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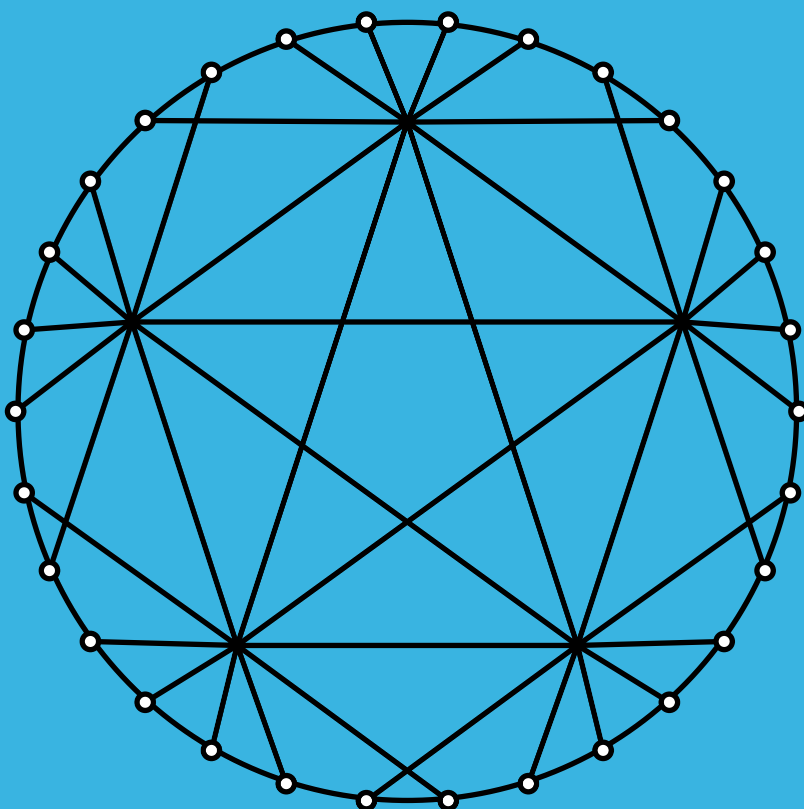

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Double jump peg solitaire on graphs

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Abstract

Peg solitaire is a game in which pegs are placed in every hole but one and the player jumps over pegs along rows or columns to remove them. Usually, the goal is to have a single peg remaining. In a 2011 paper, this game is generalized to graphs. In this paper, we consider a variation in which each peg must be jumped twice in order to be removed. For this variation, we consider the solvability of several graph families. For our major results, we characterize solvable joins of graphs and show that the Cartesian product of solvable graphs is likewise solvable.

1 Introduction and preliminary results

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (in other words, a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z. In [6], peg solitaire is generalized to graphs. A graph, $G = (V, E)$, is a set of vertices, V, and a set of edges, E. If there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z, provided that $xy, yz \in E$. Such

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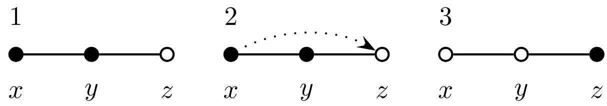

Figure 1: A typical jump in peg solitaire, $x \cdot \overrightarrow{y} \cdot z$

a jump will be denoted $x \cdot \overrightarrow{y} \cdot z$. Because of the nature of peg solitaire we assume that all graphs are connected finite graphs with no loops or multiple edges, unless specified otherwise. For more information on the traditional game, see [1, 10]. For all undefined graph theory terminology, refer to West [15].

Since the 2011 paper by Beeler and Hoilman, there have been a number of papers that consider variations of the original game (see for example [4, 8, 11, 12, 13]). Building on these papers, we consider a variation in which the pegs require two (not necessarily consecutive) jumps in order to remove them. As an analogy, we can think of these pegs as soldiers wearing armor. The first jump over the peg removes its armor and the second jump removes the peg. For convenience of exposition, we will refer to pegs that have not been jumped as *2-pegs* and pegs that have been jumped once as *1-pegs*. The goal of this paper is to explore the solvability of graphs in this variation.

Traditionally, the game begins with a *starting state* $S = (S_0, S_1, S_2)$, where S_0 is the set of vertices with holes, S_1 is the set of vertices with 1-pegs, and S_2 is the set of vertices with 2-pegs. In order to mirror the original game, we will assume that $|S_0| = 1$, $S_1 = \emptyset$, and $S_2 = V(G) - S_0$ unless otherwise noted. After a sequence of legal moves we will arrive at an associated *terminal state* where no further jumps are possible. We denote this terminal state $T = (T_0, T_1, T_2)$, where the T_i are defined analogously to the S_i. Note that the above definition implies that $T_1 \cup T_2$ is an independent set of vertices. Our goal is usually to minimize $|T_1 \cup T_2|$. A graph is *solvable* if there exists a starting state S and an associated terminal state T such that $|T_1 \cup T_2| = 1$. If the graph is not solvable, then we say that it is *k-solvable*, where k is the minimum value of $|T_1 \cup T_2|$ across all possible terminal states associated with a starting state with one hole and 2-pegs elsewhere.

In the solvable cases, we may be interested in whether the final peg is a 1-peg or a 2-peg. If we can end the game with a single 1-peg, then we say that the graph is *T_1-solvable*. Likewise, if we can end the game with a single 2-peg, then we say that the graph is *T_2-solvable*. If a graph is both T_1-solvable and T_2-solvable, then we say that it is *T_1T_2-solvable*.

We now present a few preliminary results and observations to aid us in our main results.

Observation 1.1.

(i) *If G is a graph in which there is a move available, then there is a first move, say $s'' \cdot \overrightarrow{s'} \cdot s$. This jump results in a hole in s'', a 1-peg in s', and 2-pegs elsewhere. If G is solvable from the configuration $S_0 = \{s\}$, $S_1 = \emptyset$, and $S_2 = V(G) - \{s\}$, then it is also solvable from the configuration $S'_0 = \{s''\}$, $S'_1 = \{s'\}$, and $S'_2 = V(G) - \{s', s''\}$.*

(ii) *If G is T_1-solvable, then there is a final jump, say $t'' \cdot \overrightarrow{t'} \cdot t$. Hence, we can "stop short" of this final jump. This results in a configuration where $T'_0 = V(G) - \{t', t''\}$, $T'_1 = \{t', t''\}$, and $T'_2 = \emptyset$.*

(iii) *If G is T_2-solvable, then there is a final jump, say $t'' \cdot \overrightarrow{t'} \cdot t$. As in (ii), we can "stop short" of this final jump, resulting in a configuration where $T'_0 = V(G) - \{t', t''\}$, $T'_1 = \{t'\}$, and $T'_2 = \{t''\}$.*

Using Observation 1.1, we can show that most T_2-solvable graphs are also T_1-solvable. Whether there are T_1-solvable graphs that are not T_2-solvable is unknown at this time. As usual, we let P_n, C_n, and K_n denote the path, cycle, and complete graph on n vertices, respectively.

Proposition 1.2. *Suppose that G is a T_2-solvable graph such that the final jump occurs on the last three vertices of a P_4-subgraph or a C_3-subgraph. It follows that G is also T_1-solvable.*

Proof. Suppose that G is T_2-solvable, with the final jump being $t'' \cdot \overrightarrow{t'} \cdot t$. Suppose that these three vertices are the last three vertices of a path on four vertices, t, t', t'', and t''', where $t''t''' \in E(G)$. To achieve a solution with a single 1-peg, we solve G but stop before the final jump of $t'' \cdot \overrightarrow{t'} \cdot t$ (see Observation 1.1 (iii)). We now have a 2-peg in t'', a 1-peg in t', and holes elsewhere. Making the jumps $t' \cdot \overrightarrow{t''} \cdot t'''$ and $t''' \cdot \overrightarrow{t''} \cdot t'$ ends the game with a single 1-peg in t'.

For the case where the final jump occurs on a C_3-subgraph, we let $t = t'''$ and repeat the above argument. $\square$

As in [6, 12, 13], we can restrict our solution to the edges of a solvable spanning subgraph. Hence the following proposition is immediate.

Proposition 1.3. *Suppose that H is a spanning subgraph of G and that H is T_1-solvable (T_2-solvable). It follows that G is likewise T_1-solvable (T_2-solvable).*

One of the more important notions for obtaining alternative solvable configurations as well as the *fool's solitaire problem* (see [8, 14]) is the notion of the *dual configuration*. In the original game, the dual configuration is obtained by reversing the roles of pegs and holes. The Duality Principle states that if S is a starting state with associated terminal state T and S' and T' are their respective dual configurations, then T' is a starting state with associated terminal state S'. While the dual was important for the original variant, so far the analog has yet to prove as valuable. However, we include it for completeness.

Theorem 1.4. (Analog of the Duality Principle) *Let $S = (S_0, S_1, S_2)$ be a starting state of a graph G. Let $T = (T_0, T_1, T_2)$ be a terminal state obtained from S via a sequence of moves such that a 1-peg is never used to jump another peg. Define $S' = (T_2, T_1, T_0)$ and $T' = (S_2, S_1, S_0)$. It follows that S' is a starting state of G with associated terminal state T'.*

Proof. Suppose that $S = (S_0, S_1, S_2)$ is a starting state of a graph G. Let j_1,...,j_n be a sequence of jumps such that a 1-peg is never used to jump another peg. This results in a terminal state $T = (T_0, T_1, T_2)$. Using this sequence of jumps, we transition through a sequence of states I_0, I_1,...,I_n, where $I_0 = S$, $I_n = T$, and I_k is obtained from I_{k-1} via the jump j_k. Note that we can write each of these states as $I_k = (I_0^k, I_1^k, I_2^k)$, where I_0^k is the set of vertices at step k that have holes, I_1^k is the set of vertices at step k that have 1-pegs, and I_2^k is the set of vertices that have 2-pegs at step k.

Define a sequence of configurations I'_0, I'_1,..., I'_n by $I'_k = (I_2^{n-k}, I_1^{n-k}, I_0^{n-k})$. Note that $I'_{n-m} = (I_2^m, I_1^m, I_0^m)$. Since the choice of state is arbitrary, it suffices to show that I'_{n-m+1} can be obtained from I'_{n-m} via the jump $j_m = a\cdot \overrightarrow{b} \cdot c$.

Case 1: Suppose that in I_{m-1}, $b \in I_1^{m-1}$. Thus, $I_0^m = (I_0^{m-1} \cup \{a, b\}) - \{c\}$, $I_1^m = I_1^{m-1} - \{b\}$, and $I_2^m = (I_2^{m-1} \cup \{c\}) - \{a\}$. Note that $j_n = a\cdot \overrightarrow{b} \cdot c$ is a legal move from state I'_{n-m} as $c \in I_0^{n-m'} = I_2^m$ and $a, b \in I_2^{n-m'} = I_0^m$. This move results in a state with $I_0^{n-m+1'} = (I_0^{n-m'} \cup \{a\}) - \{c\} = I_2^{m-1}$, $I_1^{n-m+1'} = I_1^{n-m'} \cup \{b\} = I_1^{m-1}$, and $I_2^{n-m+1'} = (I_2^{n-m'} \cup \{c\}) - \{a, b\} = I_0^{m-1}$. Hence proving the claim.

Case 2: Suppose that in I_{m-1}, $b \in I_2^{m-1}$. Thus, $I_0^m = (I_0^{m-1} \cup \{a\}) - \{c\}$, $I_1^m = I_0^{m-1} \cup \{b\}$, and $I_2^m = (I_2^{m-1} \cup \{c\}) - \{a, b\}$. Note that $j_m = a \cdot \overrightarrow{b} \cdot c$ is a legal move from state I'_{n-m} as $c \in I_0^{n-m'} = I_2^m$, $b \in I_1^{n-m'} = I_1^m$, and $a \in I_2^{n-m'} = I_0^m$. This move results in a state with $I_0^{n-m+1'} = (I_0^{n-m'} \cup \{a, b\}) - \{c\} = I_2^{m-1}$, $I_1^{n-m+1'} = I_1^{n-m'} - \{b\} = I_1^{m-1}$, and $I_2^{n-m+1'} = (I_2^{n-m'} \cup \{c\}) - \{a\} = I_0^{m-1}$. Hence proving the claim. □

To see why the restriction on making jumps using only the 2-pegs is necessary, consider the graph G with $V(G) = \{a, b, c, d, e, f\}$ and $E(G) = \{ab, bc, bd, ce, de, ef\}$. Beginning with the hole in d, the jumps $f \cdot \overrightarrow{e} \cdot d$, $c \cdot \overrightarrow{e} \cdot f$, $a \cdot \overrightarrow{b} \cdot c$, $b \cdot \overrightarrow{c} \cdot e$, $e \cdot \overrightarrow{d} \cdot b$, and $d \cdot \overrightarrow{b} \cdot a$ result in a terminal state with $T_0 = \{b, d, e\}$, $T_1 = \{a, c\}$, and $T_2 = \{f\}$. The dual of this terminal configuration has $S'_0 = \{f\}$, $S'_1 = \{a, c\}$, and $S'_2 = \{b, d, e\}$. Beginning with this starting state and reversing the above sequence of jumps immediately results in an attempt to jump into vertex a, which contains a 1-peg. As this is an illegal move, the sequence of moves cannot be reversed.

2 Graph families

Theorem 2.1. *For $n \in \{2, 3\}$, the path on n vertices is T_2-solvable, but not T_1-solvable. For $n \geq 4$, the path on n vertices is T_1T_2-solvable.*

Proof. Assume that the vertices of P_n are v_0, v_1,..., v_{n-1}, where the labels are assigned in the obvious way. For $n = 2$, no moves are available and the result is trivial.

For $n = 3$, suppose that the hole is in v_1. Clearly, no moves are possible. Thus, we can assume without loss of generality that the initial hole is in v_0. The moves $v_2 \cdot \overrightarrow{v_1} \cdot v_0$ and $v_0 \cdot \overrightarrow{v_1} \cdot v_2$ are then forced. This ends the game with the final 2-peg in v_2.

Suppose that $n \geq 4$. Place the initial hole in v_0. For $i = 1, ..., n-2$, jump $v_{i+1} \cdot \overrightarrow{v_i} \cdot v_{i-1}$ and $v_{i-1} \cdot \overrightarrow{v_i} \cdot v_{i+1}$. This ends with the final 2-peg in v_{n-1}. As this is the end vertex of a P_4-subgraph, the graph is also T_1-solvable by Proposition 1.2. □

Corollary 2.2. *For $n \geq 3$, the cycle and the complete graph on n vertices are T_1T_2-solvable, regardless of the placement of the initial hole.*

Proof. Note that C_n and K_n have P_n as a spanning subgraph. Hence their solvability follows from Proposition 1.2, Proposition 1.3, and Theorem 2.1. Since C_n and K_n are vertex transitive, this is true regardless of where the initial hole is placed. □

The *star* is the graph on vertices u, v_1,...,v_n, where uv_i is an edge for $i = 1, ..., n$. This graph is denoted $K_{1,n}$.

Proposition 2.3. *For $n \in \{1, 2\}$, the star $K_{1,n}$ is T_2-solvable, but not T_1-solvable. For $n \geq 3$, the star $K_{1,n}$ is $(n - 1)$-solvable.*

Proof. Note that $K_{1,1}$ and $K_{1,2}$ are isomorphic to P_2 and P_3, respectively. Hence the result follows from Theorem 2.1.

For $n \geq 3$, only the peg in the center vertex u can be removed. This is accomplished by placing the initial hole in v_1 and jumping $v_2 \cdot \overrightarrow{u} \cdot v_1$ and $v_1 \cdot \overrightarrow{u} \cdot v_2$. As these are the only available moves, the game ends with $n - 1$ 2-pegs on the graph. □

The *double star* is the tree with vertex set $V = \{x, y, x_1, ..., x_n, y_1, ..., y_m\}$ and edge set $E = \{xy, xx_i, yy_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. This graph is denoted $S_{n,m}$. Without loss of generality, we will assume that $n \geq m \geq 1$.

Theorem 2.4. *For the double star $S_{n,m}$:*

(i) *If $m = 1$, then $S_{n,m}$ is T_1T_2-solvable if $n = 1$, T_2-solvable if $n = 2$, and $(n - 1)$-solvable if $n \geq 3$.*

(ii) *If $m \geq 2$, then $S_{n,m}$ is T_1T_2-solvable if $n \leq 2m$, T_2-solvable if $n = 2m + 1$, and $(n - 2m)$-solvable if $n \geq 2m + 2$.*

Proof. We begin with the case of $m = 1$. If $n = 1$, then the graph is isomorphic to P_4, and the result follows from Theorem 2.1. If $n \geq 2$, then we prove the result by case analysis. If the hole is in y, then the initial jump $x_n \cdot \overrightarrow{x} \cdot y$ is forced. Noting that $y \cdot \overrightarrow{x} \cdot x_n$ ends the game, we instead jump $x_1 \cdot \overrightarrow{x} \cdot x_n$ and $y_1 \cdot \overrightarrow{y} \cdot x$. The jump $x \cdot \overrightarrow{y} \cdot y_1$ ends the game, so instead we jump $x_n \cdot \overrightarrow{x} \cdot x_1$. The next jump from this configuration will result $n - 1$ 2-pegs and a 1-peg. If the hole is in x, then the jumps $y_1 \cdot \overrightarrow{y} \cdot x$ and $x \cdot \overrightarrow{y} \cdot y_1$ are forced, leaving us with no available moves. If the initial hole is in y_1, then the jumps $x \cdot \overrightarrow{y} \cdot y_1$, $y_1 \cdot \overrightarrow{y} \cdot x$, and $x_n \cdot \overrightarrow{x} \cdot y$ are forced. A jump over x will end the game, so we instead jump $x \cdot \overrightarrow{y} \cdot y_1$, $y_1 \cdot \overrightarrow{y} \cdot x$, and $x_1 \cdot \overrightarrow{x} \cdot x_n$. The game ends with 2-pegs in x_2,...,x_n. Note that this proves that $S_{2,1}$ is T_2-solvable. Finally, suppose that the initial hole is in x_n. If we jump $x_1 \cdot \overrightarrow{x} \cdot x_n$, then this results in a hole in x_1, a 1-peg in x, and 2-pegs elsewhere. Up to automorphism on the vertices, this is the configuration obtained when the initial hole is in y and the (forced) jump $x_1 \cdot \overrightarrow{x} \cdot y$ has been made. So we instead jump $y \cdot \overrightarrow{x} \cdot x_n$ followed by $x_1 \cdot \overrightarrow{x} \cdot y$ and $y_1 \cdot \overrightarrow{y} \cdot x$. If we jump $y \cdot \overrightarrow{x} \cdot x_1$, then the next jump ends the game with one 1-peg and $n-1$ 2-pegs. If we jump $x \cdot \overrightarrow{y} \cdot y_1$, then we end the game with n 2-pegs. So we instead jump $x_n \cdot \overrightarrow{x} \cdot x_1$. Any further jump ends the game with one 1-peg and $n-1$ 2-pegs. As all possibilities have been examined, the minimum of $n-2$ 2-pegs is achieved by beginning with the initial hole in y_1. Further, the graph is not T_1-solvable.

We address the case of $m \geq 2$ by first establishing necessary conditions. Consider how we can remove a peg from the set $X = \{x_1, ..., x_n\}$. To remove a peg from X, we must first have a peg in x. If that peg is not there, we must place it there with the jump $y_i \cdot \overrightarrow{y} \cdot x$. Further, once a 2-peg is in x, two 2-pegs from X may jump over it and out of X before another jump of the form $y_i \cdot \overrightarrow{y} \cdot x$ is required to place an additional peg in x. Hence each 2-peg in $y_1, ..., y_m$ can "exchange" with two 2-pegs in X. Therefore, $n \leq 2m$ is necessary for the graph to be T_1T_2-solvable. Moreover, if $n \geq 2m+1$, then, at best, $n-2m$ 2-pegs remain in the graph.

We now show that the conditions described above are sufficient when $m \geq 2$. We begin by showing that $S_{n,m}$ is T_1T_2-solvable if $n = m$. Begin with the initial hole in x. For $i = 1$,...,$m-1$, jump $y_m \cdot \overrightarrow{y} \cdot x$, $y_i \cdot \overrightarrow{y} \cdot y_m$, $x_m \cdot \overrightarrow{x} \cdot y$, and $x_i \cdot \overrightarrow{x} \cdot x_m$. Then jump $y_m \cdot \overrightarrow{y} \cdot y_1$, $y_1 \cdot \overrightarrow{y} \cdot x$, $x_m \cdot \overrightarrow{x} \cdot y$, and $y \cdot \overrightarrow{x} \cdot x_m$. The game ends with a 2-peg in x_m. As this is the end vertex of the P_4-subgraph induced by the vertices y_m, y, x, and x_m, the graph is also T_1-solvable by Proposition 1.2.

Assume that $m \geq 2$ and $n \geq m+1$. Begin with the initial hole in x_n and jump $y \cdot \overrightarrow{x} \cdot x_n$ and $x_n \cdot \overrightarrow{x} \cdot y$. If $n = m+1$, then this reduces the graph to the initial state of the $n = m$ case. If $n \geq m+2$, then continue the game by jumping $y_1 \cdot \overrightarrow{y} \cdot x$ and $y_m \cdot \overrightarrow{y} \cdot y_1$. Next, for $i = 1$,...,$\min\{m-1, n-m-1\}$, jump $x_{n-2i+1} \cdot \overrightarrow{x} \cdot y$, $y_1 \cdot \overrightarrow{y} \cdot y_m$, $y_m \cdot \overrightarrow{y} \cdot y_1$, $x_{n-2i} \cdot \overrightarrow{x} \cdot y$, $y_{m-i} \cdot \overrightarrow{y} \cdot y_m$, and $y_m \cdot \overrightarrow{y} \cdot x$. Then jump $x_1 \cdot \overrightarrow{x} \cdot y$.

If $n \leq 2m-1$, then jump $x_{2m-n+1} \cdot \overrightarrow{x} \cdot x_1$. This reduces the graph to the initial state of the $n = m$ case. Thus, it is T_1T_2-solvable.

If $n = 2m$, then we instead jump $y \cdot \overrightarrow{x} \cdot x_1$ to end the game with a 2-peg in x_1. As this is the end vertex of the P_4-subgraph induced by the vertices x_1, x, y, and y_1, the graph is also T_1-solvable by Proposition 1.2.

If $n \geq 2m+1$, then jump $x \cdot \overrightarrow{y} \cdot y_1$, $y_1 \cdot \overrightarrow{y} \cdot x$, and $x_2 \cdot \overrightarrow{x} \cdot y$. If $n = 2m+1$, then this ends the game with a 2-peg in y. If $n \geq 2m+2$, then this ends the game with a total of $n - 2m$ 2-pegs in $x_3, ..., x_{n-2m+1}, y$. $\square$

It is worth noting that P_{2n+1} (where $n \geq 2$), C_{2n+1} (where $n \geq 2$), and $S_{n,m}$ (where $m+2 \leq n \leq 2m+2$) are solvable in this double jump variation, but unsolvable in the original single jump variation (see for example [6, 7]). Whether there is a graph that is solvable in the single jump variation but unsolvable in the double jump variation is unknown at this time.

3 Joins of graphs

In this section, we characterize when the join of two graphs is solvable. The *join* of graphs G and H, denoted $G \vee H$, is the graph with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. Note that in this section, our component graphs may be disconnected graphs. However, the resulting join will be connected. We first show that if both graphs have at least two vertices each, then the join is solvable.

Theorem 3.1. *If $|V(G)| \geq 2$ and $|V(H)| \geq 2$, then the join $G \vee H$ is T_1T_2-solvable.*

Proof. Let $V(G) = \{g_1, ..., g_n\}$ and $V(H) = \{h_1, ..., h_m\}$. Begin with the hole in g_n. For $i = 1, ..., m-1$, jump $g_1 \cdot \overrightarrow{h_i} \cdot g_n$ and $g_n \cdot \overrightarrow{h_i} \cdot g_1$. This results in 2-pegs in g_1,...,g_{n-1} and h_m with holes in all other vertices. For $j = 1, ..., n-1$, jump $h_m \cdot \overrightarrow{g_j} \cdot h_1$ and $h_1 \cdot \overrightarrow{g_j} \cdot h_m$. This results in a single 2-peg in h_m. This is the end vertex of the P_4-subgraph with vertex set $\{g_n, h_1, g_1, h_m\}$. Hence it is also T_1-solvable by Proposition 1.2. $\square$

Note that the complete bipartite graph $K_{n,m}$ is the join of the complements of K_n and K_m. Hence the following result is immediate.

Corollary 3.2. *The complete bipartite graph $K_{n,m}$ is T_1T_2-solvable if $n \geq 2$ and $m \geq 2$.*

Note that the star $K_{1,n}$ is the join of K_1 with the complement of K_n. So not all joins are solvable (see Proposition 2.3). In our next theorem, we show that the remaining joins are solvable.

Theorem 3.3. *If G is a graph with at least one edge, then $G \vee K_1$ is T_1T_2-solvable.*

Proof. Let the vertex of K_1 be denoted u. Let two vertices of G that share an edge be denoted b_1 and b_2. Let all other vertices of G be denoted $p_1, ..., p_k$. Begin with the initial hole in u. For $i = 1, ..., \lfloor k/3 \rfloor$, jump $b_1 \cdot \overrightarrow{b_2} \cdot u$, $p_{k-3i+2} \cdot \overrightarrow{u} \cdot b_1$, $p_{k-3i+3} \cdot \overrightarrow{u} \cdot p_{k-3i+2}$, $b_1 \cdot \overrightarrow{b_2} \cdot u$, $p_{k-3i+2} \cdot \overrightarrow{u} \cdot b_1$, and $p_{k-3i+1} \cdot \overrightarrow{u} \cdot b_2$. Then jump $b_1 \cdot \overrightarrow{b_2} \cdot u$.

If $k \equiv 0 \pmod 3$, then jump $u \cdot \overrightarrow{b_2} \cdot b_1$ to end the game.

Otherwise, we follow with the jumps $b_2 \cdot \overrightarrow{u} \cdot b_1$, $p_1 \cdot \overrightarrow{u} \cdot b_2$, and $b_2 \cdot \overrightarrow{b_1} \cdot u$. If $k \equiv 1 \pmod 3$, then this ends the game. If $k \equiv 2 \pmod 3$, then we follow with the jumps $p_2 \cdot \overrightarrow{u} \cdot b_2$ and $b_2 \cdot \overrightarrow{u} \cdot b_1$ to end the game.

Note that in all of the above cases, the game ends with a 2-peg jumping over a 1-peg on the C_3 subgraph induced by u, b_1, and b_2. Thus the graph is also T_1-solvable by Proposition 1.2. $\square$

Note that Corollary 2.2, Proposition 2.3, Theorem 3.1, and Theorem 3.3 characterize the solvability of all joins. This is summarized in the following corollary.

Corollary 3.4. *Let G and H be graphs with at least one vertex each. The join $G \vee H$ is T_1T_2-solvable if and only if $\min\{|V(G)|, |V(H)|\} \geq 2$ or $\min\{|E(G)|, |E(H)|\} \geq 1$. The join $G \vee H$ is T_2-solvable, but not T_1-solvable, if and only if $|V(G)| + |V(H)| = 3$ and $|E(G)| = |E(H)| = 0$. The join $G \vee H$ is not solvable if and only if $|V(G)| = 1$, $|V(H)| \geq 3$, and $|E(H)| = 0$.*

In [3], the question of how much any single edge addition can improve the solvability of a graph is explored in the single jump variation. If the addition of any edge changes the solvability of the graph, then we say that the graph is *edge-critical*. Note that in this double jump variation, the star is an edge-critical graph. Further, the addition of any edge to $K_{1,n}$ changes the number of remaining pegs from $n-1$ 2-pegs to either a single 2-peg or a single 1-peg. Ergo, the number of pegs can be reduced by an arbitrarily large amount by the addition of a single edge.

4 Cartesian products

In this section, we give results on the Cartesian product of two solvable graphs. For graphs G and H, the Cartesian product of G and H is denoted $G\Box H$ [15]. For $g \in V(G)$ and $h \in V(H)$, let $(g,h) \in V(G\Box H)$ denote the vertex in the Cartesian product induced by those vertices. We define G_h to be the copy of G induced by the vertex $h \in V(H)$. Per the notation used in Observation 1.1, we will also assume that G is solvable with the initial jump being $g_s''\cdot \overrightarrow{g_s'} \cdot g_s$. In the case where G is solvable, the final jump is denoted $g_t''\cdot \overrightarrow{g_t'} \cdot g_t$. Analogous definitions will be used for H.

Theorem 4.1. *If G is T_1-solvable, then the Cartesian product $G\Box P_2$ is T_1-solvable. If G is T_2-solvable, then the Cartesian product $G\Box P_2$ is T_1T_2-solvable.*

Proof. If G is isomorphic to P_2, then $G\Box P_2$ is isomorphic to C_4 which is T_1T_2-solvable by Corollary 2.2. We only consider connected graphs and (up to isomorphism) P_2 is the only connected graph on two vertices. Thus we may assume that $|V(G)| \geq 3$.

Let $V(P_2) = \{h_0, h_1\}$. Place the initial hole in (g_s, h_0) and make the jumps $(g_s', h_1)\cdot\overrightarrow{(g_s, h_1)}\cdot(g_s, h_0)$ and $(g_s, h_0)\cdot\overrightarrow{(g_s, h_1)}\cdot(g_s', h_1)$. As both G_{h_0} and G_{h_1} have holes in g_s and 2-pegs elsewhere, we solve them independently with the final pegs in (g_t, h_0) and (g_t, h_1). If G is T_1-solvable, we jump $(g_t, h_1)\cdot\overrightarrow{(g_t, h_0)}\cdot(g_t', h_0)$ to solve. If G is T_2-solvable, then we make the additional jump $(g_t', h_0)\cdot\overrightarrow{(g_t, h_0)}\cdot(g_t, h_1)$. As this final jump is on the P_4-subgraph with vertex set $\{(g_t, h_0), (g_t', h_0), (g_t', h_1), (g_t, h_1)\}$, it follows that this is also T_1-solvable by Proposition 1.2. □

Theorem 4.2. *If G and H are both T_2-solvable, then the Cartesian product $G\Box H$ is T_1T_2-solvable.*

Proof. Place the initial hole in (g_s, h_s) and solve H_{g_s} ending with the final 2-peg in (g_s, h_t). We then make the jumps $(g_s', h_t)\cdot\overrightarrow{(g_s, h_t)}\cdot(g_s, h_t')$ and $(g_s, h_t')\cdot\overrightarrow{(g_s, h_t)}\cdot(g_s', h_t)$. Now all copies of G have a hole in g_s and 2-pegs elsewhere. For $h \notin \{h_s', h_s''\}$, we solve these copies, ending with the final 2-peg in g_t. For $G_{h_s'}$, and $G_{h_s''}$ we stop before making the final jump, leaving a 1-peg in g_t', a 2-peg g_t'', and holes elsewhere. We then make the jumps $(g_t', h_s')\cdot\overrightarrow{(g_t', h_s'')}\cdot(g_t, h_s'')$, $(g_t'', h_s'')\cdot\overrightarrow{(g_t'', h_s')}\cdot(g_t', h_s')$, $(g_t'', h_s')\cdot\overrightarrow{(g_t', h_s')}\cdot(g_t, h_s')$, $(g_t, h_s')\cdot\overrightarrow{(g_t, h_s'')}\cdot(g_t', h_s'')$, and $(g_t', h_s'')\cdot\overrightarrow{(g_t', h_s')}\cdot(g_t, h_s')$. All of the remaining pegs are on the subgraph H_{g_t}. This subgraph has a hole in h''_s, a 1-peg in h'_s, and 2-pegs elsewhere. Thus it is solvable by Observation 1.1 with the final 2-peg in (g_t, h_t). As this peg is the end vertex of the P_4-subgraph with vertex set $\{(g_t, h'_t), (g'_t, h'_t), (g'_t, h_t), (g_t, h_t)\}$, it follows that this is also T_1-solvable by Proposition 1.2. $\square$

A similar technique yields our next result.

Theorem 4.3. *If G is T_2-solvable and H is T_1-solvable, then the Cartesian product $G\square H$ is T_1-solvable.*

Proof. Begin with the initial hole in (g_s, h_s). Solve H_{g_s} ending with the final 1-peg in (g_s, h_t). We then jump $(g'_s, h_t)\cdot\overrightarrow{(g_s, h_t)}\cdot(g_s, h'_t)$ followed by $(g''_s, h'_t)\cdot\overrightarrow{(g'_s, h'_t)}\cdot(g'_s, h_t)$. With the exception of $G_{h'_t}$, all copies of G have a hole in g_s and 2-pegs elsewhere. As for $G_{h'_t}$, it has a hole in g''_s, a 1-peg in g'_s, and 2-pegs elsewhere. By Observation 1.1, we solve each copy of G leaving the final 2-peg in g_t. Now jump $(g_t, h'_s)\cdot\overrightarrow{(g_t, h_s)}\cdot(g'_t, h_s)$ and $(g'_t, h_s)\cdot\overrightarrow{(g_t, h_s)}\cdot(g_t, h'_s)$. Since H_{g_t} has a hole in h_s and 2-pegs elsewhere, it is T_1-solvable with the final 1-peg in (g_t, h_t). $\square$

A natural question is when the Cartesian product of two T_1-solvable graphs is also solvable. For this, we need an additional hypothesis regarding the solvability of one of our graphs in the original "single jump" variation. In order to keep the different levels of abstraction sufficiently clear, we will assume that if a graph H is solvable in the single jump variation, then the first jump is $s''\cdot\overrightarrow{s'}\cdot s$ and the final jump is $t''\cdot\overrightarrow{t'}\cdot t$. Another useful concept is when a graph is *distance 2-solvable* in the single jump variation. A graph is distance 2-solvable if the final two pegs are located in vertices t_1 and t_3 which share a mutually adjacent vertex t_2. There have been a number of papers determining the solvability of various graphs in the single jump variation. For these results, the interested reader is referred to [2, 5, 6, 7, 9].

Theorem 4.4. *Suppose that G is T_1-solvable. Suppose that H is T_1-solvable and that H is either solvable or distance 2-solvable in the single jump variation. It follows that the Cartesian product $G\square H$ is T_1-solvable.*

Proof. As usual, we begin with the initial hole in (g_s, h_s). Solve H_{g_s}, ending with the final 1-peg in h_t. Jump $(g'_s, h_t)\cdot\overrightarrow{(g_s, h_t)}\cdot(g_s, h'_t)$ followed by $(g''_s, h'_t)\cdot\overrightarrow{(g'_s, h'_t)}\cdot(g'_s, h_t)$. With the exception of $G_{h'_t}$, each copy of G has a hole in g_s and 2-pegs elsewhere. As for $G_{h'_t}$, it has a hole in g''_s, a 1-peg in g'_s, and 2-pegs elsewhere. Per Observation 1.1, we can solve each of these copies of G independently, ending with the final 1-pegs in g'_t and g''_t. We then jump $(g'_t, s'')\cdot\overrightarrow{(g'_t, s')}\cdot(g_t, s')$, $(g''_t, s)\cdot\overrightarrow{(g'_t, s)}\cdot(g_t, s)$, and $(g_t, s')\cdot\overrightarrow{(g_t, s)}\cdot(g'_t, s)$. Note that $H_{g''_t}$ has a hole in s and 1-pegs elsewhere. Further note that $H_{g'_t}$ has holes in s' and s'' and 1-pegs elsewhere. Hence they are solvable or distance 2-solvable by the analog of Observation 1.1.

Suppose that H is solvable in the single jump variation. We solve both copies of H independently, ending with the final 1-pegs in (g''_t, t) and (g'_t, t). The jump $(g''_t, t)\cdot\overrightarrow{(g'_t, t)}\cdot(g_t, t)$ completes the solution.

Suppose that H is distance 2-solvable in the single jump variation. Distance 2-solve both copies of H independently, ending with the final 1-pegs in (g''_t, t_1), (g''_t, t_3), (g'_t, t_1), and (g'_t, t_3). To complete the solution, we make the following three jumps $(g''_t, t_1)\cdot\overrightarrow{(g'_t, t_1)}\cdot(g'_t, t_2)$, $(g'_t, t_3)\cdot\overrightarrow{(g''_t, t_3)}\cdot(g''_t, t_2)$, and $(g''_t, t_2)\cdot\overrightarrow{(g'_t, t_2)}\cdot(g_t, t_2)$. $\square$

5 Additional open problems

In this final section, we present additional problems as possible avenues for future research.

In [8, 14], the problem of maximizing the number of pegs on the graph (under the caveat that the player makes a jump whenever possible) is explored. This is called the *fool's solitaire* problem. In this variation, there are two natural analogs to the fool's solitaire problem. The first is to try and maximize $|T_1 \cup T_2|$. The second is to maximize the "weight" of the pegs, i.e., maximize $|T_1| + 2|T_2|$.

A natural extension is to consider k-jump peg solitaire on graphs. However, the solvability of many of these cases would be implied by smaller values of k. For example, any graph solvable in the single jump variation would also be solvable in the triple jump variant. To see this, replace any jump of the form $a\cdot\overrightarrow{b}\cdot c$ with the jumps $a\cdot\overrightarrow{b}\cdot c$, $c\cdot\overrightarrow{b}\cdot a$, and $a\cdot\overrightarrow{b}\cdot c$. Therefore, it is of interest to find graphs that are solvable in triple jump that are not solvable in single jump.

To continue on with this thread, we may also consider an arbitrary starting state $S = (S_0, S_1, ..., S_k)$, where S_0 is the set of vertices with holes and S_i is the set of vertices with i-pegs. Given a graph G and an initial configuration S on the vertices of G, is the graph solvable from this configuration?

Finally, we could consider a variation in which 1-pegs are allowed to jump into other 1-pegs, creating a 2-peg. We could also consider a variation in which the "top peg" of a 2-peg is allowed to jump, leaving a 1-peg behind.

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