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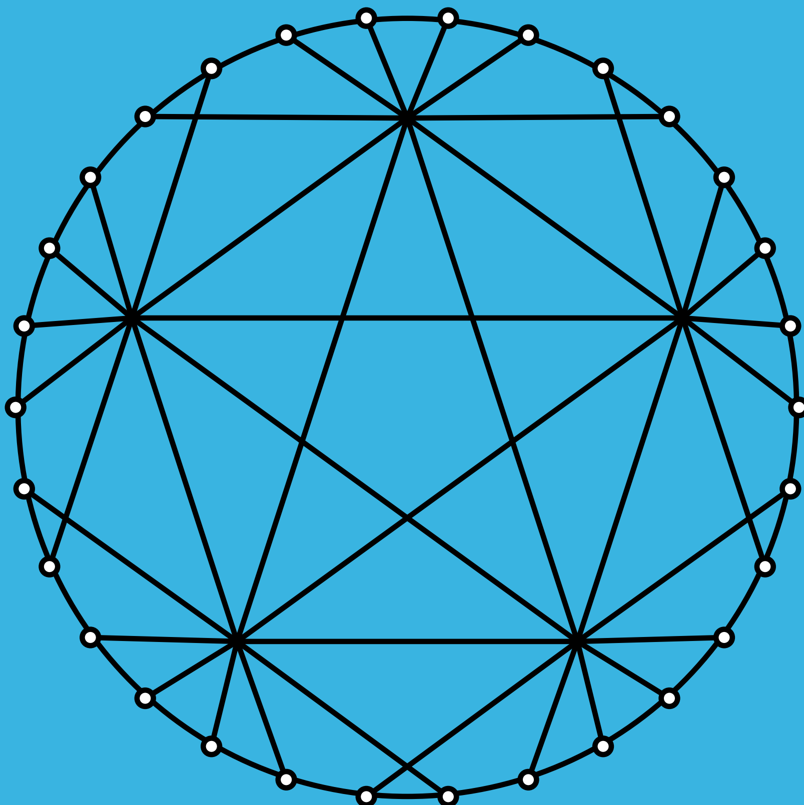

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On the Steiner Quadruple System with Ten Points

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Abstract

The Steiner quadruple system on ten points, SQS(10), is constructed with points corresponding to the ten triangle factors of the complete graph on six vertices. This construction shows that the Tutte-Coxeter graph is obtained from the SQS(10) by taking the blocks as vertices, and joining two blocks by an edge when they are disjoint.

1 Introduction

An $S(t, k, v)$ *Steiner system* consists of a set V of v *points*, and a set of k-subsets of V, called *blocks*, such that each t-subset of V is contained in exactly one block. An $S(3, 4, v)$ Steiner system is called a *Steiner quadruple system* of order v and denoted SQS(v). Hanani showed that there exists an SQS(v) if and only if $v \equiv 2$ or $4 \pmod 6$ [8], see [5, 9]. Up to isomorphism there is a unique SQS(8) and a unique SQS(10) [1], see [9]. The SQS(10) has thirty blocks.

The *Cremona-Richmond configuration* (shown below), also known as the *generalised quadrangle* GQ(2, 2), is constructed by taking the fifteen edges

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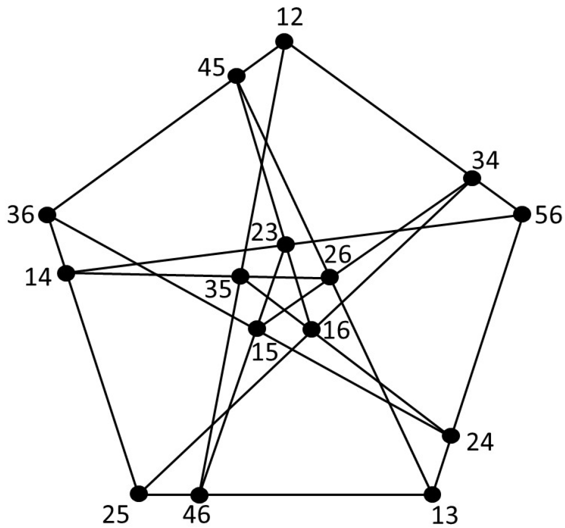
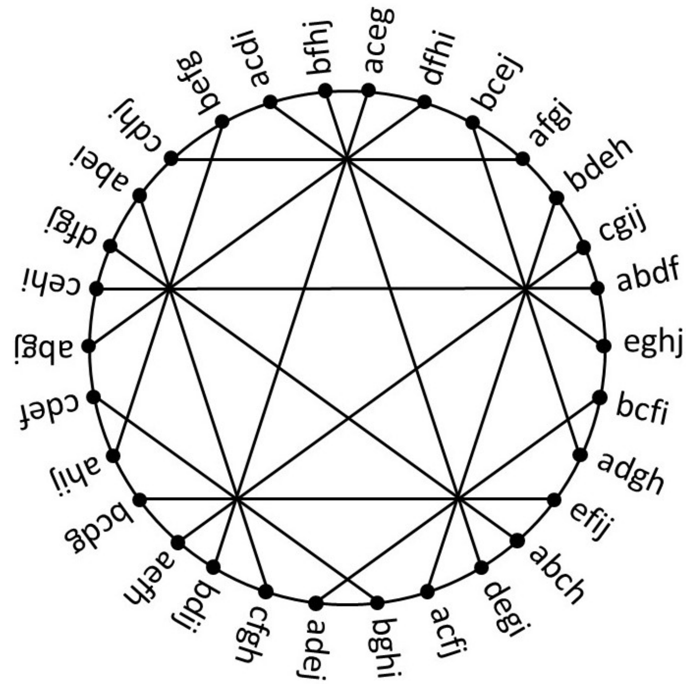
of the complete graph K_6 as points and the fifteen 1-factors of K_6 as lines – a point is on a line when the point's corresponding edge is in the line's corresponding 1-factor. In the Cremona-Richmond configuration there are three points on each line, three lines through each point, at most one line through any two points, and there are no triangles.

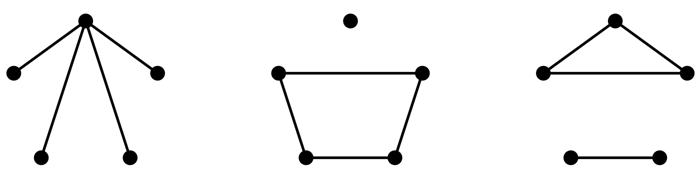


The *Tutte-Coxeter graph*, also known as *Tutte's* 8*-cage*, is the incidence graph of the Cremona-Richmond configuration. Thus, it has a vertex for each point of the configuration, a vertex for each line of the configuration, and two vertices are adjacent when they correspond with a point and an incident line. It is a bipartite cubic graph with thirty vertices, and is the smallest cubic graph with girth 8 [10], see Section 4.7 in [7].

Here, an SQS(10) is constructed where the points are the ten triangle factors of the complete graph K_6, and the blocks are defined as follows. For each edge of K_6, there is a block consisting of the four triangle factors that contain the edge, and for each 1-factor of K_6, there is a block consisting of the four triangle factors that avoid all the edges of the 1-factor. An immediate consequence of this construction is that the Tutte-Coxeter graph can be constructed from the SQS(10) by taking the blocks as vertices, and joining two blocks by an edge when they are disjoint. A drawing of the Tutte-Coxeter graph with its vertices labeled by the blocks of an SQS(10) on points $a, b, c, d, e, f, g, h, i, j$, and with disoint blocks adjacent, is shown below.



This construction of the SQS(10) somewhat resembles the construction of SQS(20)s given in [6], where the points are the twenty triangles of K_6. It is also worth mentioning the construction of the SQS(10) where the points are the edges of K_5, and the blocks are the edge sets of any subgraph that is isomorphic to any of the three shown below, see [4] or Example 6.1 of Chapter VIII in [2].



In Section 2 the construction of the SQS(10) with the triangle factors of K_6 as points is given in detail. Some further discussion on block intersections, related designs and configurations, and automorphisms is presented in Section 3.

2 Construction of the SQS(10)

A graph consisting of three pairwise adjacent vertices x, y and z will be called a *triangle*, and denoted by xyz. A *k-factor* of a graph is a spanning k-regular subgraph, and a 2-factor in which each component is a triangle is called a *triangle factor*. The complete graph K_6 has ten triangle factors. The vertex set of K_6 will be $\{1,2,3,4,5,6\}$, and $\mathcal{T}$ is defined to be the set of ten triangle factors of K_6:

$$\begin{array}{lll}\mathcal{T} = & \{ & \{123,456\},\ \{124,356\},\ \{125,346\},\ \{126,345\},\ \{134,256\}, \\ & & \{135,246\},\ \{136,245\},\ \{145,236\},\ \{146,235\},\ \{156,234\} \quad \}.\end{array}$$

An SQS(10) with point set $\mathcal{T}$ will be constructed. Define

$$\begin{array}{llll} a=\{123,456\}, & b=\{156,234\}, & c=\{126,345\}, & d=\{145,236\}, \\ e=\{125,346\}, & f=\{146,235\}, & g=\{124,356\}, & h=\{135,246\}, \\ i=\{136,245\}, & j=\{134,256\} & & \end{array}$$

so that $\mathcal{T}=\{a,b,c,d,e,f,g,h,i,j\}$. A block $\{\alpha,\beta,\gamma,\delta\}\subseteq\mathcal{T}$ will be denoted by just $\alpha\beta\gamma\delta$.

An edge joining x and y is denoted by xy, and for each edge xy of K_6, B_{xy} is defined to be the block consisting of the four triangle factors of K_6 that contain the edge xy. For example,

$$B_{34}=\{\ \{134,256\},\{156,234\},\{126,345\},\{125,346\}\ \}=bcej.$$

Let $\mathcal{F}$ be the set of fifteen 1-factors of K_6. A 1-factor of K_6 with edges uv, wx and yz will be denoted by $\{uv,wx,yz\}$. For each 1-factor $F\in\mathcal{F}$, let B_F be the set consisting of the four triangle factors of K_6 that contain none of the edges of F. For example,

$$B_{\{14,26,35\}}=\{\ \{123,456\},\{125,346\},\{136,245\},\{156,234\}\ \}=abei.$$

Define $\mathcal{B}_{\mathcal{E}}=\{B_{xy}\mid xy\in E(K_6)\}$ and $\mathcal{B}_{\mathcal{F}}=\{B_F\mid F\in\mathcal{F}\}$. The blocks of $B_{\mathcal{E}}$ and $B_{\mathcal{F}}$ are listed below. They form an SQS(10) with point set $\mathcal{T}$.

		$B_{\mathcal{E}}$					$B_{\mathcal{F}}$		
$aceg$	$ahij$	$dfgj$	$bdeh$	$bcfi$	$dfhi$	$bcdg$	$abei$	$acfj$	$eghj$
$abdf$	$bghi$	$efij$	$cdhj$	$bcej$	$adej$	$cgij$	$befg$	$abch$	$bfhj$
$cfgh$	$degi$	$acdi$	$aefh$	$abgj$	$cehi$	$afgi$	$adgh$	$cdef$	$bdij$

Since an SQS(10) has thirty blocks, to show that $\mathcal{B}_{\mathcal{E}} \cup \mathcal{B}_{\mathcal{F}}$ is indeed the block set of an SQS(10) with point set $\mathcal{T}$, it is sufficient to show that any three pairwise-distinct triangle factors α, β, γ of K_6 occur in at least one block of $\mathcal{B}_{\mathcal{E}} \cup \mathcal{B}_{\mathcal{F}}$.

If there is an edge uv that is in each of α, β and γ, then $\{\alpha, \beta, \gamma\} \subseteq B_{uv} \in \mathcal{B}_{\mathcal{E}}$. So assume that there is no such edge, let $\alpha = \{uvw, xyz\}$, and without loss of generality assume that β contains the edge uv and γ contains the edge uw. Now, the third vertex of the triangle of β that contains the edge uv, and the third vertex of the triangle of γ that contains the edge uw are distinct. To see this, note that if, for example, uvx is a triangle of β and uwx is a triangle of γ, then yz is an edge in each of α, β and γ, contradicting the assumption that no edge is in each of α, β and γ. Thus, without loss of generality $\beta = \{uvx, wyz\}$ and $\gamma = \{uwy, vxz\}$. So the edges uz, vy and wx are in none of α, β or γ, and this means that $\{\alpha, \beta, \gamma\} \subseteq B_F \in \mathcal{B}_{\mathcal{F}}$ where F is the 1-factor $\{uz, vy, wx\}$. This completes the proof that $\mathcal{B}_{\mathcal{E}} \cup \mathcal{B}_{\mathcal{F}}$ is the block set of an SQS(10) with point set $\mathcal{T}$.

3 Discussion

Block Intersection Graphs:

Consider intersections of pairs of blocks of the SQS(10). The possibilities are covered by the following cases.

- $|B_{uv} \cap B_{\{uv,wx,yz\}}| = 0$. Each triangle factor in B_{uv} contains the edge uv, and each triangle factor in $B_{\{uv,wx,yz\}}$ does not contain the edge uv.
- $|B_{uw} \cap B_{\{uv,wx,yz\}}| = 2$. The only triangle factors in $B_{uw} \cap B_{\{uv,wx,yz\}}$ are $\{uwy, vxz\}$ and $\{uwz, vxy\}$.
- $|B_{uv} \cap B_{uw}| = 1$ when uv and uw are distinct adjacent edges. The only triangle factor in $B_{uv} \cap B_{uw}$ is the triangle factor containing the triangle uvw.
- $|B_{uv} \cap B_{wx}| = 2$ when uv and wx are non-adjacent edges. The only triangle factors in both B_{uv} and B_{wx} are $\{uvy, wxz\}$ and $\{uvz, wxy\}$ where $\{u, v, w, x, y, z\} = \{1, 2, 3, 4, 5, 6\}$.

- $|B_{\{uv,wx,yz\}} \cap B_{\{u'v',w'x',y'z'\}}| = 1$ when the 1-factors $\{uv, wx, yz\}$ and $\{u'v', w'x', y'z'\}$ are edge-disjoint. Without loss of generality the union of these two 1-factors is the 6-cycle (u, v, w, x, y, z). So $\{uwy, vxz\}$, being the only triangle factor that contains no edges of the 6-cycle (u, v, w, x, y, z), is the only triangle factor in $B_{\{uv,wx,yz\}} \cap B_{\{u'v',w'x',y'z'\}}$.
- $|B_{\{uv,wx,yz\}} \cap B_{\{uv,w'x',y'z'\}}| = 2$ when the 1-factors $\{uv, wx, yz\}$ and $\{uv, w'x', y'z'\}$ are distinct and share a common edge. Without loss of generality the union of these two 1-factors consists of the edge uv and the 4-cycle (w, x, y, z). So $\{uwy, vxz\}$ and $\{uxz, vwy\}$, being the only triangle factors that do not contain the edge uv nor any edges of the 4-cycle (w, x, y, z), are the only triangle factors in $B_{\{uv,wx,yz\}} \cap B_{\{u'v',w'x',y'z'\}}$.

It follows from this characterisation of block intersections that pairs of disjoint blocks of the SQS(10) correspond precisely with occurrences of edges of K_6 in 1-factors of K_6. Thus, the graph obtained from the SQS(10) by taking the blocks as vertices and joining disjoint blocks by an edge is the Tutte-Coxeter graph. Further, a block from $\mathcal{B}_\mathcal{E}$ and a block from $\mathcal{B}_\mathcal{F}$ intersect in either zero or two points, and distinct blocks both from $\mathcal{B}_\mathcal{E}$, or both from $\mathcal{B}_\mathcal{F}$, intersect in either one or two points.

The pairs of blocks from $\mathcal{B}_\mathcal{E}$ intersecting in exactly one point correspond exactly with pairs of adjacent edges of K_6, and pairs of blocks from $\mathcal{B}_\mathcal{F}$ intersecting in exactly one point correspond exactly with pairs of edge-disjoint 1-factors of K_6. Thus, the graph obtained from the SQS(10) by taking the blocks as vertices, and joining two blocks by an edge precisely when they intersect in exactly one point, is the union of two vertex-disjoint copies of the line graph of K_6. The fact that the blocks of $\mathcal{B}_\mathcal{F}$ also give rise to a copy of the line graph of K_6 follows from the symmetry between $\mathcal{B}_\mathcal{E}$ and $\mathcal{B}_\mathcal{F}$, which is discussed later.

Related Configurations and Designs:

The blocks of the SQS(10) correspond with the points and lines of the Cremona-Richmond configuration – with the blocks of $\mathcal{B}_\mathcal{E}$ corresponding to the points and the blocks of $\mathcal{B}_\mathcal{F}$ corresponding to the lines, or vice-versa. Consider a point α of the SQS(10), which is a triangle factor in K_6. The point α is in six blocks of $\mathcal{B}_\mathcal{E}$ and six blocks of $\mathcal{B}_\mathcal{F}$. So α corresponds to a set of six points and six lines in the Cremona-Richmond configuration.

The six points are edges in the triangle factor α, and the six lines are 1-factors that avoid the edges of the triangle factor α. Thus, α corresponds to a set of six points and six lines where none of the six points is on any of the six lines. In the Tutte-Coxeter graph, this is a set of twelve independent vertices, six from each half of the vertex bipartition.

In the Cremona-Richmond configuration, the six lines corresponding to α, together with the nine points that are incident with these lines, form a 3 by 3 grid where each point is on two lines, and where the lines partition into two sets of three pairwise-parallel lines. Thus, each such grid is the trivial generalised quadrangle GQ$(2, 1)$. The ten grids corresponding to the ten points of the SQS(10) are shown below, with the rows and columns representing the lines.

14	25	36
26	34	15
35	16	24

$a = \{123, 456\}$

12	35	46
45	26	13
36	14	25

$b = \{156, 234\}$

13	24	56
25	36	14
46	15	23

$c = \{126, 345\}$

12	34	56
46	25	13
35	16	24

$d = \{145, 236\}$

13	24	56
26	35	14
45	16	23

$e = \{125, 346\}$

12	34	56
45	16	23
36	15	24

$f = \{146, 235\}$

13	25	46
26	34	15
45	16	23

$g = \{124, 356\}$

12	34	56
36	25	14
45	16	23

$h = \{135, 246\}$

12	34	56
35	26	14
46	15	23

$i = \{136, 245\}$

12	35	46
36	24	15
45	16	23

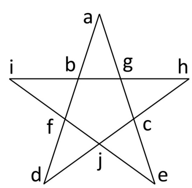
$j = \{134, 256\}$

A t-(v, k, λ)-*design* consists of a set V of v *points*, and a collection of k-subsets of V, called *blocks*, such that each t-subset of V is contained in exactly λ blocks. The SQS(10) is a 3-$(10, 4, 1)$-design. For each point α of the SQS(10), if α is removed from the twelve blocks containing α, then the resulting blocks form a 2-$(9, 3, 1)$-design, also known as a Steiner triple system of order 9, or an affine plane of order 3.

Each of $\mathcal{B}_\mathcal{E}$ and $\mathcal{B}_\mathcal{F}$ forms a 2-$(10, 4, 2)$-design on the point set $\mathcal{T}$. To see this, observe that distinct triangle factors $\alpha, \beta \in \mathcal{T}$ have exactly two common edges, one from each triangle of α, and one from each triangle of β. Thus, without loss of generality $\alpha = \{uvw, xyz\}$ and $\beta = \{uvx, wyz\}$. So B_{uv} and B_{yz} are the only blocks of $\mathcal{B}_\mathcal{E}$ that contain α and β, and $B_{\{uy,vz,wx\}}$ and $B_{\{uz,vy,wx\}}$ are the only blocks of $\mathcal{B}_\mathcal{F}$ that contain α and β. In each of these 2-$(10, 4, 2)$-designs, distinct blocks intersect in either one or two points, and the two designs are isomorphic.

Let u be a vertex of K_6. Each pair of edges incident with u is in a unique triangle factor of K_6, and the ten triangle factors given by the ten pairs of edges incident with u correspond to the ten points of the SQS(10). Thus, the configuration of five blocks of the SQS(10) corresponding to the five edges incident with u contains each point of the SQS(10) exactly twice, and any two of the five blocks intersect in exactly one point. For example, the five edges incident with vertex 2 of K_6, and the corresponding five-block configuration of the SQS(10) are shown below.

$$\begin{array}{lll}
12: & \{\{123,456\},\{124,356\},\{125,346\},\{126,345\}\} = & aceg \\
23: & \{\{123,456\},\{234,156\},\{146,235\},\{145,236\}\} = & abdf \\
24: & \{\{124,356\},\{156,234\},\{136,245\},\{135,246\}\} = & bghi \\
25: & \{\{125,346\},\{146,235\},\{136,245\},\{134,256\}\} = & efij \\
26: & \{\{126,135\},\{145,236\},\{135,246\},\{134,256\}\} = & cdhj
\end{array}$$



The set of all six vertices of K_6 thus yields a set of six copies of this five-block configuration, any two of which intersect in exactly one block, and which collectively contain each of the fifteen blocks of $\mathcal{B}_\mathcal{E}$ twice each. As described in what follows next, a similar picture arises for the blocks of $\mathcal{B}_\mathcal{F}$, with six five-block configurations in $\mathcal{B}_\mathcal{F}$ arising from the six 1-factorisations of K_6. A 1-*factorisation* of a graph is a set of pairwise edge-disjoint 1-factors that contain all the edges.

An arbitrary 1-factorisation $\mathcal{D}$ of K_6 consists of five 1-factors of K_6, each pair of these 1-factors forms a 6-cycle, which has a unique triangle factor in its complement, and each of the ten triangle factors of K_6 occurs in the complement of exactly two 1-factors of $\mathcal{D}$. To see this, let T be an arbitrary triangle factor and observe that the three edges of each triangle of T are in three distinct 1-factors of $\mathcal{D}$. Thus, there are exactly two 1-factors of $\mathcal{D}$ avoiding the triangle – this pair of 1-factors avoids every edge of T, which means that T occurs in the complement of exactly two 1-factors of $\mathcal{D}$. So the configuration of five blocks of the SQS(10) corresponding to the five 1-factors of $\mathcal{D}$ contains each point exactly twice, and any two of the five blocks intersect in exactly one point. For example, the five 1-factors of a

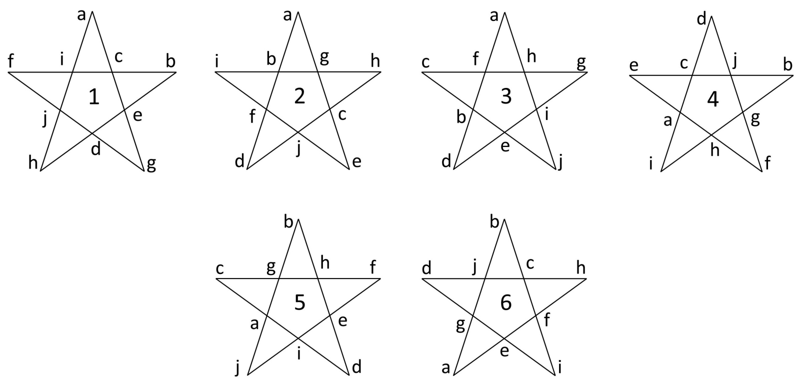
1-factorisation of K_6, and the corresponding five-block configuration from $\mathcal{B}_{\mathcal{F}}$, are given below.

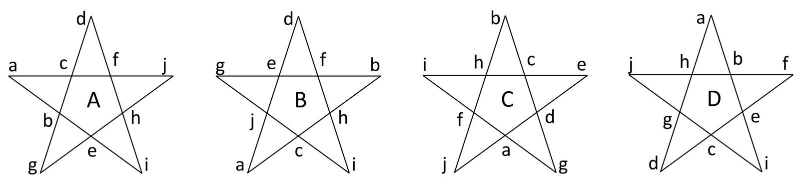
$$\begin{array}{lll}
\{12,34,56\}: & \{\{135,246\},\{136,245\},\{145,236\},\{146,235\}\} = & dfhi \\
\{13,25,46\}: & \{\{124,356\},\{126,345\},\{145,236\},\{156,234\}\} = & bcdg \\
\{14,26,35\}: & \{\{123,456\},\{125,346\},\{136,245\},\{156,234\}\} = & abei \\
\{15,24,36\}: & \{\{123,456\},\{126,345\},\{134,256\},\{146,235\}\} = & acfj \\
\{16,23,45\}: & \{\{124,356\},\{125,346\},\{134,256\},\{135,246\}\} = & eghj
\end{array}$$

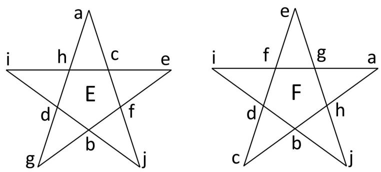
The set of all six 1-factorisations of K_6 (any two of which necessarily have exactly one 1-factor in common) thus yields a set of six copies of this five-block configuration, any two of which intersect in exactly one block, and which collectively contain each of the fifteen blocks of $\mathcal{B}_{\mathcal{F}}$ twice each.

Shown below are the six five-block configurations of $\mathcal{B}_{\mathcal{E}}$, which correspond with the six vertices 1, 2, 3, 4, 5, 6 of K_6, the six 1-factorisations of K_6, labelled A, B, C, D, E, F, and the corresponding six five-block configurations of $\mathcal{B}_{\mathcal{F}}$. The ten triangle factors are also indicated.

$$\begin{array}{llll}
a = \{123,456\}, & b = \{156,234\}, & c = \{126,345\}, & d = \{145,236\}, \\
e = \{125,346\}, & f = \{146,235\}, & g = \{124,356\}, & h = \{135,246\}, \\
i = \{136,245\}, & j = \{134,256\} & &
\end{array}$$







$\{12,34,56\}$	$\{12,34,56\}$	$\{12,36,45\}$	$\{12,36,45\}$
$\{13,25,46\}$	$\{13,26,45\}$	$\{13,25,46\}$	$\{13,24,56\}$
$\{14,26,35\}$	$\{14,25,36\}$	$\{14,23,56\}$	$\{14,26,35\}$
$\{15,24,36\}$	$\{15,23,46\}$	$\{15,26,34\}$	$\{15,23,46\}$
$\{16,23,45\}$	$\{16,24,35\}$	$\{16,24,35\}$	$\{16,25,34\}$
A	B	C	D

$\{12,35,46\}$	$\{12,35,46\}$
$\{13,26,45\}$	$\{13,24,56\}$
$\{14,23,56\}$	$\{14,25,36\}$
$\{15,24,36\}$	$\{15,26,34\}$
$\{16,25,34\}$	$\{16,23,45\}$
E	F

Observe that a 2-$(16,6,2)$-design with point set $\mathcal{T} \cup \{1,2,3,4,5,6\}$ can be obtained from $\mathcal{B}_{\mathcal{E}}$ as follows. First, add u and v to the block B_{uv} for each block $B_{uv} \in \mathcal{B}_{\mathcal{E}}$, then include a new block $\{1,2,3,4,5,6\}$. Also, for each block $B_F \in \mathcal{B}_{\mathcal{F}}$, there are exactly two 1-factorisations $\mathcal{X}_F, \mathcal{Y}_F \in \{A,B,C,D,E,F\}$ that contain the 1-factor F, and a 2-$(16,6,2)$-design with point set $\mathcal{T} \cup \{A,B,C,D,E,F\}$ can be obtained from $\mathcal{B}_{\mathcal{F}}$ by first adding $\mathcal{X}_F$ and $\mathcal{Y}_F$ to the block B_F for each block $B_F \in \mathcal{B}_{\mathcal{F}}$, and then including a new block $\{A,B,C,D,E,F\}$.

There is a well-known construction of a 2-$(21,5,1)$-design, also known as a projective plane of order 4, from the Cremona-Richmond configuration, see Chapter 8 of [3]. A 2-$(21,5,1)$-design can be obtained directly from the SQS(10) as follows. The points are the fifteen blocks of $\mathcal{B}_{\mathcal{E}}$ and the six five-block configurations of $\mathcal{B}_{\mathcal{F}}$. There are two types of blocks. First, each of the six five-block configurations of $\mathcal{B}_{\mathcal{E}}$ forms a block of the 2-$(21,5,1)$-design. For example, the five-block configuration corresponding to vertex 2 gives rise to the block $\{abdf, aceg, bghi, cdhj, efij\}$. Second, for each block B of $\mathcal{B}_{\mathcal{F}}$, the three blocks of $\mathcal{B}_{\mathcal{E}}$ that are disjoint from B together with the two five-block configurations of $\mathcal{B}_{\mathcal{F}}$ that contain B form a block of the 2-$(21,5,1)$-design. For example, the block $adgh \in \mathcal{B}_{\mathcal{F}}$ gives rise to the block $\{bcej, bcfi, efij, D, E\}$.

Automorphisms:

It is well known that for $n \neq 6$, the automorphism group $\mathrm{Aut}(S_n)$ of the symmetric group S_n is composed entirely of inner automorphisms, and that the outer automorphisms of S_6 correspond to symmetries that interchange edges of K_6 with 1-factors of K_6, and vertices of K_6 with 1-factorisations of K_6, see Chapter 8 of [3]. However, less attention seems to have been paid to the fact that the 3-transitive action of $\mathrm{Aut}(S_6)$ on ten points can be viewed as an action on the set of triangle factors of K_6.

Let G be the automorphism group of the SQS(10). It is known that G has order $2 \cdot 6! = 1440$, is isomorphic to $\mathrm{Aut}(S_6)$, acts 3-transitively on the points, and acts transitively on the blocks. Let H be the subgroup of G given by the natural induced action of S_6 on $\mathcal{T}$. So $H \cong S_6$, is a normal subgroup of G with index 2, and corresponds with the group of inner automorphisms of S_6. The group H is the set-wise stabilizer of $\mathcal{B}_{\mathcal{E}}$ (and $\mathcal{B}_{\mathcal{F}}$) and the elements of $G \setminus H$ interchange the blocks of $\mathcal{B}_{\mathcal{E}}$ with the blocks of $\mathcal{B}_{\mathcal{F}}$. The group G is generated by H and any automorphism, for example $\theta = (a\ f)(b\ g)(c\ h)(d\ i)(e\ j)$, which swaps the blocks of $\mathcal{B}_{\mathcal{E}}$ with the blocks of $\mathcal{B}_{\mathcal{F}}$.

The outer automorphisms of S_6 correspond to conjugations of H by elements of $G \setminus H$, and can be described via the induced action of G on the twelve five-block configurations of the SQS(10). For example, consider the SQS(10) automorphisms

$$\phi = (b\ j)(d\ i)(f\ h) \in H \quad \text{and} \quad \theta = (a\ f)(b\ g)(c\ h)(d\ i)(e\ j) \in G \setminus H.$$

The automorphism ϕ induces the permutation $(1\ 2)$ of the six five-block configurations of $\mathcal{B}_{\mathcal{E}}$, and the permutation $(A\ B)(C\ D)(E\ F)$ of the six five-block configurations of $\mathcal{B}_{\mathcal{F}}$, while θ induces the bijection

$$1 \leftrightarrow D, \qquad 2 \leftrightarrow C, \qquad 3 \leftrightarrow F, \qquad 4 \leftrightarrow A, \qquad 5 \leftrightarrow B, \qquad 6 \leftrightarrow E$$

between the six five-block configurations of $\mathcal{B}_{\mathcal{E}}$ and the six five-block configurations of $\mathcal{B}_{\mathcal{F}}$. So the outer automorphism of S_6 given by θ maps $(1\ 2)$ to $(1\ 2)(3\ 6)(4\ 5)$.

The stabilizer of each five-block configuration is a subgroup of H and is isomorphic to S_5. For example, the stabilizer of the five-block configuration corresponding to the 1-factorisation A acts on the set $\{B, C, D, E, F\}$ of five-block configurations of $\mathcal{B}_{\mathcal{F}}$, and also on the set $\{1, 2, 3, 4, 5, 6\}$ of six five-block configurations of $\mathcal{B}_{\mathcal{E}}$. The action on $\{1, 2, 3, 4, 5, 6\}$ is the well-known 3-transitive action of S_5 on six points.

The stabilizer of each block of the SQS(10) has order 48, is a subgroup of H, and is isomorphic to $S_4 \times S_2$. The stabilizer of B_{uv} is generated by permutations of the four vertices not incident with uv and the transposition $(u\ v)$. The point-wise stabilizer of B_{uv} is the group of order 2 generated by $(u\ v)$. The stabilizer of the block $B_{\{uv,wx,yz\}}$ is the automorphism group of the 1-factor $\{uv, wx, yz\}$. The point-wise stabilizer of $B_{\{uv,wx,yz\}}$ is the group of order 2 generated by $(u\ v)(w\ x)(y\ z)$.

Let H' be the subgroup of H generated by the action of the alternating group A_6. Then $H' \trianglelefteq G$, and the quotient group $G/H' = \{H', \theta H', \phi H', \theta\phi H'\}$ where, as previously, $\theta = (a\ f)(b\ g)(c\ h)(d\ i)(e\ j) \in G \setminus H$ and $\phi = (b\ j)(d\ i)(f\ h) \in H \setminus H'$. Also, $\langle H', \phi\rangle$ is isomorphic to S_6, $\langle H', \theta\rangle$ is isomorphic to the projective linear group PGL$(2, 9)$, and $\langle H', \theta\phi\rangle$ is isomorphic to the Mathieu group M_{10}.

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