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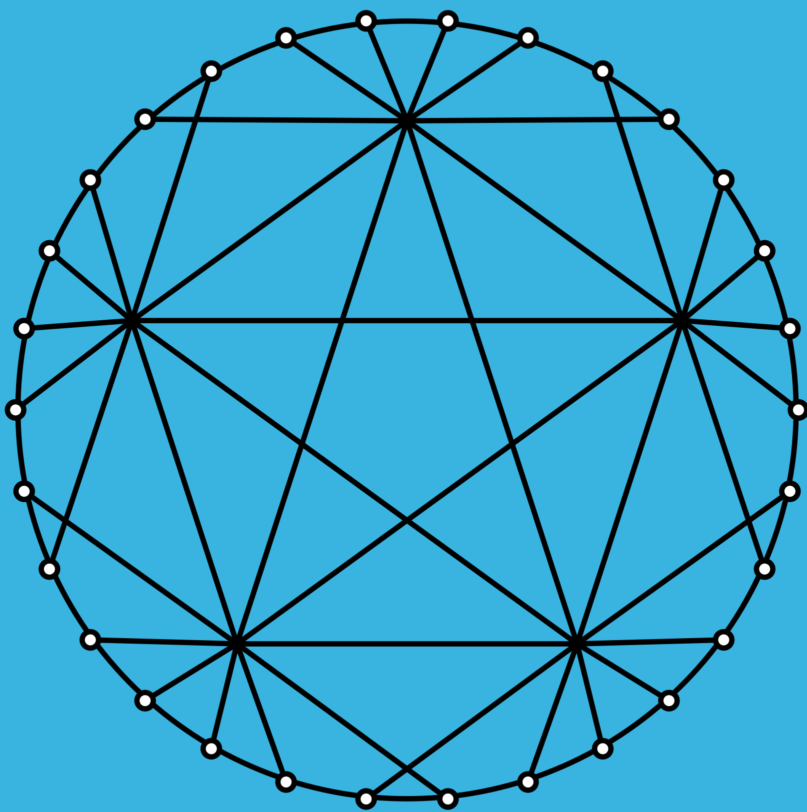

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Chaotic and euclidean rhythms

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Abstract

The article shows a dichotomy between even rhythms and chaotic rhythms from a mathematical point of view. The main purpose of this article is to acquaint the reader with some mathematical structures inside rhythm; it should help music composers and performers to make them aware of the idea that mathematics can simplify complex visions.

1 Introduction

It is known that rhythms, built in such a way that their accents are distributed over pulses in the most possible uniform way, cover the greatest part of rhythms in Western music and they are related to the concepts of order and periodicity [6]; now the next developing point would be that of creating a geometric structure modelling chaos. To reach that purpose, we propose the identification of the perfect mathematical structure - which is a difference set - with a chaotic rhythm. We provide some examples from music literature in which we are sure that music composers were actually looking for patterns that give the idea of chaos and disorder.

A composer could be more conscious about choosing an appropriate rhythmical pattern - or a specific subset of notes - according to his musical ideas if he knows which mathematical structure can be used. On the other hand, a performer could recognize mathematical aspects mainly in rhythms, changing his interpretation's view according to the rhythm's features.

Key words and phrases: Rhythm, chaotic rhythm, music, Erdös, Paley difference set

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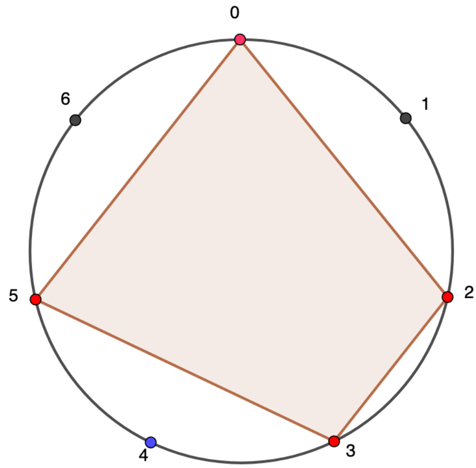

Figure 1: The rhythm $R = \{0, 2, 3, 5\}_7$ where pulses are regularly set on the circumference of a circle clockwise, and beats are red colored.

2 Euclidean rhythms

A rhythm is a set of *accents* or beats, selected from a set of pulses. The pulses are evenly distributed over time in the greatest part of cases. To this end, addictive groups $\mathbb{Z}_n$ can describe this set of rhythmic pulses. Let us consider for instance $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$, then we can make a selection of some of the elements of this group and get a rhythm, called R; a concrete example is the following (Figure 1):

$$R = \{0, 2, 3, 5\}_7.$$

We use $R(n, k)$ to denote a rhythm of k beats selected from n regularly spaced pulses. We write $R = \{a_1, a_2, ..., a_k\}_n$ to denote the rhythm for which the beats occur at the time pulses $a_1, a_2, ..., a_k$. It is known that the greatest part of rhythms inside the *World music* are the ones built such that they are distributed in the most possible uniform way as shown in [5] [6]. Obviously uniform distributions are easy to perform if the number n is a multiple of k. We get a more complex situation when we try to distribute k accents on n pulses when n and k are relative prime numbers. The problem can be seen also from a pure geometric point of view: we would like to maximize the area of a polygon inscribed on the circumference of a circle where its vertices are the beats of the rhythm selected from the n distributed uniformly on the circumference of a circle. To this end, it is possible to get such a rhythm using some interesting algorithms. Let us start by considering the euclidean algorithm. This algorithm, purely recursive, is simplified as follows. The reader may think of k elements as ones and the remaining $n-k$ ones as zeroes. As an example, we set $n=13$ and $k=5$, and begin with the string [1111100000000]. We distribute the eight zeroes on the right of the five ones so that the lengths of the substrings of zeros differ at most 1. Thus, the substrings have lenghts 2,2,2,1,1 and we insert them in decreasing size. The resulting string is [100] [100] [100] [10] [10] and the two new elements are [100] and [10].

We insert the two [10] elements as equally as possible to the right of the three [100] elements yielding the string [[100][10]] [[100][10]] [100]. We now have the two elements [[100][10]] and [100]. The algorithm concludes by inserting the single element [100] to the right of the first [100][10] yielding the string [1001010010010].

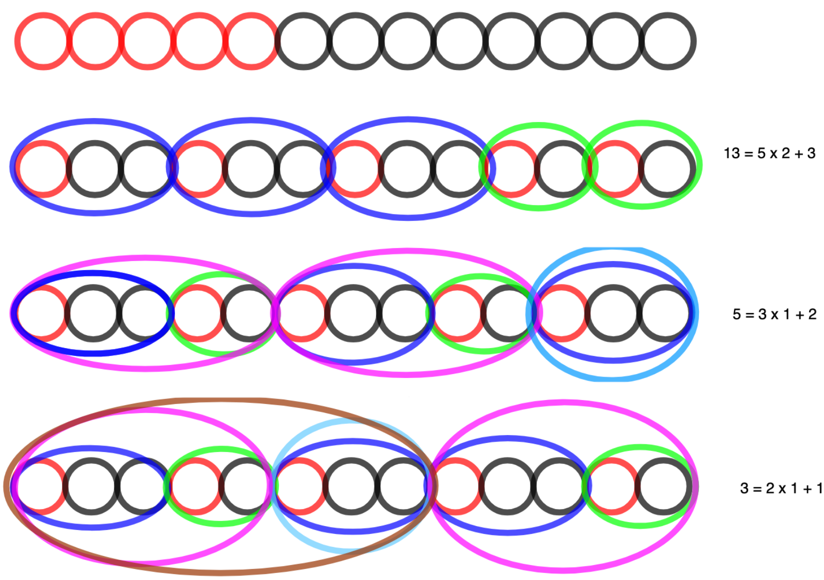

Figure 2: The euclidean algorithm for $n=13$ and $k=5$.

A euclidean rhythm is a rhythm obtained with the euclidean algorithm described above, or a rotation of it. A rotation of a rhythm $R=\{a_1, a_2, ..., a_k\}_n$ is

$$R_1 = \{a_1 + r, a_2 + r, ..., a_k + r\}_n$$

where $r \in \mathbb{Z}_n$. In a euclidean rhythm, it can be proved that the polygon obtained considering its beats as the vertices on the circumference of a circle is one of the polygons with maximal area possible, where n and k are fixed.

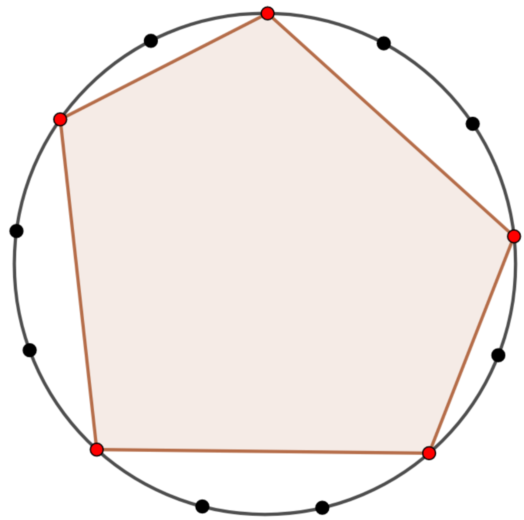

Figure 3: The euclidean rhythm $R = \{0, 3, 5, 8, 11\}_{13}$ described as the string [1001010010010]. The area of the polygon is the greatest area possible for a polygon with 5 vertices, distributed in 13 possible places on the circumference of a circle. Equivalently the sum of all $\binom{k}{2} = \binom{5}{2}$ possible chordal distances between vertices is maximal, up to rotation.

Godfried Toussaint's work (see [6]) states clearly that euclidean rhythms are the most common rhythmic pattern of the *World music*, providing an exhaustive number of examples.

3 Erdös-deep rhythms

If we place n points uniformly around the circumference of a circle and cyclically label them clockwise with the elements of $\mathbb{Z}_n$, then for two points $a_i, a_j \in \mathbb{Z}_n$, we define the *chordal distance* between them to be the value of $\{a_i - a_j, a_j - a_i\}$ belonging to $\{0, 1, 2, ..., \frac{n}{2}\}$. We use $\delta(a_i, a_j)$ to denote the chordal distance between a_i and a_j.

Let R be a rhythm with n pulses and k beats. We represent the n pulses by uniformly distributing the elements of $\mathbb{Z}_n$ on the circumference of a circle in clockwise order $0, 1, ..., n-1$. The k beats are represented as a k-element subset $\{a_1, a_2, ..., a_k\}$ of $\mathbb{Z}_n$. For each pair of elements a_i, a_j in R, we know that $\delta(a_i, a_j) \in \mathbb{Z}_n$. The *multiplicity* of a chordal distance δ in R is the number of distinct pairs in R whose chordal distance is δ.

A rhythm $R(n, k)$ is called an Erdös deep rhythm if for every multiplicity $1, 2, ..., k-1$ there exists a chordal distance with that multiplicity. For instance, the reader may check the rhythm $E = \{0, 2, 3, 4\}_6$: there is a couple of points at chordal distance 3, two couples at chordal distance 1 and three couples at chordal distance 2. More details are present in [5].

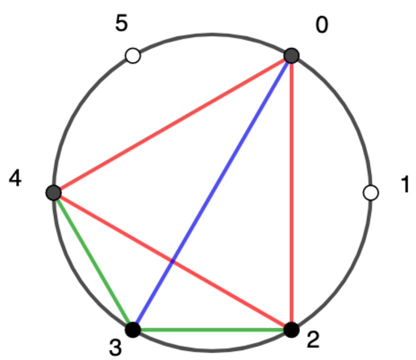

Figure 4: The rhythm $E = \{0, 2, 3, 4\}_6$ represented on the circumference of a circle. One may check that for all numbers from $1, 2, 3 = 4 - 1 = k - 1$ there exists a chordal distance with that exact multiplicity.

It is possible to characterize Erdös-deep rhythms. A theorem states that a rhythm $R(n,k)$ is Erdös-deep if and only if $E = \{0,2,3,4\}_6$ or a rotation of

$$D_{n,k,m} = \{im \mod n \mid i = 0, ..., k-1\}_n,$$

where $k \leq \lfloor \frac{n}{2} \rfloor$, $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$, and m and n are relatively prime, [1]. For instance, let us take the rhythm $R = D_{12,5,5}$ which means that we have $n = 12$ pulses, $k = 5$ beats, and parameter $m = 5$ is relatively prime with 12. The rhythm D becomes: $D_{12,5,5} = \{5 \cdot i \mod 12 \mid i = 0,1,2,3,4\} = \{0,5,10,3,8\}_{12}$.

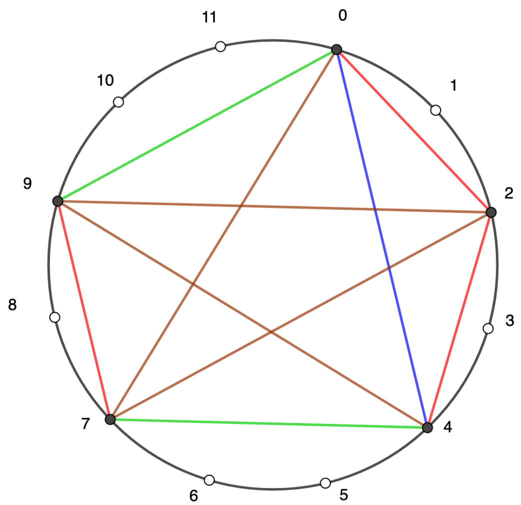

Figure 5: A rotation of the rhythm $D_{12,5,5}$: for all multiplicities 1,2,3,4, there exists a chordal distance 2,3,4,5 with a specific multiplicity. This is euclidean too: the reader may check it, exploiting the euclidean algorithm and starting from the string [111110000000].

The example of Figure 5 opens a new perspective in music because it defines not only a rhythm, but it could represent a major scale in a tempered system with 12 semitones. The reader could just consider white and black keys of the piano in order to recognize the pattern of Figure 5. Therefore we can say that a major scale is identified with a specific Erdös-deep rhythm.

There is a connection between Erdös-deep rhythms and euclidean ones: a rhythm $R_{n,k}$ with its beats distributed over the pulses in the most possible uniform way is Erdös-deep if and only if n and k are relatively prime.

4 Chaotic rhythms

We ask ourselves what happens if we change the condition on the multiplicity of chordal distances of a rhythm, changing it in a sort of dual perspective. While an Erdös-deep rhythm has a specific chordal distance for all multiplicities, now we will study a family of rhythms in which every distinct chordal distance has a same multiplicity λ. To this purpose, let us give two useful definitions.

1) Let G be a group. A subset S of G is called a *difference set* if the multiset
$$\{gh^{-1} \ : \ g, h \in S\}$$
contains every non-identity element of G precisely λ times.

2) An *almost difference set* has the same definition as a difference set, with a difference that some differences have multiplicity λ and some others have multiplicity $\lambda + 1$; more details on this definition in [3]. A rhythm generated by a difference set up to rotation, or its complement, is said to be *chaotic rhythm*.

Some famous difference sets are the so-called *Paley difference sets* [2]. A Paley difference set is a set on a Galois field $\mathbb{F}_q$:
$$P = \{x^2 | x \in \mathbb{F}_q \setminus \{0\}\} \quad q \equiv 3 \pmod 4$$

A method to generate a Paley difference set is the following: consider triangular numbers modulo a prime number q such that $q \equiv 3 \pmod 4$; the set of elements calculated is the complement of the Paley difference set with the same parameters. On the contrary, if $q \equiv 1 \pmod 4$, we get the complement of an almost difference set. Summarizing let us consider $R_n = \{S_j \mid j = 0, ..., n\}$, where

$$S_j = \begin{cases} \sum_{i=1}^{j} i \pmod n, & \text{if } j \in [1, n] \\ 0, & \text{if } j = 0 \end{cases} .$$

Then n is an odd prime number if and only if $|R_n| = \lceil \frac{n}{2} \rceil$. The reader should know that counting triangular numbers is related to counting squares modulo n. To count squares modulo n, one may read [4]. The rhythm is a

$(n, \frac{n+1}{2}, \lambda)$ Paley difference set or a $(n, \frac{n+1}{2}, \lambda, \frac{n-1}{4})$ almost Paley difference set - the last parameter indicates how many differences have multiplicity λ. Consider $F = F(7, 3, 2)$: triangular numbers modulo 7 are 0,1,3,6. The reader can check that the complement - the set with elements 2,4,5 - is a Paley difference set.

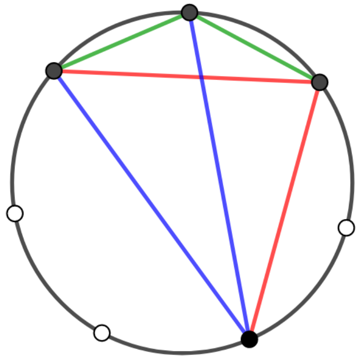

Figure 6: The chaotic rhythm with $n = 7$, $k = 4$, $\lambda = 2$. The multiplicity is $\lambda = 2$ for all chordal distances: a sort of "dual definition" compared with Erdös-deep rhythms.

We provide here concrete examples from music literature: in the first bar of the third movement - *Precipitato* - of the Sonata n.7 opus 83 by Sergey Prokofiev there are two complementary chaotic rhythms. Indeed, the two hands of the pianist have to perform two Paley difference sets: we can highlight a correspondence between the eight notes of the score of Figure 7 and the points on the circumference - the ones of Figure 6 - moving counterclockwise. What the audience perceives is a rhythmic pattern where beats are difficult to be foreseen: this perfectly combines with the purpose of Prokofiev's music who wanted to recreate the violent bombing on the city of St. Petersburg during the battle of Stalingrad (1942-1943). Almost difference sets are also used by György Ligeti in his famous Etudes for piano. On the contrary euclidean rhythms are the most common rhythms of music, from classical music to *World music* - the reader can just check the table of Greek ancient rhythms discovering that the greatest part of them are euclidean up to rotation. In Toussaint's article, there is a huge list of euclidean rhythms from *World music*.

Figure 7: S. Prokofiev, Piano Sonata n. 7 op.83, Precipitato - first bar.

To conclude, we have studied two opposite classes of rhythms: the Erdös-deep rhythms which are connected to a concept of order, of recursion; and the Difference set rhythms which are connected to the concept of chaos and disorder, having opposite geometric properties from the other class. These concepts of order and disorder are defined from a human perception point of view. Nowadays we still do not know why human perception tends to connect Erdös deep rhythms with the concept of order, and difference set rhythms with the concept of chaos.

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