Q_n)$, is minimum size of all connected dominating sets.

In 1988, Kabatyanskii and Panchenko [4] showed

$$\lim_{n\to\infty} \frac{\gamma(Q_n)}{2^n/n} = 1.$$

In an upcoming paper, Griggs [3] utilizes this result to show that

$$\lim_{n\to\infty} \frac{\gamma_c(Q_n)}{2^n/n} = 1.$$

We use this last result to develop an upper bound for the Steiner diameter of subsets of $V(Q_n)$.

Lemma 1. *Suppose that $S \subset V(Q_n)$. Then,*

$$d(S) \leq |S| + \frac{2^n}{n}(1 + o(1)).$$

Proof. Begin with a spanning tree of a minimum connected dominating set of Q_n. Add edges as needed to connect each element of S to this tree. The resulting subgraph spans S and contains at most $|S| + \gamma_c(Q_n) - 1$ edges. Using [3], we then have that $d(S) \leq |S| + \frac{2^n}{n}(1 + o(1))$. $\square$

3 Lower bound

To bound the Steiner distance of large vertex subsets of Q_n from below, we partition the vertices of the hypercube into two sets. Identifying each vertex of Q_n into a binary string of length n, we let vertices with an even number of 1's make up the set of even vertices and denote this set by $\mathcal{E}_n$. Similarly, we let the vertices with an odd number of 1's make up the set of odd vertices and denote this set by $\mathcal{O}_n$. We refer to changing the value of the ith entry of a binary string $v = v_0 \cdots v_i \cdots v_n$ as "flipping" the ith entry of v. Given an entry v_i, we let $\bar{v}_i = 1 - v_i$. That is, $\bar{v}_i$ is the flipped value of v_i. For the proof of Theorem 2, we use probabilistic methods similar to those found in [1].

Theorem 2. *If $S \subset \mathcal{E}_n$ with $|S| \geq 2$, then*

$$d(S) \geq |S| + \frac{|S|^2}{n2^n} - \frac{(n+1)}{2}.$$

Proof. Suppose that $S \subset \mathcal{E}_n$. Let S' be a subset of the odd vertices which is the image of S under some automorphism of Q_n. That is, $S \subset \mathcal{E}_n$, $S' \subset \mathcal{O}_n$, and $S' = \gamma(S)$ for some $\gamma \in \mathrm{Aut}(Q_n)$. To show that such a subset S' exists, consider the set of all vertices in S with the first entry flipped. Since S' is the image of S under the automorphism γ, we have that $d(S) = d(S')$.

Now suppose that λ_1 and λ_2 are automorphisms of Q_n which preserve the parity of their inputs. Then, $\lambda_1(S) \subset \mathcal{E}_n$ and $\lambda_2(S') \subset \mathcal{O}_n$. Since λ_1 and λ_2 are automorphisms, we have that $d(\lambda_1(S)) = d(\lambda_2(S')) = d(S)$.

We now bound $d(\lambda_1(S) \cup \lambda_2(S'))$ above and below in terms of $|S|$ and $d(S)$. For the lower bound, note that $\lambda_1(S)$ and $\lambda_2(S')$ are disjoint. Hence, we have the naive bound

$$2|S| - 1 \leq d(\lambda_1(S) \cup \lambda_2(S')). \tag{1}$$

For the upper bound, suppose that $\lambda_1(T)$ and $\lambda_2(T')$ are Steiner trees of $\lambda_1(S)$ and $\lambda_2(S')$ in Q_n, respectively. Denote the respective edge sets of the Steiner trees by $E(\lambda_1(T))$ and $E(\lambda_2(T'))$. Using no more than n edges (Since $diam(Q_n) = n$), we may connect $\lambda_1(T)$ and $\lambda_2(T')$ to form a subgraph of Q_n which contains $\lambda_1(S) \cup \lambda_2(S')$. Hence,

$$d(\lambda_1(S) \cup \lambda_2(S')) \leq |E(\lambda_1(T)) \cup E(\lambda_2(T'))| + n. \tag{2}$$

Linking inequalities (1) and (2) together and applying the principle of inclusion and exclusion, we have

$$\begin{aligned} 2|S| - 1 &\leq |E(\lambda_1(T)) \cup E(\lambda_2(T'))| + n \\ &= |E(\lambda_1(T))| + |E(\lambda_2(T'))| - |E(\lambda_1(T)) \cap E(\lambda_2(T'))| + n \\ &= 2d(S) - |E(\lambda_1(T)) \cap E(\lambda_2(T'))| + n, \end{aligned}$$

which implies that

$$2d(S) - |E(\lambda_1(T)) \cap E(\lambda_2(T'))| \geq 2|S| - (n+1). \tag{3}$$

Let $\Gamma = \langle \alpha, \beta_{i,j} : 1 \leq 0 < j \leq n-1 \rangle$ be the subgroup of the group of automorphisms of Q_n generated by the automorphisms

$$\begin{aligned} \alpha :& v_0 v_1 \cdots v_{n-1} \mapsto v_1 \cdots v_{n-1} v_0 \\ \beta_{i,j} :& v_0 v_1 \cdots v_i \cdots v_j \cdots v_{n-1} \mapsto v_0 v_1 \cdots \bar{v_i} \cdots \bar{v_j} \cdots v_{n-1}. \end{aligned}$$

In words, α shifts each entry of its input to the left by 1 (modulo n), while $\beta_{i,j}$ flips only the values of the ith and jth entries of its input. Note that each element of Γ preserves the parity of its input. We now verify the following claim:

Claim: For any two edges $e_1, e_2 \in E(Q_n)$, there exists a *unique* element of $\lambda \in \Gamma$ such that $\lambda(e_1) = e_2$.

Proof. Suppose that $e_1 = ab$ and $e_2 = uv$ where a and u are even vertices while b and v are odd vertices. Without loss of generality, we may assume that $a = \mathbf{0}$, the vertex of all zeros. This implies that the string b contains a single 1. We shall first prove existence of an automorphism $\lambda \in \Gamma$ mapping e_1 to e_2.

Since $u \in \mathcal{E}_n$, using a composition of automorphisms of the form $\beta_{i,j}$ we may map uv to $\mathbf{0}\hat{v}$, where $\hat{v}$ has a single 1. Then, using some power of the automorphism α, we may then map the edge $\mathbf{0}\hat{v}$ to the edge $\mathbf{0}b = e_1$. Let λ be the composition of these automorphisms in Γ.

To show that this automorphism is unique, we show that $|\Gamma| = n2^{n-1}$. Since $\alpha \circ \beta_{ij} = \beta_{i-1,j-1} \circ \alpha$ (where the indexes are taken modulo n), any

$\lambda \in \Gamma$ can be described as first applying an appropriate power of α and then flipping an even number of digits. As we have n choices for the power of α and 2^{n-1} choices for the subset of digits we flip, we conclude that $|\Gamma| = n2^{n-1}$.

Since Q_n has $n2^{n-1}$ edges, any $\lambda \in \Gamma$ maps the edge $\mathbf{0}b$ to an edge in such a way that $\mathbf{0}$ is mapped the edge's vertex in $\mathcal{E}_n$, and all edges of Q_n will be the image of $\mathbf{0}b$ under some $\lambda \in \Gamma$, the claim follows. □

We now consider the experiment of selecting elements $\lambda_1, \lambda_2 \in \Gamma$ independently with uniform probability, and applying them to T and T', respectively. Consider the random variable $X = |E(\lambda_1(T)) \cap E(\lambda_2(T'))|$. For the expected value of X, $\mathbb{E}(X)$, we have that

$$\max_{\lambda_1,\lambda_2}\{|E(\lambda_1(T) \cap \lambda_2(T'))|\} \geq \mathbb{E}(X).$$

Using our claim, we observe that

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{f \in E(Q_n)} P[(f \in E(\lambda_1(T))) \text{ and } (f \in E(\lambda_2(T')))] \\
&= \sum_{f \in E(Q_n)} \frac{|E(\lambda_1(T))|}{n2^{n-1}} \cdot \frac{|E(\lambda_2(T'))|}{n2^{n-1}} \\
&= \frac{|E(\lambda_1(T))|^2}{n2^{n-1}} \\
&= \frac{d(S)^2}{n2^{n-1}},
\end{aligned}$$

which implies

$$\max_{\lambda_1,\lambda_2}\{|E(\lambda_1(T)) \cap E(\lambda_2(T'))|\} \geq \frac{d(S)^2}{n2^{n-1}}.$$

Using λ_1 and λ_2 which achieve this maximum and applying inequality (3), we see that

$$2d(S) - \frac{d(S)^2}{n2^{n-1}} \geq 2|S| - (n+1).$$

Using the above inequality, we now bound $d(S)$ from below. Since $|S| \geq 2$ and $S \subset \mathcal{E}_n$, we have that $n \geq 2$. Hence, $d(S) = |S| + x$ for some $x \geq 0$.

So,

$$\begin{aligned}
2(|S|+x) - \frac{(|S|+x)^2}{n2^{n-1}} &\geq 2|S| - (n+1) \\
2|S| + 2x - \frac{|S|^2 + 2|S|x + x^2}{n2^{n-1}} &\geq 2|S| - (n+1) \\
2x - \frac{2|S|x}{n2^{n-1}} + 2|S| - \frac{|S|^2 + x^2}{n2^{n-1}} &\geq 2|S| - (n+1) \\
2x\left(1 - \frac{|S|}{n2^{n-1}}\right) &\geq \frac{|S|^2 + x^2}{n2^{n-1}} - (n+1) \\
x &\geq \frac{|S|^2}{n2^n} - \frac{(n+1)}{2},
\end{aligned}$$

and the result is proven. □

Remark: In the above theorem, we assumed $|S| \geq 2$. If $|S| = 1$, we have that $d(S) = 0$.

With these results in hand, we can determine the asymptotic growth of $sdiam_k(Q_n)$ for large k. In particular, we can determine the first and second order terms if $k = \Omega(2^n)$, while we can determine the first order term if $2^n/n = o(k)$.

Corollary 3. *If $k = k(n)$ is regarded as a function n, then*

1. *if $k = \Omega(2^n)$, then $sdiam_k(Q_n) = k + \Theta(2^n/n)$, and*
2. *if $2^n/n = o(k)$, then $\lim_{n\to\infty} \frac{sdiam_k(Q_n)}{k} = 1$.*

Proof. If $k \leq 2^{n-1}$, let $S \subset V(Q_n)$ be a subset of the even vertices of size k. If $k > 2^{n-1}$, let S contain all even vertices and choose the remaining odd vertices randomly. Applying the bounds determined in Lemma 1 and Theorem 2, we see that

$$k + \frac{k^2}{n2^n} - \frac{n+1}{2} \leq d(S) \leq sdiam_k(Q_n) \leq k + \frac{2^n}{n}(1 + o(1)).$$

If $k = \Omega(2^n)$, then $sdiam_k(Q_n)$ is bounded above and below by $k+\Theta(2^n/n)$. More precisely, we have that $k+c_1(2^n/n) \leq sdiam_k(Q_n) \leq k+c_2(2^n/n)$ for some positive constants c_1 and c_2. On the other hand, if only $2^n/n = o(k)$, we have $sdiam_k(Q_n) = k(1 + o(1))$, giving $\lim_{n\to\infty} \frac{sdiam_k(Q_n)}{k} = 1$. □

Acknowledgement

The last two authors were supported in part by the National Science Foundation contract DMS-1600811.

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