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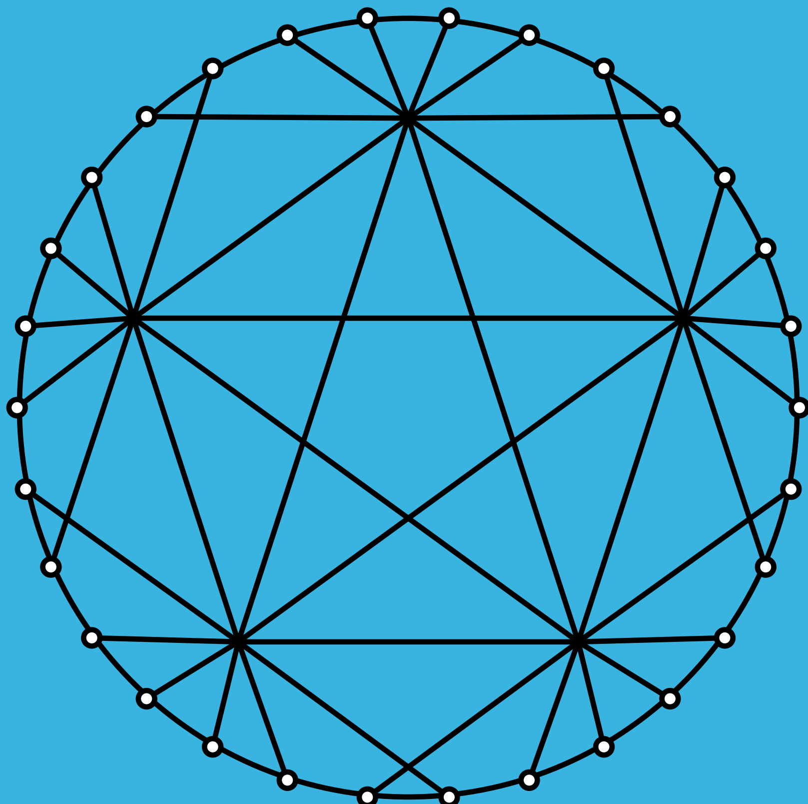

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Regular dominating functions in regular graphs

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Abstract

A vertex v in a graph G is said to dominate a vertex u if either $u = v$ or $uv \in E(G)$. A set S of vertices in G is a dominating set of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the domination number $\gamma(G)$ of G. Domination has also been looked at in terms of functions. A function $f : V(G) \to \{0, 1\}$ is a dominating function of G if $c_f(v) = \sum_{u \in N[v]} f(u) \geq 1$ for every vertex v of G. For a dominating function f of G, let $\gamma(f) = \sum_{v \in V(G)} f(v)$. Thus, the domination number of G is $\gamma(G) = \min\{\gamma(f) : f$ is a dominating function of $G\}$. We use dominating functions to investigate graphs all of whose vertices can be dominated by the same number of vertices. In this paper, our emphasis is on regular graphs having this property.

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1 Introduction

A vertex v in a graph G is said to *dominate* a vertex u if either $u = v$ or $uv \in E(G)$. That is, a vertex v dominates the vertices in its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the *domination number* $\gamma(G)$ of G. We refer to the books [1, 2, 4] for graph theory notation and terminology not described in this paper.

There is another way that domination and the domination number of a graph G has been looked at (see [1, 6, 8]). Each function $f : V(G) \to \{0, 1\}$ gives rise to another function $c_f : V(G) \to \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers, defined by $c_f(v) = \sum_{u\in N[v]} f(u)$. If $c_f(v) \in \mathbb{N}$ for every vertex v of G, then f is a *dominating function* of G. Dominating functions have been studied extensively by many (see [3, 6, 7, 8], for example). If f is a dominating function of G, then the set $\mathcal{I}_f(G) = \{v \in V(G) : f(v) = 1\}$ is a dominating set of G. (The set $\mathcal{I}_f(G)$ is also denoted by $\mathcal{I}_f$ if the graph G under consideration is understood.) The *domination number* $\gamma(f)$ *of a dominating function* f of a graph G is $\gamma(f) = \sum_{v\in V(G)} f(v) = |\mathcal{I}_f|$ and so the domination number $\gamma(G)$ of G can be defined as

$$\gamma(G) = \min\{\gamma(f) : \ f \text{ is a dominating function of } G\}.$$

If f is a dominating function of a graph G and $c_f(v)$ equals the same positive integer k for every vertex v of G, then f is called a *regular* (or k*-regular*) *dominating function* of G. Consequently, if G has a k-regular dominating function, then there is a dominating set S of G such that every vertex of G is dominated by exactly k vertices of S. Since a vertex v whose degree $\deg v$ is the minimum degree $\delta(G)$ of G can be dominated by at most $1 + \delta(G)$ vertices of S, it follows that $1 \le k \le 1 + \delta(G)$. For example, Figures 1(a) and 1(b) show two dominating functions f and g of a tree T, respectively. For the dominating function f of T in Figure 1(a), $c_f(v) = 2$ for every vertex v of T; while for the dominating function g of T in Figure 1(b), $c_g(v) = 1$ for every vertex v of T. Thus, f is 2-regular and g is 1-regular.

While the functions f_1 and f_2 defined on the graph H in Figures 2(a) and 2(b), respectively, are both dominating functions, neither f_1 nor f_2 is a regular dominating function. In fact, this graph H has no regular dominating functions. To see this, assume, to the contrary, that H has a regular dominating function $f : V(H) \to \{0, 1\}$ such that f gives rise to a constant function $c_f : V(G) \to \mathbb{N} \cup \{0\}$. Since $c_f(r) = f(r) + f(s) + f(w)$ and $c_f(w) = f(r) + f(s) + f(w) + f(x)$, it follows that $f(x) = 0$. Similarly,

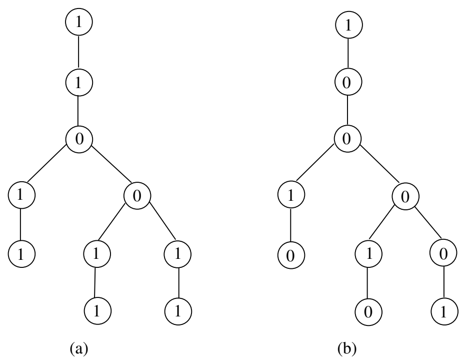

Figure 1: Two dominating functions of a tree

$f(t) = 0$. Since $c_f(v) = f(v) + f(z)$ and $c_f(z) = f(v) + f(z) + f(u) + f(y)$, it follows that $f(u) = f(y) = 0$. Now, the fact that $c_f(x) = c_f(w)$ forces $f(r) = f(s) = 0$. However then, $c_f(t) = 0$, which is impossible.

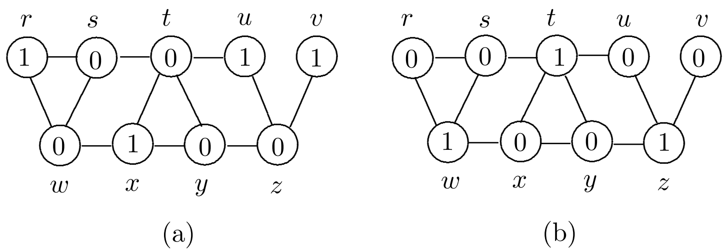

Figure 2: Two dominating functions of a graph H

2 Properties of regular dominating functions

We begin with some preliminary results dealing with regular dominating functions of graphs. For a dominating function f of a nontrivial connected graph G, let

$$\mathcal{I}_f(G) = \{v \in V(G) : \ f(v) = 1\}$$

be the set of all vertices x of G for which $f(x) = 1$. Thus, $\overline{\mathcal{I}}_f(G) = V(G) - \mathcal{I}_f(G)$ is the set of all vertices y of G for which $f(y) = 0$. The sum $\sum_{v \in V(G)} c_f(v)$ counts $f(v) = 1$ exactly $\deg v + 1$ times, once in $c_f(v)$ and once in $c_f(u)$ for each $u \in N(v)$. Consequently, we have the following observation.

Observation 2.1. *Let G be a nontrivial connected graph and let $f : V(G) \to \{0,1\}$ be a dominating function of G. Then*

$$\sum_{v\in V(G)} c_f(v) = \sum_{x\in \mathcal{I}_f(G)} (\deg x + 1) f(x) = \sum_{x\in \mathcal{I}_f(G)} (\deg x + 1).$$

The following is an immediate consequence of Observation 2.1.

Proposition 2.2. *Let G be a nontrivial connected graph of order n and let $f : V(G) \to \{0,1\}$ be a dominating function of G where $|\mathcal{I}_f(G)| = s$.*

(1) *If G is an r-regular graph for some integer $r \geq 2$, then*

$$\textstyle\sum_{v\in V(G)} c_f(v) = (r+1)s.$$

(2) *If f is a k-regular dominating function for some integer $k \geq 1$, then*

$$\textstyle\sum_{v\in V(G)} c_f(v) = nk.$$

Proposition 2.3. *Let G be a nontrivial connected graph. Then G has a k-regular dominating function for some positive integer k if and only if either G is $(k-1)$-regular or G consists of two induced vertex-disjoint subgraphs F and H, where the vertex set of G is partitioned into $V(F)$ and $V(H)$, such that F is $(k-1)$-regular and each vertex of H is adjacent to exactly k vertices in F.*

Proof. First, suppose that G has a k-regular dominating function $f : V(G) \to \{0,1\}$ for some positive integer k. Let $F = G[\mathcal{I}_f]$ and $H = G[\overline{\mathcal{I}}_f]$ (if $\overline{\mathcal{I}}_f \neq \emptyset$). Since $c_f(v) = k$ for each $v \in V(G)$, it follows that every vertex in F is adjacent to exactly $k-1$ vertices in F and so F is $(k-1)$-regular and every vertex in H is adjacent to exactly k vertices in F. Next, we verify the converse. Since the statement is true if G is $(k-1)$-regular, we may assume that G is constructed from two induced vertex-disjoint subgraphs F and H such that F is $(k-1)$-regular and each vertex of H is adjacent to exactly k vertices in F. Then a k-regular dominating function of G can be defined by assigning 0 to each vertex of H and assigning 1 to each vertex of F. $\square$

The following useful observations are consequences of Proposition 2.3. The distance between two vertices u and v in a connected graph is denoted by $d(u,v)$.

Corollary 2.4. *A nontrivial connected graph G has a 1-regular dominating function if and only if G has a dominating set W such that $d(u,v) \geq 3$ for every two vertices u and v of W.*

Corollary 2.5. *Let f be a 2-regular dominating function of a nontrivial connected graph G. Then the edge set of the subgraph $G[\mathcal{I}_f]$ induced by $\mathcal{I}_f$ is a matching of G and so $|\mathcal{I}_f|$ is even.*

We now apply the results obtained above to study regular dominating functions in the grid graph $P_n \,\square\, K_2$ (the Cartesian product of P_n and K_2) for $n \geq 2$. By Corollaries 2.4 and 2.5, we see that, while the 4-cycle $C_4 = P_2 \,\square\, K_2$ has a 3-regular dominating function, it has neither a 1-regular nor a 2-regular dominating function.

Proposition 2.6. *For each integer $n \geq 3$, the grid graph $P_n \,\square\, K_2$ has a k-regular dominating function for $k \in \{1,2\}$ if and only if n is odd. Furthermore, $P_n \,\square\, K_2$ has no 3-regular dominating function for any integer $n \geq 3$.*

Proof. Let $G = P_n \,\square\, K_2$ be constructed from the two copies (u_1, u_2, …, u_n) and (v_1, v_2, …, v_n) of the path P_n of order n by adding the edges $u_i v_i$ for $1 \leq i \leq n$. First, we show that if $n \geq 4$ is even, then G does not have a k-regular dominating function for each $k \in \{1,2\}$. We consider two cases.

Case 1. $\boldsymbol{k = 1}$. Suppose that $f : V(G) \to \{0,1\}$ is a 1-regular dominating function of G. Observe that exactly one of u_1 and v_1 must be assigned 1 by f, for otherwise, if $f(u_1) = f(v_1) = 1$, then $c_f(u_1) \geq 2$; while if $f(u_1) = f(v_1) = 0$, then $f(u_2) = f(v_2) = 1$ and so $c_f(u_2) \geq 2$, which is impossible in either case. Similarly, exactly one of u_n and v_n must be assigned 1 by f. Let $\mathcal{I}_f = \{v \in V(G) : \ f(v) = 1\}$. Then $s = |\mathcal{I}_f| \geq 2$. By Observation 2.1,

$$2n = \sum_{v \in V(G)} c_f(v) = \sum_{v \in \mathcal{I}_f} (\deg(v) + 1) = 3 \cdot 2 + 4(s-2).$$

Consequently, $n = 3 + 2(s-2)$ is odd.

Case 2. $\boldsymbol{k = 2}$. Suppose that $f : V(G) \to \{0,1\}$ is a 2-regular dominating function of G. Observe that both of u_1 and v_1 must be assigned 1 by f, for otherwise, suppose that $f(u_1) = 0$. Then $f(v_1) = f(u_2) = f(v_2) = 1$ and so $c_f(v_2) \geq 3$, which is impossible. Similarly, both of

u_n and v_n must be assigned 1 by f. Then $s = |\mathcal{I}_f| \geq 4$. Since the edge set of $G[\mathcal{I}_f]$ is a matching, s is even. By Observation 2.1,

$$4n = \sum_{v \in V(G)} c_f(v) = \sum_{v \in \mathcal{I}_f} (\deg(v) + 1) = 3 \cdot 4 + 4(s - 4).$$

Consequently, $n = 3 + (s - 4)$ is odd.

For the converse, assume that $n \geq 3$ is odd. We show that there is both a 1-regular and a 2-regular dominating function of G. Define $f_1 : V(G) \to \{0,1\}$ by $f_1(u_i) = 1$ for $i \equiv 1 \pmod 4$ and $f_1(v_j) = 0$ for $j \equiv 3 \pmod 4$ and let $f_1(w) = 0$ for any other vertex $w \in V(G)$. Define $f_2 : V(G) \to \{0,1\}$ by $f_2(u_i) = f_2(v_i) = 1$ if i is odd and $f_2(u_i) = f_2(v_i) = 0$ if i is even. Then f_1 is a 1-regular dominating function of G and f_2 is a 2-regular dominating function of G.

Next, we show that G does not have 3-regular dominating function for each integer $n \geq 3$. Assume, to the contrary, that there is an integer $n \geq 3$ such that $G = P_n \,\square\, K_2$ has a 3-regular dominating function $g : V(G) \to \{0,1\}$. Since $c_g(u_1) = c_g(v_1) = 3$, it follows that $g(x) = 1$ for each $x \in \{u_1, u_2, v_1, v_2\}$. This forces $g(u_3) = g(v_3) = 0$ so that $c_g(u_3) \leq 2$, which is impossible. $\square$

The following lemma will be useful in studying regular dominating functions in connected graphs containing many vertices with the same neighborhood.

Lemma 2.7. *Let G be a nontrivial connected graph and f a regular dominating function of G. If $u, v \in V(G)$ such that $N(u) = N(v)$, then $f(u) = f(v)$.*

Proof. Let $u, v \in V(G)$ such that $N(u) = N(v) = X$. Since f is a regular dominating function of G, it follows that $c_f(u) = f(u) + \sum_{x \in X} f(x) = f(v) + \sum_{x \in X} f(x) = c_f(v)$ and so $f(u) = f(v)$. $\square$

With the aid of Lemma 2.7, we now present a result on regular dominating functions of complete multipartite graphs.

Proposition 2.8. *A complete multipartite graph G has a regular dominating function if and only if either G is regular or $\gamma(G) = 1$.*

Proof. Let G be a complete multipartite graph of order n. Thus, G has two or more partite sets. Suppose, first, that G is r-regular for some integer $r \geq 1$. Then G has an $(r+1)$-regular dominating function. If $\gamma(G) = 1$, then G contains a vertex that is adjacent to every other vertex in G and so $\Delta(G) = n - 1$. Thus, G has a 1-regular dominating function. For the converse, suppose that G has a k-regular dominating function for some positive integer k. Assume, to the contrary, that G is not regular and $\Delta(G) \neq n - 1$, but G has a regular dominating function f. Thus, every partite set of G contains at least two vertices and there are two partite sets U and W such that $|U| \neq |W|$. Let $X = V(G) - (U \cup W)$ and $A = \sum_{x \in X} f(x)$ (where $A = 0$ if $X = \emptyset$). As a consequence of Lemma 2.7, we consider two cases.

Case 1. $\boldsymbol{f(u) \neq f(w)}$ **when** $\boldsymbol{u \in U}$ **and** $\boldsymbol{w \in W}$**.**

We may assume that $f(u) = 1$ for all $u \in U$ and $f(w) = 0$ for all $w \in W$. If $u \in U$ and $w \in W$, then $c_f(u) = 1 + A$ and $c_f(w) = |U| + A$. Since $|U| \geq 2$, it follows that $c_f(u) \neq c_f(w)$, which is impossible.

Case 2. $\boldsymbol{f(u) = f(w)}$ **when** $\boldsymbol{u \in U}$ **and** $\boldsymbol{w \in W}$**.**

First, suppose that $f(v) = 1$ for all $v \in U \cup W$. If $u \in U$ and $w \in W$, then $c_f(u) = 1 + |W| + A$ and $c_f(w) = 1 + |U| + A$. Since $|U| \neq |W|$, it follows that $c_f(u) \neq c_f(w)$, which is impossible. Next, suppose $f(v) = 0$ for all $v \in U \cup W$. Then $X \neq \emptyset$ and so $A > 0$. Since $c_f(v) = A > 0$ for each $v \in U \cup W$, there is a partite set Y of G such that $f(y) = 1$ for each $y \in Y$. If $y \in Y$, then $c_f(y) = 1 + (A - |Y|)$. Since $|Y| \geq 2$, it follows that $c_f(v) \neq c_f(y)$ for each $v \in U \cup W$ and $y \in Y$, which is a contradiction. □

The following two results describe certain properties possessed by 1-regular dominating functions.

Theorem 2.9. *Let G be a nontrivial connected graph and suppose that f_1 and f_2 are 1-regular dominating functions of G, where $\mathcal{I}_{f_i} = \{v \in V(G) : f_i(v) = 1\}$ for $i = 1, 2$. Then $|\mathcal{I}_{f_1}| = |\mathcal{I}_{f_2}|$, that is, the number of vertices assigned 1 by a 1-regular dominating function of G is unique.*

Proof. Observe that $\mathcal{I}_{f_1}$ and $\mathcal{I}_{f_2}$ are independent dominating sets in G. Additionally, every vertex in $V(G) - \mathcal{I}_{f_1}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_1}$ and every vertex in $V(G) - \mathcal{I}_{f_2}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_2}$. Since $\mathcal{I}_{f_1}$ and $\mathcal{I}_{f_2}$ are independent sets, each vertex in $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_2} - \mathcal{I}_{f_1}$ and each vertex in $\mathcal{I}_{f_2} - \mathcal{I}_{f_1}$ is adjacent to exactly one vertex in $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$. This implies that the set of edges between $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$ and $\mathcal{I}_{f_1} - \mathcal{I}_{f_2}$ form a matching between these sets. It therefore follows that $|\mathcal{I}_{f_1} - \mathcal{I}_{f_2}| = |\mathcal{I}_{f_2} - \mathcal{I}_{f_1}|$. Since $|\mathcal{I}_{f_1}| = |\mathcal{I}_{f_1} - \mathcal{I}_{f_2}| + |\mathcal{I}_{f_1} \cap \mathcal{I}_{f_2}|$ and $|\mathcal{I}_{f_2}| = |\mathcal{I}_{f_2} - \mathcal{I}_{f_1}| + |\mathcal{I}_{f_1} \cap \mathcal{I}_{f_2}|$, it follows that $|\mathcal{I}_{f_1}| = |\mathcal{I}_{f_2}|$. $\square$

Theorem 2.10. *A nontrivial connected graph G has a 1-regular dominating function if and only if there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ where $|V_i| = n_i$ for $1 \le i \le t$ such that $K_{1,n_i-1} \subseteq G[V_i]$ and the center of K_{1,n_i-1} has degree $n_i - 1$ in G.*

Proof. First, suppose that G has a 1-regular dominating function $f : V(G) \to \{0,1\}$. Let $\mathcal{I}_f = \{v_1, v_2, \ldots, v_t\}$ for some positive integer t. Since $c_f(v) = 1$ for all $v \in V(G)$, it follows that $f(u) = 0$ for each $u \in N(v_i)$. For $1 \le i \le t$, let $V_i = N[v_i]$ be the closed neighborhood of v_i. Then $|V_i| = 1 + \deg_G v_i = n_i$ and $K_{1,n_i-1} \subseteq G[V_i]$ for $1 \le i \le t$. Furthermore, $\{V_1, V_2, \ldots, V_t\}$ is a partition of $V(G)$.

For the converse, suppose that $\{V_1, V_2, \ldots, V_t\}$ is a partition of $V(G)$ such that $|V_i| = n_i$, $K_{1,n_i-1} \subseteq G[V_i]$, and the center v_i of K_{1,n_i-1} has degree $n_i - 1$ in G for $1 \le i \le t$. Define the function $f : V(G) \to \{0,1\}$ by $f(v_i) = 1$ for $1 \le i \le t$ and $f(u) = 0$ otherwise. We show that f is a 1-regular dominating function of G. Let $x \in V(G)$. Then $x \in V_i$ for some integer i with $1 \le i \le t$. If $x = v_i$, then $f(x) = 1$ and $f(y) = 0$ for each $y \in N(x) = V_i - \{x\}$, implying that $c_f(x) = 1$. If $x \ne v_i$, then x is adjacent to v_i and x is not adjacent to v_j for each integer j with $1 \le j \le t$ and $j \ne i$, which implies that $c_f(x) = 1$. Consequently, f is a 1-regular dominating function of G. $\square$

By Theorem 2.9, the cardinality of a dominating set W of a graph in Corollary 2.4 or the size t of a partition of the vertex set of a graph in Theorem 2.10 is unique. On the other hand, the partition of the vertex set in Theorem 2.10 may not be unique. For example, the graph G of order 8 in Figure 3 has two distinct 1-regular dominating functions, which give rise to two distinct partitions of $V(G)$ as described in Theorem 2.10.

3 Regular graphs

We saw that if G is a connected r-regular graph with a k-regular dominating function, then $1 \le k \le r + 1$. Furthermore, every connected r-regular

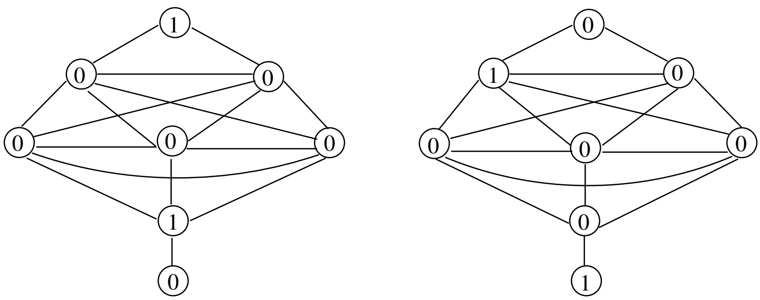

Figure 3: The partition of the vertex set in Theorem 2.10 is not unique

graph G has an $(r+1)$-regular dominating function, while there is no guarantee that G has a k-regular dominating function for a given integer k with $1 \leq k \leq r$. In this section, we study those connected r-regular graphs having a k-regular dominating function for a given integer k with $1 \leq k \leq r$.

Let G be a nontrivial connected graph and let $f : V(G) \to \{0,1\}$ be a function of G. The *complementary function* $\overline{f} : V(G) \to \{0,1\}$ is defined by

$$\overline{f}(v) = 1 - f(v) \text{ for every vertex } v \text{ of } G.$$

Observation 3.1. *Let G be a nontrivial connected graph. If $f : V(G) \to \{0,1\}$ is a function of G, then $c_f(v) + c_{\overline{f}}(v) = 1 + \deg v$ for each vertex v of G.*

Proof. Let $v \in V(G)$ where $f(v) = i \in \{0,1\}$ and $c_f(v) = k$. Then $\overline{f}(v) = 1 - i$. Since $c_f(v) = k$ and $f(v) = i$, it follows that v is adjacent to $k - i$ vertices labeled 1 by f and so v is adjacent to $\deg v - k + i$ vertices labeled 0 by f. Thus, v is adjacent to $\deg v - k + i$ vertices labeled 1 by $\overline{f}$. Hence, $c_{\overline{f}}(v) = (1-i)+(\deg v - k + i) = 1 + \deg v - k$. Consequently, $c_f(v) + c_{\overline{f}}(v) = \deg v + 1$. $\square$

The complementary function of a dominating function of a graph may or may not be a dominating function of the graph. For example, for the dominating function f of the tree T in Figure 1(a), its complementary function is not a dominating function of T; while for the dominating function g of the tree T in Figure 1(b), its complementary function is a dominating function of T. The following is an immediate consequence of Observation 3.1.

Corollary 3.2. *Let G be a nontrivial connected graph. Suppose that f is a dominating function of G such that $\overline{f}$ is also a dominating function of G.*

Then f and $\overline{f}$ are both regular if and only if G is regular. Furthermore, if G is an r-regular graph and f is a k-regular dominating function of G where $1 \leq k \leq r+1$, then $\overline{f}$ is an $(r+1-k)$-regular dominating function of G.

For example, Figure 4(a) shows a 1-regular dominating function f of C_6. Its complementary function $\overline{f}$, shown in Figure 4(b), is a 2-regular dominating function of C_6. This example illustrates the following result concerning connected 2-regular graphs, namely cycles. Since a cycle C_n of order $n \geq 3$ is 2-regular, it has a 3-regular dominating function. We now determine other values of k for which cycles have a k-regular dominating function.

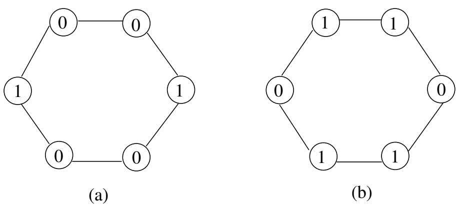

Figure 4: Two regular dominating functions of C_6

Proposition 3.3. *For $k \in \{1,2\}$, a cycle C_n of order $n \geq 3$ has a k-regular dominating function if and only if $n \equiv 0 \pmod 3$.*

Proof. First, suppose that C_n has a k-regular dominating function f with $k \in \{1,2\}$. We show that $n \equiv 0 \pmod 3$. Assume that $|\mathcal{I}_f| = s$. By Proposition 2.2, $3s = nk$. If $k = 1$, then $3s = n$; while if $k = 2$, then $3s = 2n$. In either case, $3 \mid n$ and so $n \equiv 0 \pmod 3$. For the converse, suppose that $n \equiv 0 \pmod 3$. We show that C_n has a k-regular dominating function for each $k \in \{1,2\}$. For $C_n = (v_1, v_2, \ldots, v_n, v_1)$, define a 1-regular dominating function $f : V(G) \to \{0,1\}$ by

$$f(v_i) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{f} : V(G) \to \{0,1\}$ is a 2-regular dominating function of C_n by Corollary 3.2. $\square$

Next, we consider cubic (3-regular) connected regular graphs. As we saw, every such graph has a 4-regular dominating function. In order to characterize those connected cubic graphs having a k-regular dominating function for each $k \in \{1,2,3\}$, we first present a necessary condition on the order of such cubic graphs.

Proposition 3.4. *Let G be a connected cubic graph of order $n \geq 4$. If G has a k-regular dominating function for some integer $k \in \{1, 2, 3\}$, then $n \equiv 0 \pmod 4$.*

Proof. Let G be a connected cubic graph of order n and let $f : V(G) \to \{0, 1\}$ be a k-regular dominating function where $k \in \{1, 2, 3\}$. Since n is even, $n = 2p$ for some integer $p \geq 2$. Suppose that $|\mathcal{I}_f| = s$. Then $4s = nk = 2pk$ and so $2s = pk$ by Proposition 2.2. If $k = 1$, then $2s = p$ and so $n = 4s$; if $k = 2$, then $s = p$ is even and so $n = 2s$; and if $k = 3$, then $2s = 3p$ and so $3 \mid s$. Let $s = 3r$ for some positive integer r. Then $n = 4r$. □

Consequently, the famous Petersen graph of order 10 has a k-regular dominating function only if $k = 4$. By Proposition 3.4, we consider only those connected cubic graphs of order n with $n \equiv 0 \pmod 4$ and determine which such graphs have a k-regular dominating function for some integer $k \in \{1, 2, 3\}$. The following is a consequence of Theorem 2.10.

Corollary 3.5. *Let G be a connected cubic graph of order $n = 4t$ for some positive integer t. Then G has a 1-regular dominating function if and only if there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ such that $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$.*

The following observation is a consequence of Corollary 3.5 (and Corollary 3.2).

Observation 3.6. *For a connected cubic graph G of order $n \geq 4$ where $n \equiv 0 \pmod 4$, a function $f : V(G) \to \{0, 1\}$ is a 1-regular dominating function of G if and only if its complementary function $\overline{f} : V(G) \to \{0, 1\}$ is a 3-regular dominating function of G.*

By Corollary 3.5 and Observation 3.6, we then have the following.

Corollary 3.7. *Let G be a connected cubic graph of order $n = 4t$ for some positive integer t. Then G has a 3-regular dominating function if and only if there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ such that $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$.*

Next, we characterize connected cubic graphs having a 2-regular dominating function. For two disjoint sets U and W of vertices in a graph G, let $[U, W]$ be the set of edges joining a vertex of U and a vertex of W in G. The bipartite subgraph of G induced by the set $[U, W]$ of edges in G is denoted by $G[U, W]$.

Theorem 3.8. *Let G be a connected cubic graph of order $n = 4t$ for some positive integer t. Then G has a 2-regular dominating function if and only if there exists a partition $\{U, W\}$ of $V(G)$ where $|U| = |W| = 2t$ such that $G[U] \cong G[W] \cong tK_2$ and $G[U, W]$ is a 2-regular subgraph of G.*

Proof. First, suppose that G has a 2-regular dominating function $f : V(G) \to \{0, 1\}$. Let $U = \mathcal{I}_f$ and $W = V(G) - U$. Then the edge set of $G[U]$ is a matching. Let $w \in W$. Since $c_f(w) = 2$, it follows that w is adjacent to exactly two vertices of U and exactly one vertex in W. This implies that $G[W]$ is a 1-regular subgraph of G and $G[U, W]$ is a 2-regular bipartite subgraph of G. Consequently, $|U| = |W| = 2t = n/2$ and $G[U] \cong G[W] \cong tK_2$.

For the converse, suppose that there exists a partition $\{U, W\}$ of $V(G)$ where $|U| = |W| = 2t$ such that $G[U] \cong G[W] \cong tK_2$ and $G[U, W]$ is a 2-regular subgraph of G. Then the function $f : V(G) \to \{0, 1\}$ defined by $f(u) = 1$ for each $u \in U$ and $f(w) = 0$ for each $w \in W$ is a 2-regular dominating function of G. $\square$

The following is a consequence of Theorem 3.8. A 1-*factor* in a graph G is a 1-regular spanning subgraph of G.

Corollary 3.9. *If G is a connected cubic graph with a 2-regular dominating function, then G has a 1-factor.*

The converse of Corollary 3.9 is not true, however. In fact, there is an infinite class of connected cubic graphs having a 1-factor but no 2-regular dominating functions, as we show next. For an integer $\ell \geq 3$, let $C_\ell = (z_1, z_2, \ldots, z_\ell, z_1)$ be the cycle of order ℓ. For $1 \leq i \leq \ell$, let L_i be the graph of order 5 obtained from the 5-cycle $(u_i, v_i, w_i, x_i, y_i, u_i)$ by adding two chords $u_i x_i$ and $w_i y_i$. The cubic graph F_ℓ of order 6ℓ is constructed by joining v_i to z_i for $1 \leq i \leq \ell$. The graph F_4 of order 24 is shown in Figure 5.

For each integer $\ell \geq 3$, the graph F_ℓ has a 1-factor. If $\ell \geq 3$ is odd, then $6\ell \not\equiv 0 \pmod 4$ and so it follows by Proposition 3.4 that F_ℓ does not have any k-regular dominating function for each $k \in \{1, 2, 3\}$. If $\ell \geq 4$ is even, then $6\ell \equiv 0 \pmod 4$. We show, even in this case, that F_ℓ does not have any k-regular dominating function for each $k \in \{1, 2, 3\}$.

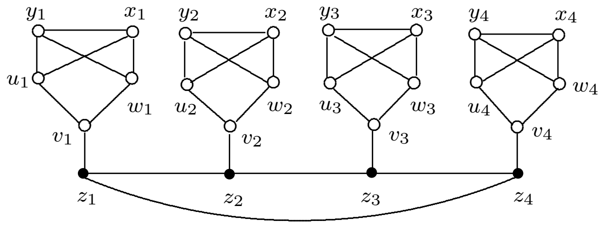

Figure 5: The cubic graph F_4

Theorem 3.10. *For each even integer $\ell \geq 4$, the graph F_ℓ of order $6\ell \equiv 0 \pmod 4$ has no k-regular dominating function for any $k \in \{1, 2, 3\}$.*

Proof. First, we show that F_ℓ does not have a 1-regular dominating function. Assume, to the contrary, that there is an even integer $\ell \geq 3$ such that F_ℓ has a 1-regular dominating function $f : V(F_\ell) \to \{0, 1\}$. Since $N(u_i) = N(w_i)$ and $v_i \in N(u_i) \cap N(w_i)$ for $1 \leq i \leq \ell$, it follows by Lemma 2.7 that $f(u_i) = f(w_i) = 0$. In particular, $f(u_1) = f(w_1) = 0$. Hence, exactly one of v_1, x_1, y_1 has the f-value 1. If $f(v_1) = 1$, then $f(x_1) = f(y_1) = 0$ and so $c_f(x_1) = c_f(y_1) = 0$, which is impossible. Thus, we may assume that $f(x_1) = 1$ and $f(y_1) = f(v_1) = 0$. This forces $f(z_1) = 1$ and so $f(z_2) = 0$. Since $f(z_2) = f(u_2) = f(w_2) = 0$, it follows that $f(v_2) = 1$. However then, $f(x_2) = f(y_2) = 0$ and so $c_f(x_2) = c_f(y_2) = 0$, which is impossible. Consequently, F_ℓ has no 1-regular dominating function and so F_ℓ has no 3-regular dominating function by Observation 3.6.

Next, we show that that F_ℓ does not have a 2-regular dominating function. Assume, to the contrary, that there is an even integer $\ell \geq 3$ such that F_ℓ has a 2-regular dominating function $g : V(H_\ell) \to \{0, 1\}$. Again, it follows by Lemma 2.7 that $g(u_i) = g(w_i)$ for $1 \leq i \leq \ell$. In particular, $g(u_1) = g(w_1)$.

- ⋆ First, suppose that $g(u_1) = g(w_1) = 1$. Since v_1 is adjacent to u_1 and w_1 and $c_f(v_1) = 2$, it follows that $g(v_1) = g(z_1) = 0$. This forces $g(z_2) = g(z_\ell) = 1$. Since $c_g(v_2) = 2$, it follows that $g(u_2) = g(w_2) = 0$ and $g(v_2) = 1$. Since $c_g(u_2) = c_g(w_2) = 2$, it follows that $\{g(y_2), g(x_2)\} = \{0, 1\}$. We may assume that $g(y_2) = 1$ and $g(x_2) = 0$. However then, $c_g(x_2) = 1$, which is impossible.

- ⋆ Next, suppose that $g(u_1) = g(w_1) = 0$. Since $c_g(v_1) = 2$, it follows that $f(v_1) = f(z_1) = 1$. Since $c_g(u_1) = c_g(w_1) = 2$, it follows that $\{g(y_1), g(x_1)\} = \{0, 1\}$. We may assume that $g(y_1) = 1$ and $g(x_1) = 0$. However then, $c_g(x_1) = 1$, which is impossible.

Therefore, F_ℓ does not have a 2-regular dominating function. □

By Theorem 3.10, there are connected cubic graphs of order n with $n \equiv 0 \pmod 4$, without any k-regular dominating function for each $k \in \{1, 2, 3\}$. On the other hand, for each integer $n \geq 4$ with $n \equiv 0 \pmod 4$, there is a connected cubic graph of order n with k-regular dominating function for each $k \in \{1, 2, 3\}$.

Theorem 3.11. *For each integer $n \geq 4$ with $n \equiv 0 \pmod 4$, there is a connected cubic graph of order n with a k-regular dominating function for each $k \in \{1, 2, 3\}$.*

Proof. Let $n = 4t$ for some positive integer t. Since K_4 is the only cubic graph of order 4 and K_4 has a k-regular dominating function for each $k \in \{1, 2, 3\}$, we may assume that $n \geq 8$. We begin with the n-cycle

$$C_n = (v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}, v_{2,1}, v_{2,2}, v_{2,3}, v_{2,4}, \ldots, v_{t,1}, v_{t,2}, v_{t,3}, v_{t,4}, v_{1,1}).$$

The cubic graph G is constructed by adding the edges $v_{i,1}v_{i,3}$ and $v_{i,2}v_{i,4}$ for $1 \leq i \leq t$. That is, if $V_i = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}\}$ for $1 \leq i \leq t$, then $G[V_i] = K_4 - v_{i,1}v_{i,4}$.

- ⋆ For $k = 1, 3$, let $\{V_1, V_2, \ldots, V_t\}$ be a partition of $V(G)$. Since $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i] = K_4 - e$ for $1 \leq i \leq t$, it follows by Corollary 3.5 that G has a k-regular dominating function for $k = 1, 3$. In fact, a 1-regular dominating function $f_1 : V(G) \to \{0, 1\}$ can be defined by $f_1(v_{i,2}) = 1$ for $1 \leq i \leq t$ and $f(x) = 0$ otherwise. Then $f_3 = \overline{f}_1$ is a 3-regular dominating function of G.

- ⋆ For $k = 2$, let $U = \{v_{i,1}, v_{i,2} : 1 \leq i \leq t\}$ and $W = \{v_{i,3}, v_{i,4} : 1 \leq i \leq t\}$. Then $\{U, W\}$ is a partition of $V(G)$. Since $G[U] \cong G[W] \cong tK_2$ and $G[U, W] \cong C_n$, it follows by Theorem 3.8 that G has a 2-regular dominating function. In fact, a 2-regular dominating function $f_2 : V(G) \to \{0, 1\}$ can be defined by $f_2(u) = 0$ for each $u \in U$ and $f_2(w) = 1$ for each $w \in W$. The complementary function $\overline{f}_2$ of f_2 is a also 2-regular dominating function of G. □

By Theorems 3.10 and 3.11, there are connected cubic graphs of order $n \equiv 0 \pmod 4$ that have no k-regular dominating function for any $k \in \{1, 2, 3\}$ and there are connected cubic graphs of order $n \equiv 0 \pmod 4$ that have k-regular dominating function for each $k \in \{1, 2, 3\}$. We saw that if a connected cubic graph G has a 1-regular dominating function, then G has a 3-regular dominating function and vice versa. However, there are connected cubic graphs (i) having 1-regular or 3-regular dominating functions but no 2-regular dominating functions or (ii) having 2-regular dominating functions but neither a 1-regular nor a 3-regular dominating function. As an example, we consider a class of well-known cubic graphs, namely prisms.

Theorem 3.12. *For $n \geq 3$, let $G = C_n \,\square\, K_2$ be the prism of order $2n$. Then*

(*i*) *G has a 1-regular or 3-regular dominating function only if $n \equiv 0 \pmod 4$ and*

(*ii*) *G has a 2-regular dominating function only if $n \equiv 0 \pmod 2$.*

Proof. Let $G = C_n \,\square\, K_2$ be constructed from two copies $(u_1, u_2, \ldots, u_n, u_1)$ and $(v_1, v_2, \ldots, v_n, v_1)$ of the n-cycle by adding the edges $u_i v_i$ for $1 \leq i \leq n$.

To verify (i), first suppose that $n \equiv 0 \pmod 4$. Then a 1-regular dominating function $f : V(G) \to \{0, 1\}$ of G can be defined by

$$f(w) = \begin{cases} 1 & \text{if } w = u_i \text{ where } i \equiv 1 \pmod 4 \\ & \text{or } w = v_j \text{ where } j \equiv 3 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

By Observation 3.6, its complementary function $\overline{f} : V(G) \to \{0, 1\}$ is a 3-regular dominating function of G.

Conversely, suppose that G has a 1-regular dominating function. By Proposition 3.4, n is even and so G is a bipartite graph. We claim that $n \equiv 0 \pmod 4$. Assume, to the contrary, that $n \equiv 2 \pmod 4$. Thus, $n = 4p + 2$ for some positive integer p and so the order of G is $n_G = 8p+4$. Since G has a 1-regular dominating function, it follows by Theorem 3.5 that there exists a partition $\{V_1, V_2, \ldots, V_t\}$ of $V(G)$ such that $|V_i| = 4$ and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$. Thus, $t = n_G/4 = 2p + 1$. Since G is a bipartite cubic graph and $K_{1,3} \subseteq G[V_i]$ for $1 \leq i \leq t$, it follows that $G[V_i] \cong K_{1,3}$. We may assume that $V_1 = \{u_1, v_1, u_2, u_n\}$. For $1 \leq i \leq t$, let a_i be the subscript of the center of $G[V_i]$. Thus, $a_1 = 1$. We may further assume that

$1 = a_1 < a_2 < \cdots < a_t$. Observe that $a_2 = v_3$, $a_3 = u_5$, $a_4 = v_7$, and so on. In general, $a_i = u_{1+4(i-1)}$ if i is odd and $1 \le i \le t$ and $a_i = v_{3+4(i-1)}$ if i is even and $2 \le i \le t-1$. In particular, $a_t = u_{8p+3} = u_{n-1}$. However then, $u_n \in V_{t-1} \cap V_t$, which is impossible. Thus, (i) holds.

To verify (ii), first suppose that $n \ge 4$ is even. Then a 2-regular dominating function $f : V(G) \to \{0,1\}$ of G can be defined by

$$f(w) = \begin{cases} 1 & \text{if } w \in \{u_i, v_i\} \text{ where } i \text{ is odd} \\ 0 & \text{if } w \in \{u_i, v_i\} \text{ where } i \text{ is even.} \end{cases}$$

Conversely, suppose that G has a 2-regular dominating function. Then $2n \equiv 0 \pmod 4$ by Proposition 3.4 and so n is even. □

The following is a consequence of Theorem 3.12.

Proposition 3.13. *For each integer $n \ge 12$ with $n \equiv 4 \pmod 8$, there is a connected cubic graph of order n that has a 2-regular dominating function but has neither a 1-regular nor a 3-regular dominating function.*

Proof. For $n \ge 12$ with $n \equiv 4 \pmod 8$, let $n = 8p + 4$ for some positive integer p and let $G = C_{4p+2} \,\square\, K_2$ be the prism graph of order n. It then follows by Theorem 3.12 that G has a 2-regular dominating function, but G has neither a 1-regular nor a 3-regular dominating function. □

The cubic graph G of order 16 in Figure 6 has both a 1-regular and 3-regular dominating function but no 2-regular dominating function. The dominating function f of G shown in Figure 6 is 1-regular and $\overline{f}$ is a 3-regular dominating function. It therefore remains to show that G has no 2-regular dominating function. Since G does not have a 1-factor, there is no partition $\{U, W\}$ of $V(G)$ where $|U| = |W| = 8$ such that $G[U] \cong G[W] \cong 4K_2$. Thus, G has no 2-regular dominating function by Theorem 3.8.

While the cubic graph in Figure 6 contains a cut-vertex, there are 2-connected cubic graphs of order n with $n \equiv 0 \pmod 4$ having a 1-regular and a 3-regular dominating function but no 2-regular dominating function. For example, let $C_\ell = (a_1, a_2, \ldots, a_\ell, a_1)$ and $C'_\ell = (b_1, b_2, \ldots, b_\ell, b_1)$ be two copies of the cycle of order $\ell \ge 3$. For $1 \le i \le \ell$, let L_i be the graph of order 6 obtained from the 6-cycle $(u_i, v_i, w_i, x_i, y_i, z_i, u_i)$ by adding two chords $u_i x_i$ and $w_i z_i$. The cubic graph G_ℓ of order 8ℓ is constructed by joining v_i to a_i and joining y_i to b_i for for $1 \le i \le \ell$. The graph G_3 of order 24 is shown in Figure 7.

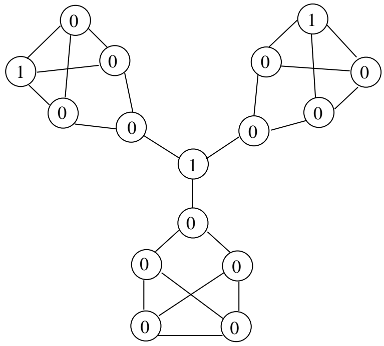

Figure 6: A cubic graph with no 2-regular dominating function

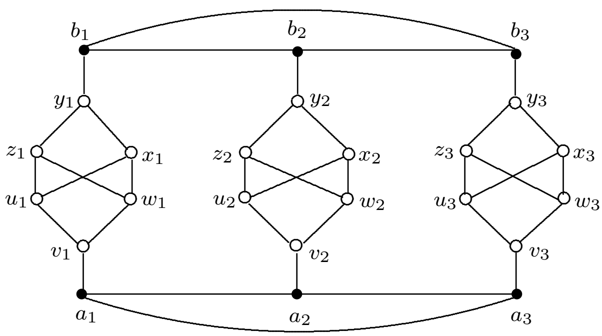

Figure 7: A 2-connected cubic graph G_3 of order 24

Proposition 3.14. *For each integer $\ell \geq 3$, the 2-connected cubic graph G_ℓ of order 8ℓ has 1-regular and 3-regular dominating functions but no 2-regular dominating function.*

Proof. First, a 1-regular dominating function $g : V(G_\ell) \to \{0,1\}$ can be defined such that $g(v_i) = g(y_i) = 1$ for $1 \leq i \leq \ell$ and $g(x) = 0$ otherwise. Then $\overline{g}$ is a 3-regular dominating function of G by Observation 3.6 . It remains to show that H_ℓ does not have a 2-regular dominating function. Assume, to the contrary, that there is an integer $\ell \geq 3$ such that G_ℓ has a 2-regular dominating function $f : V(G_\ell) \to \{0,1\}$. Since $N(u_i) = N(w_i)$ and $N(x_i) = N(z_i)$ for $1 \leq i \leq \ell$, it follows by Lemma 2.7 that $f(u_i) = f(w_i)$ and $f(x_i) = f(z_i)$. In particular, $f(u_1) = f(w_1)$ and $f(x_1) = f(z_1)$.

- ⋆ First, suppose that $f(u_1) = f(w_1) = 1$. Since v_1 is adjacent to u_1 and w_1 and $c_f(v_1) = 2$, it follows that $f(v_1) = f(a_1) = 0$. Since $c_f(a_1) = 2$, it follows that $f(a_2) = f(a_\ell) = 1$. If $f(v_2) = 1$, then $f(u_2) = f(w_2) = f(x_2) = f(z_2) = 0$. However then, $c_f(u_2) = c_f(w_2) = 1$, which is impossible. Thus, $f(v_2) = 0$ and $f(u_2) = f(w_2) \in \{0, 1\}$. If $f(u_2) = f(w_2) = 0$, then $c_f(v_2) = 1$; while if $f(u_2) = f(w_2) = 1$, then $c_f(v_2) = 3$. A contradiction is produced in either case.

- ⋆ Next, suppose that $f(u_1) = f(w_1) = 0$. Since v_1 is adjacent to u_1 and w_1 and $c_f(v_1) = 2$, it follows that $f(v_1) = f(a_1) = 1$. This forces $f(x_1) = f(z_1) = 0$. However then, $c_f(u_1) = c_f(w_1) = 1$, which is impossible.

Therefore, G_ℓ does not have a 2-regular dominating function. □

We are therefore left with the following question: *Does there exist a* 3*-connected cubic graph having a* 1*-regular and a* 3*-regular dominating function but no* 2*-regular dominating function?*

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