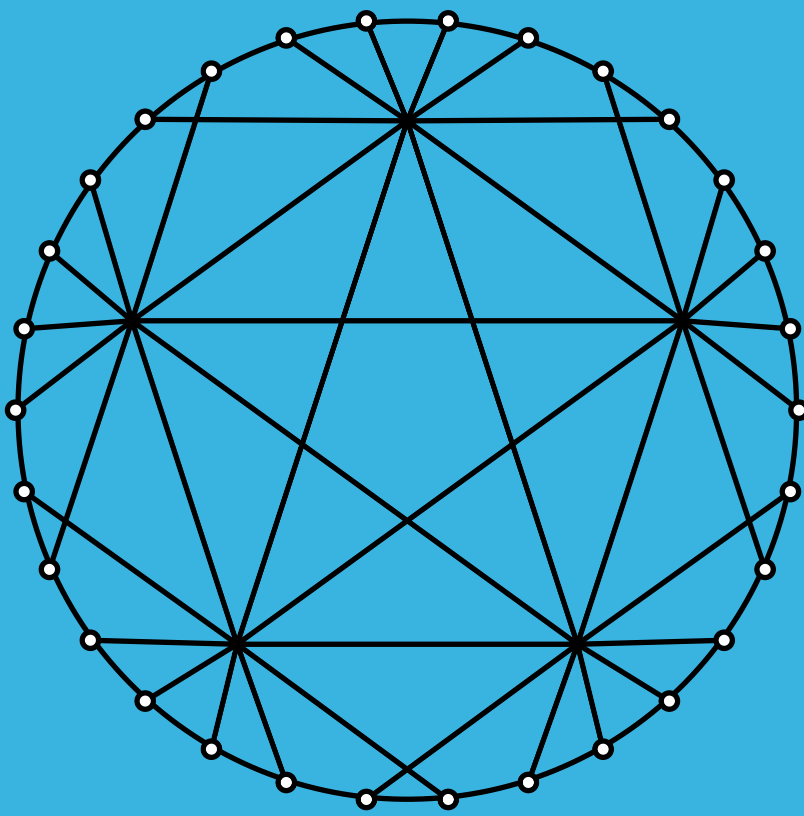
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 92
June 2021



Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **92** (2021), 50–61

Decompositions of complete graphs and complete bipartite graphs into bipartite cubic graphs of order at most 12

Peter Adams[1], Ryan C. Bunge[2], Roger B. Eggleton[3], Saad I. El-Zanati*[2], Uğur Odabaşı[2], and Wannasiri Wannasit[4]

[1]University of Queensland, Brisbane, Queensland, Australia
p.adams@uq.edu.au
[2]Illinois State University, Normal, Ilinois , USA
rcbunge@ilstu.edu, saad@ilstu.edu, odabasi.ugur@gmail.com
[3]University of Newcastle, Callaghan, New South Wales, Australia
roger.eggleton@newcastle.edu.au
[4]Chiang Mai University, Chiang Mai, Thailand
wannasiri.w@cmu.ac.th

Abstract

There are ten bipartite cubic graphs of order $n \leq 12$. For each such graph G we give necessary and sufficient conditions for the existence of decompositions of K_n and of $K_{m,n}$ into copies of G.

1 Introduction

A *decomposition* of a graph H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of subgraphs of H such that each edge of H appears in exactly one G_i. If each G_i in Δ

*Corresponding author.
Key words and phrases: Graph decompositin, bipartite cubic graphs
AMS (MOS) Subject Classifications: 05C51

Received: 11 August 2020
Accepted: 16 December 2020

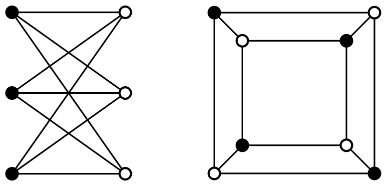


Figure 1: The bipartite cubic graphs of order 6 or 8.

is isomorphic to a given graph G, then Δ is called a *G-decomposition* of H and the copies of G in Δ are called *G-blocks*. A G-decomposition of H is also known as an *(H, G)-design*, and a (K_n, G)-design is often known as a *G-design of order n*.

Given a graph G, a classic problem in combinatorics is to find necessary and sufficient conditions on n for the existence of a (K_n, G)-design. This is known as the *spectrum problem* for G. It has been investigated and settled for numerous classes of simple graphs (see [2] and [7] for summaries and the website maintained by Bryant and McCourt [9] for more up-to-date results). If in particular G is bipartite, it is also of interest to find necessary and sufficient conditions for the existence of $(K_{m,n}, G)$-designs. For compactness, let us call this the *bispectrum problem* for G.

Let $\mathcal{F}_k$ be the set of bipartite cubic graphs of order k, for even $k \geq 6$. Then $|\mathcal{F}_6| = 1$, $|F_8| = 1$, $|F_{10}| = 2$ and $|F_{12}| = 6$ (see Figures 1–3 and [16]).

We consider the bispectrum and spectrum problems for each $G \in \mathcal{F}_k$ with $k \leq 12$. The bispectrum problem has been settled for each $G \in \mathcal{F}_k$ with $k \leq 8$; we extend this to each $G \in \mathcal{F}_{10} \cup \mathcal{F}_{12}$. Again, the spectrum problem has been settled for each $G \in \mathcal{F}_k$ with $k \leq 10$; we extend this to each $G \in \mathcal{F}_{12}$.

2 Known results

The original spectrum problem, for $K_3 = C_3$, was posed by Woolhouse [20] and settled by Kirkman in 1847 [13]. The spectra for K_4 and K_5 were determined by Hanani a little more than a century later [12]. Various authors investigated the spectrum for cycles C_n, $n \geq 4$; this was fully settled by Alspach and Gavlas [6], and by Šajna [17] in the early 2000s.

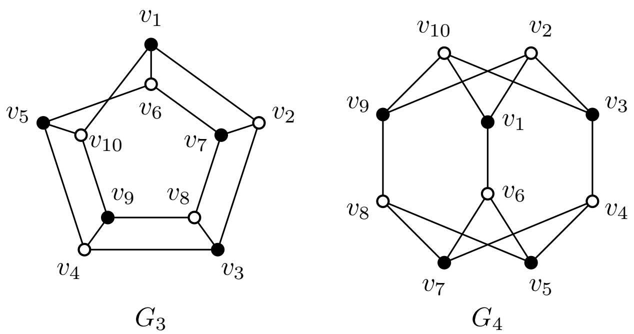


Figure 2: The two bipartite cubic graphs of order 10.

Thus the case of connected 2-regular graphs is settled. The connected 3-regular (that is, cubic) graphs are the next challenge, and our knowledge in this case remains limited to specific instances.

Let us review known results for cubic graphs. The number of connected cubic graphs of each order is sequence A002851 in OEIS [18]. The two cubic graphs of order 6 are $K_{3,3} \in \mathcal{F}_6$ and the 3-prism $D_3 = C_3 \times K_2$. The spectrum for $K_{3,3}$ was established by Guy and Beineke [11]; for D_3 the spectrum was determined by Carter [10]. For both cubic graphs of order 6 the spectrum is

$$\{n \geq 10 : n \equiv 1 \pmod{9}\}.$$

The spectrum for the 3-cube $Q_3 = D_4 = C_4 \times K_2 \in \mathcal{F}_8$ was found by Maheo [14]; for the other four cubic graphs of order 8 the spectrum was determined by three of the present authors [5]. For all five cubic graphs of order 8, the spectrum is

$$\{n \geq 17 : n \equiv 1, 16 \pmod{24}\}.$$

There are 19 connected cubic graphs of order 10. Among these graphs, spectrum results were found for the Petersen graph by Adams and Bryant [1], for the 5-prism D_5 and the Möbius 5-ladder $M_5 \in \mathcal{F}_{10}$ by Meszka, Nedela, Rosa and Skoviera [15], and for three others by Adams et al. [3]. More recently four of the present authors [4] have shown that the spectrum for each connected cubic graph of order 10 is

$$\{n \geq 16 : n \equiv 1, 10 \pmod{15}\} \cup X,$$

where $X = \varnothing$ for five specified graphs and $X = \{10\}$ for the other 14.

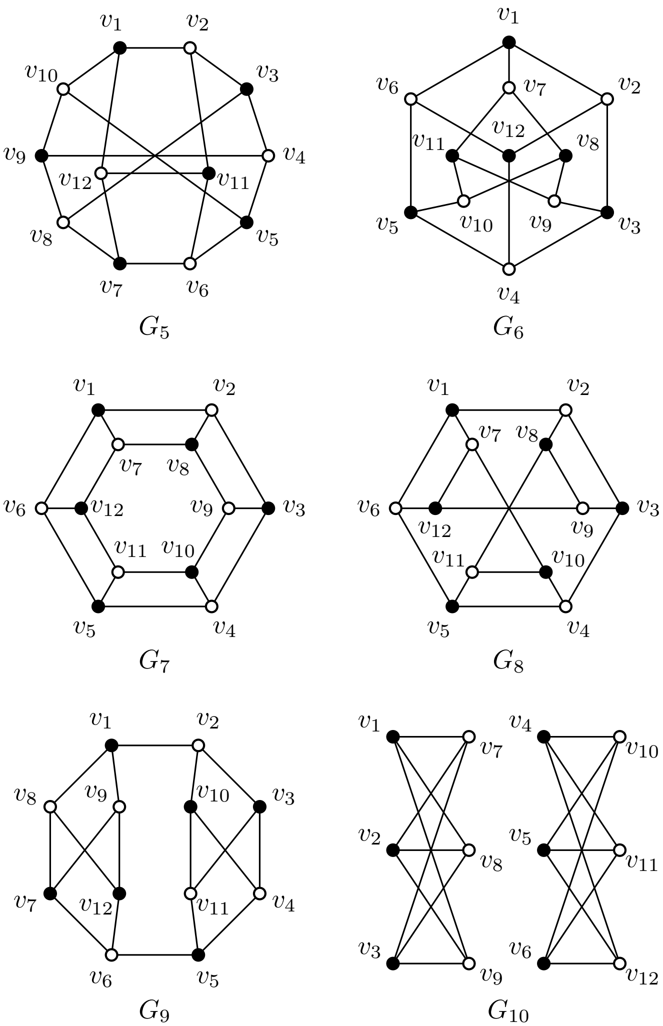


Figure 3: The six bipartite cubic graphs of order 12.

For the bispectrum problem, clearly $K_{3,3} \in \mathcal{F}_6$ has bispectrum

$$\big\{(m,n) : m,n \geq 3;\ m \equiv n \equiv 0 \pmod 3\big\}.$$

For the 3-cube $Q_3 \in \mathcal{F}_8$ the bispectrum has been shown [8] to be

$$\big\{(m,n) : m,n \geq 6;\ m \equiv n \equiv 0 \pmod 3;\ mn \equiv 0 \pmod 4\big\}.$$

3 Decompositions of complete bipartite graphs

We use the standard interval notation $[a,b]$ for the set $\{n \in \mathbb{Z} : a \leq n \leq b\}$. For the graphs $G_i \in \mathcal{F}_k$ with $i \in [3,10]$ labeled as in Figures 2 and 3, we shall use the notation $G_i = G_i(v_1, v_2, \ldots, v_k)$. For instance, the vertex set V and edge set E of the graph $G_3(0,6,1,7,2,8,3,9,4,10)$ are

$$\begin{aligned} V &= \{0,6,1,7,2,8,3,9,4,10\} = [0,4] \cup [6,10],\\ E &= \big\{\{0,6\},\{6,1\},\{1,7\},\{7,2\},\{2,8\},\{8,3\},\{3,9\},\{9,4\},\{4,10\},\\ &\qquad \{10,0\},\{0,8\},\{6,3\},\{1,9\},\{7,4\},\{2,10\}\big\}. \end{aligned}$$

We now specify decompositions of $K_{6,15}$ and $K_{9,15}$ for each $G \in \mathcal{F}_{10}$.

Decompositions of $K_{6,15}$

Let $V(K_{6,15}) = [0,5] \cup [6,20]$ with the implied vertex partition and let

$$\begin{aligned} \Delta_3 = \{&G_3(0,6,1,7,2,8,3,9,4,10),\ G_3(0,7,3,11,1,12,5,13,2,14),\\ &G_3(0,9,2,12,3,15,5,6,4,16),\ G_3(0,11,4,15,1,17,5,18,2,19),\\ &G_3(0,13,4,14,3,18,1,8,5,20),\ G_3(1,10,3,17,2,16,5,19,4,20)\},\\ \Delta_4 = \{&G_4(0,6,1,7,2,8,3,9,4,10),\ G_4(0,7,4,11,1,12,2,13,5,14),\\ &G_4(0,9,1,14,2,15,3,6,5,16),\ G_4(0,11,3,16,2,17,4,18,5,19),\\ &G_4(1,8,4,13,0,18,3,20,5,15),\ G_4(1,19,2,10,3,17,5,12,4,20)\}. \end{aligned}$$

For $i \in \{3,4\}$, a G_i-decomposition of $K_{6,15}$ consists of the G_i-blocks in Δ_i.

Decompositions of $K_{9,15}$

Let $V(K_{9,15}) = [0,8] \cup [9,23]$ with the implied vertex partition and let

$$\begin{aligned}\Delta_3 = \{&G_3(0,9,1,10,2,11,3,12,4,13),\ G_3(0,10,3,13,1,14,5,15,6,16),\\ &G_3(0,12,2,9,4,15,7,14,6,17),\ G_3(0,18,1,11,4,19,2,15,8,20),\\ &G_3(0,21,1,17,2,22,3,19,5,23),\ G_3(1,20,3,14,8,22,7,16,4,23),\\ &G_3(2,16,8,10,6,20,5,9,7,21),\ G_3(3,17,8,12,6,23,7,13,5,18),\\ &G_3(4,18,7,11,5,21,8,19,6,22)\},\\ \Delta_4 = \{&G_4(0,9,1,10,2,11,3,12,4,13),\ G_4(0,10,4,11,1,12,5,14,6,15),\\ &G_4(0,14,2,9,3,16,5,13,7,17),\ G_4(0,18,1,15,2,19,3,20,4,21),\\ &G_4(0,20,1,16,2,22,6,13,8,23),\ G_4(2,18,5,20,6,23,7,9,8,21),\\ &G_4(3,18,6,19,4,14,8,16,7,21),\ G_4(5,10,7,22,3,23,4,17,8,15),\\ &G_4(6,11,7,19,1,17,5,22,8,12)\}.\end{aligned}$$

For $i \in \{3,4\}$, a G_i-decomposition of $K_{9,15}$ consists of the G_i-blocks in Δ_i.

Bispectrum for order 10

We now have the starter decompositions needed to establish the bispectrum for graphs in $\mathcal{F}_{10}$.

Theorem 3.1. *The bispectrum of each $G \in \mathcal{F}_{10}$ is*

$$\big\{(m,n) : m,n \geq 6;\ m \equiv n \equiv 0 \pmod 3;\ mn \equiv 0 \pmod 5\big\}.$$

Proof. To establish necessity, suppose there is a $(K_{m,n}, G)$-design for $G \in \mathcal{F}_{10}$. The "local condition" is that the vertex degrees of $K_{m,n}$ must be multiples of the vertex degrees of G, so $m \equiv n \equiv 0 \pmod 3$. The "global conditions" are that the size of $K_{m,n}$ must be a multiple of the size of G, and the partite sets of $K_{m,n}$ must be large enough to accommodate the partite sets of G. This requires $15 \mid mn$ and $m, n \geq 5$. These conditions consolidate to restricting (m,n) to the set $\big\{(m,n) : m,n \geq 6;\ m \equiv n \equiv 0 \pmod 3;\ mn \equiv 0 \pmod 5\big\}$.

Now we prove sufficiency. Without loss of generality, assume $n \equiv 0 \pmod{15}$.

First suppose m is even, so $m \equiv 0 \pmod 6$. Then $m = 6x$, $n = 15y$ where x, y are positive integers, and xy edge-disjoint copies of the $(K_{6,15}, G)$-design produce a $(K_{6x,15y}, G)$-design.

Now suppose m is odd, so $m \equiv 3 \pmod 6$. Then $m = 6x+3$, $n = 15y$ where x, y are positive integers. Then y edge-disjoint copies of the $(K_{9,15}, G)$-design and $(x-1)y$ edge-disjoint copies of the $(K_{6,15}, G)$-design produce a $(K_{6x+3,15y}, G)$-design. □

Decompositions of $K_{6,6}$

Let $V(K_{6,6}) = [0,5] \cup [6,11]$ with the implied vertex partition and let

$$\begin{aligned}
\Delta_5 &= \{G_5(0,6,1,7,2,8,3,9,4,10,5,11),\ G_5(0,9,2,11,1,10,3,6,4,8,5,7)\},\\
\Delta_6 &= \{G_6(0,6,1,7,2,8,9,3,10,11,4,5),\ G_6(0,10,2,9,1,11,7,3,6,8,4,5)\},\\
\Delta_7 &= \{G_7(0,6,1,7,2,8,9,3,10,4,11,5),\ G_7(0,10,2,9,1,11,7,5,6,4,8,3)\},\\
\Delta_8 &= \{G_8(0,6,1,7,2,8,9,3,10,4,11,5),\ G_8(0,10,2,9,1,11,7,4,6,3,8,5)\},\\
\Delta_9 &= \{G_9(0,6,1,7,2,8,3,9,10,4,11,5),\ G_9(0,8,1,9,2,6,3,7,11,4,10,5)\},\\
\Delta_{10} &= \{G_{10}(0,1,2,3,4,5,6,7,8,9,10,11),\ G_{10}(0,1,2,3,4,5,9,10,11,6,7,8)\}.
\end{aligned}$$

For $i \in [5,10]$, a G_i-decomposition of $K_{6,6}$ consists of the G_i-blocks in Δ_i.

Decompositions of $K_{9,6}$

Let $V(K_{9,6}) = [0,8] \cup [9,14]$ with the implied vertex partition and let

$$\begin{aligned}
\Delta_5 = \{&G_5(0,9,1,10,2,11,3,12,4,13,5,14), G_5(0,10,3,9,4,14,6,13,7,11,8,12),\\
&G_5(1,13,5,12,2,9,6,10,7,14,8,11)\},\\
\Delta_6 = \{&G_6(9,0,10,1,11,2,3,12,4,5,13,14), G_6(9,4,11,3,10,5,6,12,7,8,13,14),\\
&G_6(9,7,10,6,11,8,1,12,2,0,13,14)\},\\
\Delta_7 = \{&G_7(0,9,1,10,2,11,12,3,13,4,14,5), G_7(0,10,6,9,2,13,14,3,11,7,12,8),\\
&G_7(1,11,8,10,7,14,12,4,9,5,13,6)\},\\
\Delta_8 = \{&G_8(0,9,1,10,2,11,12,3,13,4,14,5), G_8(0,10,3,11,4,13,14,6,12,7,9,8),\\
&G_8(1,12,2,9,5,14,11,7,13,8,10,6)\},\\
\Delta_9 = \{&G_9(0,9,1,10,2,11,3,12,13,4,14,5), G_9(0,11,1,12,6,9,3,10,14,7,13,8),\\
&G_9(2,9,5,10,6,11,4,12,13,7,14,8)\},\\
\Delta_{10} = \{&G_{10}(0,1,2,3,4,5,9,10,11,12,13,14),\\
&G_{10}(6,7,8,0,1,2,9,10,11,12,13,14),\\
&G_{10}(3,4,5,6,7,8,9,10,11,12,13,14)\}.
\end{aligned}$$

For $i \in [5,10]$, a G_i-decomposition of $K_{9,6}$ consists of the G_i-blocks in Δ_i.

Bispectrum for order 12

We now have the starter decompositions needed to establish the bispectrum for graphs in $\mathcal{F}_{12}$.

Theorem 3.2. *The bispectrum of each $G \in \mathcal{F}_{12}$ is*

$$\big\{(m,n) : m,n \geq 6;\ m \equiv n \equiv 0 \pmod 3;\ mn \equiv 0 \pmod 2\big\}.$$

Proof. To establish necessity, suppose there is a $(K_{m,n}, G)$-design for $G \in \mathcal{F}_{12}$. The "local condition" is that the vertex degrees of $K_{m,n}$ must be multiples of the vertex degrees of G, so $m \equiv n \equiv 0 \pmod 3$. The "global conditions" are that the size of $K_{m,n}$ must be a multiple of the size of G and that the partite sets of $K_{m,n}$ must be large enough to accommodate the partite sets of G. This requires $18 \mid mn$ and $m, n \geq 6$. These conditions consolidate to restricting (m,n) to the set $\big\{(m,n) : m,n \geq 6;\ m \equiv n \equiv 0 \pmod 3;\ mn \equiv 0 \pmod 2\big\}$.

Now we prove sufficiency. Without loss of generality, assume n is even, so $n \equiv 0 \pmod 6$.

First suppose m is even, so $m \equiv 0 \pmod 6$, too. Then $m = 6x$, $n = 6y$ where x, y are positive integers, and xy edge-disjoint copies of the $(K_{6,6}, G)$-design produce a $(K_{6x,6y}, G)$-design.

Now suppose m is odd, so $m \equiv 3 \pmod 6$. Then $m = 6x+3$, $n = 6y$ where x, y are positive integers. Then y edge-disjoint copies of the $(K_{9,6}, G)$-design and $(x-1)y$ edge-disjoint copies of the $(K_{6,6}, G)$-design produce a $(K_{6x+3,6y}, G)$-design. $\square$

For $t \geq 2$, a complete t-partite graph K decomposes into $\binom{t}{2}$ edge-disjoint complete bipartite graphs with partite sets that coincide with pairs of partite sets of K. Hence Theorem 3.2 implies the following corollary.

Corollary 3.3. *Let $G \in \mathcal{F}_{12}$ and let K be a complete multipartite graph. If the order of each partite set of K is a multiple of 3 that is at least 6 and if at most one of the partite sets has odd order, then K admits a G-decomposition.*

4 Decompositions of complete graphs

For any $G \in \mathcal{F}_{12}$ the existence of a (K_n, G)-design requires the "local condition" (i) that the vertex degrees of K_n must be a multiple of the vertex degrees of G, along with the "global conditions" (ii) that the size of K_n must be a multiple of the size of G and (iii) that the order of K_n must be at least the order of G. These conditions respectively require that $3 \mid n-1$, $18 \mid \binom{n}{2}$ and $n \geq 12$. Then $9 \mid n-1$ and either $4 \mid n$ or $4 \mid n-1$, which consolidate to

$$\{n \geq 28 : n \equiv 1, 28 \pmod{36}\}.$$

In 2012 two of the present authors [19] proved the following.

Theorem 4.1. *For each $G \in \mathcal{F}_{12}$ there is a (K_n, G)-design whenever $n \equiv 1 \pmod{36}$, $n \geq 37$.*

It remains to settle the case for $n \equiv 28 \pmod{36}$, $n \geq 28$. As in the previous section, we first determine starter decompositions.

Decompositions of K_{28}

Let K_{28} be the complete graph on the vertex set $V = \mathbb{Z}_7 \times \mathbb{Z}_4$. For brevity we shall write r_s for $(r, s) \in \mathbb{Z}_7 \times \mathbb{Z}_4$.

$$\begin{aligned}
\Delta_5 = \{&G_5(0_0, 1_0, 3_0, 6_0, 0_1, 2_0, 1_1, 3_1, 2_1, 0_2, 3_2, 4_1),\\
&G_5(0_0, 2_1, 3_2, 4_0, 0_2, 2_0, 6_2, 2_2, 0_3, 1_3, 2_3, 5_3),\\
&G_5(0_0, 4_3, 4_2, 0_1, 3_2, 0_3, 4_1, 6_2, 1_3, 6_3, 5_1, 2_3)\},\\
\Delta_6 = \{&G_6(0_0, 1_0, 3_0, 6_0, 0_1, 2_1, 3_1, 4_0, 1_1, 4_1, 0_2, 2_2),\\
&G_6(0_0, 0_2, 1_0, 3_2, 5_0, 0_3, 4_2, 2_1, 2_3, 1_3, 6_1, 1_2),\\
&G_6(0_0, 4_3, 6_1, 3_3, 2_3, 5_3, 6_3, 1_2, 0_1, 5_2, 1_3, 3_2)\},\\
\Delta_7 = \{&G_7(0_0, 1_0, 3_0, 6_0, 0_1, 2_1, 3_1, 6_1, 0_2, 5_1, 3_2, 2_2),\\
&G_7(0_0, 0_1, 3_0, 1_2, 1_0, 2_2, 3_2, 6_1, 0_3, 2_0, 1_3, 0_2),\\
&G_7(0_0, 1_3, 1_1, 0_3, 0_2, 2_3, 3_3, 4_3, 6_3, 3_1, 5_3, 6_2)\},\\
\Delta_8 = \{&G_8(0_0, 1_0, 3_0, 6_0, 0_1, 2_1, 3_1, 1_1, 0_2, 4_1, 1_2, 5_2),\\
&G_8(0_0, 4_1, 5_0, 0_2, 1_0, 1_2, 3_2, 0_1, 0_3, 4_2, 1_3, 2_3),\\
&G_8(0_0, 5_2, 0_1, 2_3, 3_0, 4_3, 3_3, 5_3, 6_3, 5_1, 1_3, 1_2)\},
\end{aligned}$$

$$\begin{aligned}\Delta_9 = \{&G_9(0_0,1_0,3_0,6_0,0_1,2_0,1_1,2_1,3_1,0_2,1_2,2_2),\\ &G_9(0_0,0_1,3_0,0_2,4_0,6_2,1_1,0_3,1_3,3_1,6_3,1_2),\\ &G_9(0_0,4_3,0_1,2_2,5_2,2_3,4_2,5_3,6_3,6_1,1_3,3_3)\},\\ \Delta_{10} = \{&G_{10}(0_0,1_0,2_0,4_0,5_0,6_0,3_0,0_1,3_1,0_2,3_2,0_3),\\ &G_{10}(0_0,0_1,2_1,1_1,3_1,1_2,4_1,0_2,0_3,2_2,4_3,5_3),\\ &G_{10}(0_0,6_2,0_3,3_1,0_2,1_2,4_3,5_3,6_3,4_1,5_2,2_3)\}.\end{aligned}$$

The bijection $\theta\colon V \to V$ such that $\theta(r,s) = (r+1,s)$ is a period-7 automorphism on K_{28}. Each of the starter sets Δ_i, $i \in [5,10]$, has three edge-disjoint copies of $G_i \in \mathcal{F}_{12}$ with the property that the set

$$\Delta_i^* = \bigcup_{t\in\mathbb{Z}_7} \theta^t(\Delta_i)$$

comprises 21 edge-disjoint copies of G_i and is a (K_{28}, G_i)-design.

Spectrum for order 12

The starter decompositions let us now settle the spectrum for graphs in $\mathcal{F}_{12}$.

Theorem 4.2. *For each $G \in \mathcal{F}_{12}$ there is a (K_n, G)-design whenever $n \equiv 28 \pmod{36}$, $n \geq 28$.*

Proof. When $G = G_i$ and $n = 28$, a suitable decomposition is provided by Δ_i^*. Now suppose $n = 36x + 28$ for some integer $x \geq 1$. Partition the vertex set of K_{36x+28} into a singleton V_∞, a subset V_0 of cardinality 27, and x subsets V_j, $j \in [1,x]$ each of cardinality 36. Evidently K_{36x+28} has a decomposition into a complete subgraph $H_0 \cong K_{28}$ on the vertex set $V_0 \cup V_\infty$, a complete subgraph $H_j \cong K_{37}$ on the vertex set $V_j \cup V_\infty$, for each $j \in [1,x]$, and a complete multipartite graph $H_\infty \cong K_{27,36,\ldots,36}$ with partite sets V_j, $j \in [0,x]$.

Now Δ_i^* is a (H_0, G_i)-design. By Theorem 4.1, there are (H_j, G_i)-designs for each $j \in [1,x]$. Finally, Corollary 3.3 ensures the existence of an (H_∞, G_i)-design. Together these constitute a (K_{36x+28}, G)-design. □

Combining Theorems 4.1 and 4.2 with the necessary conditions established at the beginning of this section, we have the following spectrum.

Theorem 4.3. *The spectrum of each $G \in \mathcal{F}_{12}$ is*

$$\{n \geq 28 : n \equiv 1, 28 \pmod{36}\}.$$

References

[1] P. Adams and D. Bryant, The spectrum problem for the Petersen graph, *J. Graph Theory* **22** (1996), 175–180.

[2] P. Adams, D. Bryant and M. Buchanan, A survey on the existence of G-designs, *J. Combin. Des.* **16** (2008), 373–410.

[3] P. Adams, C. Chan, S. I. El-Zanati, E. Holdaway, U. Odabaşı and J. Ward, The Spectrum Problem for 3 of the Cubic Graphs of Order 10, *J. Combin. Math. Combin. Comput.* **110** (2019), 89–101.

[4] P. Adams, S. I. El-Zanati, U. Odabaşı and W. Wannasit, The Spectrum Problem for the Connected Cubic Graphs of Order 10, *Discuss. Math. Graph Theory*, to appear.

[5] P. Adams, S. I. El-Zanati and W. Wannasit, The spectrum problem for the cubic graphs of order 8, *Ars Combin.* **137** (2018), 345–354.

[6] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory, Ser. B* **81** (2001), 77–99.

[7] D. Bryant and S. El-Zanati, Graph decompositions, in: *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz (Editors), 2nd ed., Chapman & Hall/CRC, Boca Raton, 2007, pp. 477–485.

[8] D. E. Bryant, S. I. El-Zanati and R. B. Gardner, Decompositions of $K_{m,n}$ and K_n into cubes, *Australas. J. Combin.* **9** (1994), 285–290.

[9] D. E. Bryant and T. A. McCourt, Existence results for G-designs, http://g-designs.smp.uq.edu.au/

[10] J. E. Carter, Designs on Cubic Multigraphs, Ph.D. thesis, McMaster University, Canada (1989).

[11] R. K. Guy and L. W. Beineke, The coarseness of the complete graph, *Can. J. Math.* **20** (1968), 888–894.

[12] H. Hanani, The existence and construction of balanced incomplete block designs, *Ann. Math. Statist.* **32** (1961), 361–386.

[13] T. P. Kirkman, On a Problem in Combinatorics, *Cambridge Dublin Math. J.* **2** (1847), 191–204.

[14] M. Maheo, Strongly graceful graphs, *Discrete Math.* **29** (1980), 39–46.

[15] M. Meszka, R. Nedela, A. Rosa and M. Skoviera, Decompositions of complete graphs into circulants, *Discrete Math.* **339** (2016), 2471–2480.

[16] R. C. Read and R. J. Wilson, *An Atlas of Graphs*, Oxford University Press, Oxford, 1998.

[17] M. Šajna, Cycle decompositions III: complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27–78.

[18] N. J. A Sloane, The On-Line Encyclopedia of Integer Sequences, http://oeis.org/A002851

[19] W. Wannasit and S. I. El-Zanati, On free α-labelings of cubic bipartite graphs, *J. Combin. Math. Combin. Comput.* **82** (2012), 269–293.

[20] W. S. B. Woolhouse, Prize Question 1733, *Lady's and Gentleman's Diary* (1844), 63–66.