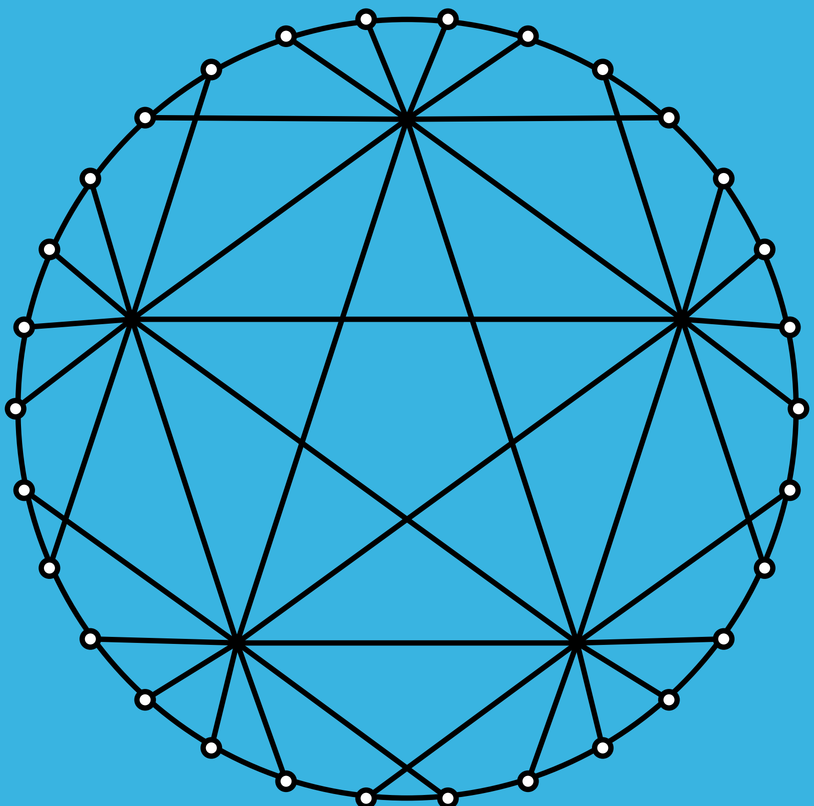
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 89
June 2020



Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **89** (2020), 75–92

Contacts and returns in 2-watermelons without wall

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Abstract: In this paper we will analyze the expected value, variance and distribution of the number of contacts as well as returns in 2-watermelons without wall (i.e. two nonintersecting lattice paths with Dyck-steps). This can be achieved via a bijection with weighted Motzkin paths and standard techniques for generating functions.

We will also analyze the asymptotic behaviour of these distributions and show that they behave like shifted negative binomial distributions with the help of singularity analysis.

1 Introduction

The concept of vicious walkers, i.e. tuples of lattice paths where no two paths touch each other, has been introduced by Fisher [4] to study wetting and melting processes. These combinatorial objects have been of much interest because they can also be used to model other phenomena, for example DNA denaturation, as can be seen in [10, 11]. A similar notion are friendly walkers, where the paths are allowed to touch but not to cross each other. Watermelons are a special case of vicious (respectively friendly) walkers where the underlying step set is the Dyck step set (i.e. (1,1) and (1,-1)) and there are certain conditions on the start- and endpoints of the paths.

Key words and phrases: lattice paths, Motzkin paths, watermelons, vicious walkers, generating functions, distribution, asymptotic analysis
AMS (MOS) Subject Classifications: 05Axx

Received: 14 January 2020
Accepted: 22 April 2020

Several parameters of watermelons have already been studied. Exact and asymptotic results for the number of p-watermelons, p denoting the number of paths, have been studied by Guttmann, Owczarek and Viennot in [7]. These results were later extended to the number of p-watermelons in a half plane (often also called wall constraint) in [8] and in a horizontal strip in [9] by Krattenthaler, Guttmann and Viennot. The average height and range of watermelons have been described by Fulmek and Feierl in [6, 2, 3].

The number of contacts between one Dyck path and a wall is also well known. In [1] Deutsch showed that the average number of contacts of a Dyck path of length n is $\frac{3n}{n+2}$ and thus tends to a constant asymptotically. We will see that the number of contacts between two paths behaves similarly.

In this paper we want to compute the average number of contacts and returns occurring in a 2-watermelon and also the distributions of these numbers. We impose no conditions on the region where the paths are allowed to be. In Section 2 we will deal with expected value and variance for the number of contacts, in Section 3 we will do the same for returns. In Section 4 we will analyze the distribution of these two parameters and in Section 5 we will study their asymptotic behaviour.

Definition 1.1. *A lattice path is a polygonal line in $\mathbb{Z} \times \mathbb{Z}$. Typically from each point there is only a finite set of allowed moves.*

In the following we will assume that all steps are of the form $(1, b)$ with $b \in \mathbb{Z}$. Hence, our paths are essentially one-dimensional objects.

In some settings there are also constraints imposed on the region the path is not allowed to leave. A common example is that the path is not allowed to go below the x-axis, which is called a wall condition. In this paper we will impose no such conditions on the paths.

Definition 1.2. *A p-watermelon of length n is a family of Dyck paths (lattice paths with step set $S = \{(1,1), (1,-1)\}$) such that all paths start at $(0,0)$ and end at (n,h) (where $n \equiv h \bmod 2$), where these paths may touch but not cross each other. In other words, if (m, y_i) denotes the coordinates of the i-th path after m steps, we have that $y_1 \leq y_2 \leq \cdots \leq y_p$ for all m.*

Definition 1.3. *The y-coordinate of the endpoint (n,h) is called the deviation of the watermelon.*

Many authors also use the following definition of watermelons (see for example [7, 8, 9, 6, 2, 3]).

Definition 1.4. *A p-watermelon of length n is a family of p nonintersecting Dyck paths $P_1, \dots, P_p$ in $\mathbb{Z}^2$ with*

1. *P_i starts at $(0, 2i-2)$ and ends at $(n, h+2i-2)$ where $n \equiv h$ mod 2.*
2. *no two paths have points in common.*

By a simple shift argument (moving the i-th path $2i-2$ units up) we can see that Definition 1.2 and 1.4 are equivalent. In this paper we will use Definition 1.2, because with this definition it becomes more visible what a contact is.

For the rest of this paper, we will only consider 2-watermelons with no wall and arbitrary deviation.

2 The average number and variance of contacts

Definition 2.1. *A contact in a 2-watermelon is a point (not counting the starting point) where both paths occupy the same lattice point, i.e. all points (m, y) such that (m, y) lies both on the lower path P_1 and the upper path P_2.*

Note that for more than two paths there are several possible ways to define contacts – either as points lying on all of the paths or as a point lying on two (or more) of the paths, but not necessarily on all of them. The first version is much more restrictive than the second one, each point that is a contact in the first sense is also a contact in the second sense. One could also count weighted contacts, i.e. if two paths meet it is counted as a simple contact with weight c, if three paths meet it is counted as double contact with weight d and so on.

2.1 Average number of contacts

Theorem 2.2. *Let X_n be the random variable counting the number of contacts in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length n and arbitrary deviation. Then*

$$\mathbb{E}X_n = \frac{(7n+13)n}{(n+4)(n+3)} = 7 - \frac{36}{n} + \frac{168}{n^2} + O\left(\frac{1}{n^3}\right).$$

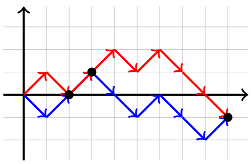
Before giving the proof of Theorem 2.2 let us observe the following bijection, which turns out to be very helpful for this proof, but also for other proofs in this paper. We can construct a bijection between 2-watermelons with arbitrary deviation and weighted Motzkin paths that start and end on the x-axis, but never cross the x-axis in the following way:

step of the upper path step of the lower path	$\nearrow$ $\searrow$	$\nearrow$ $\nearrow$	$\searrow$ $\searrow$	$\searrow$ $\nearrow$
step of the Motzkin path	$\nearrow$	$\xrightarrow{u}$	$\xrightarrow{d}$	$\searrow$

These are weighted Motzkin paths where there are two different kinds of level steps. The height of the Motzkin path corresponds to (half of) the distance of the paths in the watermelon. Because the paths may not cross, this distance may not be negative, i.e. the Motzkin path may never cross the x-axis. The condition that both paths end on the same height corresponds to the condition that the Motzkin path has to end on the x-axis. A contact between the two paths occurs each time the Motzkin path touches the x-axis. Thus we want to count the number of returns of the Motzkin path to the x-axis.

Now, let F denote the generating function. A Motzkin path can be constructed as a sequence of the following objects: a level step with weight u, a level step with weight d, an up-step and a down step and a Motzkin path in-between.

Figure 1 illustrates the bijection between 2-watermelons with arbitrary deviation and Motzkin paths with wall. Contacts are marked with black dots.



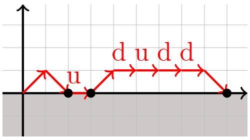


Figure 1: The bijection between 2-watermelons and weighted Motzkin paths

Using this bijection and the decomposition "a Motzkin path is a sequence of arches (i.e., lattice paths of size > 0 that touch the x-axis only at the beginning and the end and stay above the x-axis otherwise) and level steps" we get a functional equation:

$$F(z) = \frac{1}{1 - z^2F(z) - 2z}.$$

Multiplying with the denominator and solving the quadratic equation we get

$$F(z) = \frac{1 - 2z - \sqrt{1-4z}}{2z^2} \tag{1}$$

Technically we would get two solutions, but the solution with $+\sqrt{1-4z}$ does not make sense from a combinatorial point of view.

Proof of Theorem 2.2: Using the above bijection and introducing a new variable u, counting the number of contacts of the Motzkin-path with the x-axis, we get

$$F(z,u) = \frac{1}{1 - u(z^2F(z,1) + 2z)}.$$

Using $F(z,1) = F(z)$ and (1) we obtain

$$F(z,u) = \frac{2}{2 - u - 2uz + u\sqrt{1-4z}} \tag{2}$$

Differentiating with respect to u and evaluating at 1 we get

$$\partial_u F(z,u)|_{u=1} = \frac{2(1 + 2z - \sqrt{1-4z}}{(1 - 2z - \sqrt{1-4z})^2}. \tag{3}$$

By rationalizing the fration we can rewrite this as

$$\partial_u F(z,u)|_{u=1} = \frac{P(z) + (2z^2 + 4z - 2)\sqrt{1-4z}}{4z^4},$$

where $P(z) = 2 - 8z + 2z^2 + 4z^3$.

The average number of contacts of a watermelon of length n (with deviation, without wall) is given by

$$\mathbb{E}X_n = \frac{[z^n]\partial_u F(z,u)|_{u=1}}{[z^n]F(z,1)}.$$

By expanding $\sqrt{1-4z}$ with the binomial series we can read off coefficients from (1) and obtain

$$[z^n]F(z,1) = C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}, \tag{4}$$

where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

To obtain $[z^n]\partial_u F(z,u)|_{u=1}$ we use that

$$[z^n]\sqrt{1-4z} = -2C_{n-1}.$$

We have

$$\begin{aligned}
[z^n]\partial_u F(z,u)|_{u=1} &= [z^n]\frac{2-8z+2z^2+4z^3}{4z^4} + [z^n]\frac{(2z^2+4z-2)\sqrt{1-4z}}{4z^4} \\
&= \frac{1}{2}[z^{n+2}]\sqrt{1-4z} + [z^{n+3}]\sqrt{1-4z} - [z^{n+4}]\frac{1}{2}\sqrt{1-4z} \\
&= -C_{n+1} - 2C_{n+2} + C_{n+3}.
\end{aligned}$$

Now we can compute

$$\mathbb{E}X_n = \frac{C_{n+3} - 2C_{n+2} - C_{n+1}}{C_{n+1}}$$

Using the definition of the Catalan numbers and pulling out the common factor $\frac{1}{n+2}\binom{2n+2}{n+1}$ this becomes after some simplifications

$$\mathbb{E}X_n = \frac{(7n+13)n}{(n+4)(n+3)}. \tag{5}$$

Expanding (5) as a series we get the assertion of Theorem 2.2. ☐

Corollary 2.3. *Let* $X := \lim_{n\to\infty} X_n$. *Then*

$$\mathbb{E}X = 7,$$

i.e. the average number of contacts in a 2-watermelon is asymptotically constant.

2.2 Variance of the number of contacts

Theorem 2.4. *Let X_n be defined as in Theorem 2.2. Then the variance of the number of contacts in a 2-watermelon is given by*

$$\begin{aligned}\mathbb{V}X_n &= \frac{12n(2n^5+13n^4+17n^3-7n^2-19n-6)}{(n+3)^2(n+4)^2(n+5)(n+6)}\\ &= 24-\frac{444}{n}+\frac{5136}{n^2}+O\left(\frac{1}{n^3}\right).\end{aligned}$$

Proof: The variance of the number of contacts in a watermelon of length n is given by

$$\mathbb{V}X_n = \left.\frac{[z^n]\partial^2_{uu}F}{[z^n]F}+\frac{[z^n]\partial_u F}{[z^n]F}-\left(\frac{[z^n]\partial_u F}{[z^n]F}\right)^2\right|_{u=1},$$

where F is shorthand for $F(z,u)$.

Since the last two terms can be computed via (5) it remains to compute

$$G(z) := \partial^2_{uu}F(z,u)|_{u=1} = \frac{P_1(z)\sqrt{1-4z}-P_2(z)}{z^6}.$$

where

$$P_1(z) = -z^4-4z^3-z^2+4z-1$$

and

$$P_2(z) = 2z^5-3z^4+6z^3+7z^2-6z+1.$$

Reading off coefficients we get

$$[z^n]G(z) = 2C_{n+1}+8C_{n+2}+2C_{n+3}-8C_{n+4}+2C_{n+5}.$$

Hence

$$\frac{[z^n]\partial^2_{uu}F(z,1)}{[z^n]F(z,1)} = \frac{66n^4+276n^3+54n^2+396n}{(n+3)(n+4)(n+5)(n+6)}. \tag{6}$$

Combining (5) and (6) we get

$$\begin{aligned}\mathbb{V}X_n &= \frac{12n(2n^5+13n^4+17n^3-7n^2-19n-6)}{(n+3)^2(n+4)^2(n+5)(n+6)}\\ &= 24-\frac{444}{n}+\frac{5136}{n^2}+O\left(\frac{1}{n^3}\right)\end{aligned}$$

which completes the proof. □

3 Returns and common steps

Definition 3.1. *A* return *is a point where the two paths of a watermelon meet, but have been apart one step before. In the Motzkin path setting this corresponds to a step that ends on the x-axis but does not start on the x-axis. In our case, the only possible return is being at height 1 and then making a down step. Level steps at height 0 do not count as return.*

Definition 3.2. *A* common step *occurs if both paths of a watermelon are at the same height and then take either an up-step or a down-step together. In the Motzkin path setting this corresponds to a level step at height 0.*

Obviously we have that the number of returns plus the number of common steps is the number of contacts. Thus it is sufficient to analyze only one of these numbers. We will consider returns. Their average number and variance can be computed in a similar manner as in the previous section.

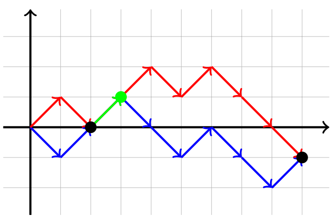


Figure 2: Returns and common steps in a 2-watermelon (with deviation -1). Returns are marked in black, common steps are marked in green.

3.1 Average number of returns

Theorem 3.3. *Let Y_n be the random variable counting the number of returns in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length n and arbitrary deviation. Then*

$$\mathbb{E}Y_n = \frac{3n(n-1)}{(n+4)(n+3)} = 3 - \frac{24}{n} + \frac{132}{n^2} + O\left(\frac{1}{n^3}\right).$$

Proof: The generating function that counts returns is given by

$$F(z,x) = \frac{1}{1-(xz^2F(z,1)+2z)}$$

where x encodes the number of returns to the x-axis. Here we only have to mark contacts occurring from a down step, thus the $+2z$-part remains unmarked. Plugging in the expression of $F(z,1)$ we computed in (1) we obtain

$$F(z,x) = \frac{2}{2-x(1-2z-\sqrt{1-4z})-4z}. \tag{7}$$

Derivating with respect to x and plugging in $x=1$ this becomes

$$\partial_x F(z,x)|_{x=1} = \frac{2(1-2z-\sqrt{1-4z})}{(1-2z+\sqrt{1-4z})^2}. \tag{8}$$

This looks very similar to what we had in formula (3) when computing contacts. The only difference is that we now have $(1-2z-\sqrt{1-4z})$ instead of $(1+2z-\sqrt{1-4z})$ in the numerator.
Multiplying out and then rationalizing the fraction, we get

$$\partial_x F(z,x)|_{x=1} = \frac{(-3z^2+4z-1)\sqrt{1-4z}+P_3(z)}{2z^4},$$

where $P_3(z) = 1-6z+9z^2-2z^3$. Now we can read off coefficients:

$$\begin{aligned}[z^n]\partial_x F(z,x)|_{x=1} &= [z^n]\frac{(-3z^2+4z-1)\sqrt{1-4z}}{2z^4} \\ &= 3C_{n+1}-4C_{n+2}+C_{n+3}\end{aligned}$$

Now we can compute the average number of returns in a watermelon of length n via

$$\mathbb{E}_r = \frac{[z^n]\partial_x F(z,x)|_{x=1}}{[z^n]F(z,1)} = \frac{\frac{3}{n+2}\binom{2n+2}{n+1} - \frac{4}{n+3}\binom{2n+4}{n+2} + \frac{1}{n+4}\binom{2n+6}{n+3}}{\frac{1}{n+2}\binom{2n+2}{n+1}}.$$

Pulling out common factors we get that the above is

$$\mathbb{E}_r = \frac{3n(n-1)}{(n+4)(n+3)} = 3 - \frac{24}{n} + \frac{132}{n^2} + O\left(\frac{1}{n^3}\right), \tag{9}$$

which completes the proof. □

Corollary 3.4. *A 2-watermelon has asymptotically on average 7 contacts and 3 returns. Thus it has on average asymptotically $7-3=4$ common steps.*

3.2 Variance of the number of returns

Theorem 3.5. *Let Y_n be defined as in Theorem 3.3. Then the variance of the number of returns in a 2-watermelon is given by*

$$\begin{aligned}\mathbb{V}Y_n &= \frac{4n(n-1)(n^4-4n^3+4n^2+279n+450)}{(n+3)^2(n+4)^2(n+5)(n+6)} \\ &= 4 - \frac{120}{n} + \frac{2004}{n^2} + O\left(\frac{1}{n^3}\right).\end{aligned}$$

Proof: The variance of the number of returns is given by

$$\mathbb{V}_{Y_n} = \frac{[z^n]\partial^2_{xx}F(z,1)}{[z^n]F(z,1)} + \frac{[z^n]\partial_x F(z,1)}{[z^n]F(z,1)} - \left(\frac{[z^n]\partial_x F(z,1)}{[z^n]F(z,1)}\right)^2.$$

Since the last two terms in this expression can be obtained with the help of (9) it remains to compute

$$\partial^2_{xx}F(z,x)|_{x=1} = \frac{P_4(z)+P_5(z)\sqrt{1-4z}}{z^6},$$

where

$$P_4(z) = 1 - 10z + 35z^2 - 50z^3 + 25z^4 - 2z^5$$

and

$$P_5(z) = -5z^4 + 20z^3 - 21z^2 + 8z - 1.$$

Thus we get that

$$\frac{[z^n]\partial^2_{xx}F(z,1)}{[z^n]F(z,1)} = \frac{10n^4 - 60n^3 + 110n^2 - 60n}{(n+3)(n+4)(n+5)(n+6)}. \tag{10}$$

Combining (9) and (10) we get that the variance of the number of returns in a watermelon of size n is given by

$$\mathbb{V}_{Y_n} = \frac{4n(n-1)(n^4-4n^3+4n^2+279n+450)}{(n+3)^2(n+4)^2(n+5)(n+6)}.$$

Asymptotic expansion of this expression finishes the proof of this theorem. □

4 Distributions

4.1 The number of contacts

Theorem 4.1. *Let X_n be the random variable counting the number of contacts in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length n and arbitrary deviation. Then the probability that such a watermelon has exactly k contacts is given by*

$$\mathbb{P}(X_n = k) = \frac{\frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k} S(k,l,m)}{\frac{1}{n+2}\binom{2n+2}{n+1}},$$

where

$$S(k,l,m) = \binom{k}{m}\binom{m}{n-\ell}2^{n+\ell}(-1)^{k-m+\ell}\binom{\frac{k-m}{2}}{\ell}.$$

Proof: In order to figure out the distribution of the number of contacts we need to consider

$$\mathbb{P}(X_n = k) = \frac{[z^n u^k]F(z,u)}{[z^n]F(z,1)}$$

where X_n is the random variable counting the number of contacts in a 2-watermelon of length n (without wall). We rationalize (2) to get rid of the square root in the denominator

$$\begin{aligned} F(z,u) &= \frac{2}{2-u-2uz+u\sqrt{1-4z}} \\ &= \frac{1}{2}\cdot\frac{2-u-2uz-u\sqrt{1-4z}}{u^2z^2+2u^2z-2uz-u+1}. \end{aligned}$$

The idea is to decompose

$$\begin{aligned} R(z,u) &:= \frac{1}{u^2z^2+2u^2z-2uz-u+1} \\ &= \frac{a(z)}{1-\alpha(z)u}+\frac{b(z)}{1-\beta(z)u} \end{aligned}$$

by partial fraction decomposition. From this expression we then can read off the coefficient of u^k.

The zeroes of the denominator (as a quadratic polynomial in u) are:

$$u_1(z) = \frac{1+2z+\sqrt{1-4z}}{2z(z+2)} \qquad \text{and} \qquad u_2(z) = \frac{1+2z-\sqrt{1-4z}}{2z(z+2)}.$$

Thus we have

$$\begin{aligned} R(z,u) &= \frac{1}{\sqrt{1-4z}}\left(\frac{1}{u_2(z)-u} - \frac{1}{u_1(z)-u}\right) \\ &= \frac{1}{\sqrt{1-4z}}\left(\frac{1}{u_2(z)(1-\frac{1}{u_2(z)}u)} - \frac{1}{u_1(z)(1-\frac{1}{u_1(z)}u)}\right). \end{aligned}$$

Now we can read off coefficients using $[u^k]\frac{1}{1-cu} = c^k$ and obtain

$$[u^k]R(z,u) = \frac{1}{\sqrt{1-4z}}\left(\frac{1}{u_2(z)^{k+1}} - \frac{1}{u_1(z)^{k+1}}\right).$$

After plugging in $u_1(z)$ and $u_2(z)$ and some simplifciations we obtain

$$\begin{aligned} [u^k]F(z,u) &= \frac{2}{2}[u^k]R(z,u) - \frac{1+2z+\sqrt{1-4z}}{2}[u^{k-1}]R(z,u) \\ &= \frac{(1+2z-\sqrt{1-4z})^k}{2^k}. \end{aligned} \tag{11}$$

To read off the coefficient of $[z^n]$ of this expression, the expansion into a binomial series turns out to be helpful:

$$\begin{aligned} \frac{(1+2z-\sqrt{1-4z})^k}{2^k} &= \frac{1}{2^k}\sum_{m=0}^{k}\binom{k}{m}(1+2z)^m(-1)^{k-m}\sqrt{1-4z}^{\,k-m} \\ &= \frac{1}{2^k}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}\left(\sum_{r=0}^{m}\binom{m}{r}2^r z^r\right)\left(\sum_{\ell\geq 0}\binom{\frac{k-m}{2}}{\ell}(-4z)^\ell\right). \end{aligned}$$

If we want to read off $[z^n]$, the variables r and ℓ have to add up to n. Thus

$$[z^n u^k]F(z,u) = \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k} S(k,l,m). \tag{12}$$

Dividing (12) by the number of all watermelons of length n as given by (4) we obtain the statement of the theorem. $\square$

4.2 The number of returns

Theorem 4.2. *Let Y_n be the random variable counting the number of contacts in a 2-watermelon without wall, where the watermelon is chosen uniformly at random among all possible 2-watermelons of length n and arbitrary deviation. Then the probability that such a watermelon has exactly k returns is given by*

$$\mathbb{P}(Y_n = k) = \frac{\sum_{j=0}^{n} A_j^{(k)} B_{n-j}^{(k)}}{\frac{1}{n+2}\binom{2n+2}{n+1}}.$$

where

$$A_n^{(k)} := \sum_{\substack{0 \le \ell \le n \\ 0 \le m \le k}} \binom{k}{m} \binom{m}{n-\ell} \binom{\frac{k-m}{2}}{\ell} 2^{n+l-k} (-1)^{k-m+n}$$

and

$$B_n^{(k)} := [z^n] \frac{1}{(1-2z)^{k+1}} = \binom{n+k}{n} 2^n.$$

Proof: We want to compute

$$\mathbb{P}(X_n = k) = \frac{[z^n x^k] F(z,x)}{[z^n] F(z,1)}.$$

By rationalizig the fraction (7) we get

$$F(z,x) = \frac{2 - 4z - x(1 - 2z + \sqrt{1-4z})}{2(x^2z^2 - 4xz^2 + 4xz + 4z^2 - x - 4z + 1)}.$$

Again, we apply a partial fraction decomposition to

$$R(z,x) = \frac{1}{x^2z^2 - 4xz^2 + 4xz + 4z^2 - x - 4z + 1}$$

and read off coefficients from that. The zeros of the denominator are

$$x_1(z) = \frac{(1-2z)(1-2z-\sqrt{1-4z})}{2z^2}]$$

and

$$x_2(z) = \frac{(1-2z)(1-2z+\sqrt{1-4z})}{2z^2}.$$

We obtain

$$R(z,x) = \frac{1}{(1-2z)\sqrt{1-4z}} \left(\frac{1}{x_1(z)(1-\frac{x}{x_1(z)})} - \frac{1}{x_(z)(1-\frac{x}{x_2(z)})} \right).$$

Reading off coefficients we obtain

$$\begin{aligned} [x^k]R(z,x) &= \frac{1}{(1-2z)\sqrt{1-4z}} \left(\frac{1}{x_1(z)^{k+1}} - \frac{1}{x_2(z)^{k+1}} \right) \\ &= \frac{(1-2z+\sqrt{1-4z})^{k+1} - (1-2z-\sqrt{1-4z})^{k+1}}{2^{k+1}(1-2z)^{k+2}\sqrt{1-4z}}. \end{aligned}$$

Rewrite (7) as

$$F(z,x) = (1-2z)R(z,x) - \frac{1-2z+\sqrt{1-4z}}{2} xR(z,x),$$

reading off the coefficient $[x^k]$ and simplifying, we obtain

$$[x^k]F(z,x) = \frac{(1-2z-\sqrt{1-4z})^k}{2^k(1-2z)^{k+1}} = A^{(k)}(z)B^{(k)}(z),$$

where

$$A^{(k)}(z) = \frac{(1-2z-\sqrt{1-4z})^k}{2^k}$$

and

$$B^{(k)}(z) = \frac{1}{(1-2z)^k}.$$

Expanding

$$B_n^{(k)} := [z^n]\frac{1}{(1-2z)^{k+1}} = \binom{n+k}{n}2^n$$

in a binomial series and a similar reasoning as in the previous subsection yields

$$A_n^{(k)} := [z^n]A^{(k)}(z) = \frac{1}{2^k}\sum_{\ell=0}^{n}\sum_{m=0}^{k}\binom{k}{m}\binom{m}{n-\ell}\binom{\frac{k-m}{2}}{\ell}2^{n+\ell}(-1)^{k-m+n}.$$

Note that $A_n^{(k)}$ looks quite similar to (12), the only difference between these expressions are the powers of -1, namely $(-1)^{k-m+n}$ and $(-1)^{k-m+\ell}$ respectively.

Using the Cauchy-Product of A and B we obtain

$$[z^n x^k]F(z,x) = \sum_{j=0}^{n} A_j^{(k)} B_{n-j}^{(k)}$$

Dividing this by the number of all watermelons of length n given by (4) we get the assertion of the theorem. $\square$

5 Asymptotic behaviour of the distributions

In this section we are going to analyze the asymptotic behaviour of the distributions of contacts and returns. The theoretical background of this section are the methods for coefficient asymptotics from Flajolet and Sedgewick [5], the proof of the following theorem and more details can be found there.

Theorem 5.1. *Let $f(z) = (1-z)^{-\alpha}$ with $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Then the n-th coefficient of f is asymptotically equal to*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k}{n^k}\right)$$

where e_k is a polynomial in α of degree $2k$. In particular

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right).$$

5.1 Contacts

Theorem 5.2. *Let the random variable X_n counting the average number of contacts be defined as in Theorem 2.2. Then $X := \lim_{n\to\infty} X_n$ is distributed as follows*

$$\mathbb{P}(X = 0) = 0$$

and

$$\mathbb{P}(X = k) = \mathbb{P}(B = k-1) \quad \text{for } k \geq 1$$

where B is a negative binomial distributed random variable with parameters $r = 2$ and $p = \frac{3}{4}$, i.e.

$$\mathbb{P}(B = k) = \binom{r+k-1}{k} p^k (1-p)^r.$$

Proof: We want to compute $[z^n u^k]F(z,u)$ for $n \to \infty$ and k fixed. The function

$$[u^k]F(z,u) = \frac{(1+2z-\sqrt{1-4z})^k}{2^k}$$

has its dominant singularity at $z = \frac{1}{4}$. Using Theorem 5.1, we obtain

$$[z^n u^k]F(z,u) \sim -\frac{k}{2^k}\left(\frac{3}{2}\right)^{k-1}\frac{n^{-3/2}4^n}{\Gamma(-\frac{1}{2})} = \frac{k3^{k-1}4^n}{4^k\sqrt{\pi n^3}}$$

for $n \to \infty$. Using $C_n \sim \frac{4^n}{\sqrt{\pi n^3}}\left(1+O\left(\frac{1}{n}\right)\right)$ we obtain

$$\mathbb{P}(X_n = k) = \frac{[z^n u^k]F(z,u)}{C_{n+1}} \sim \frac{k3^{k-1}}{4^{k+1}}\left(1+O\left(\frac{1}{n}\right)\right)$$

for $n \to \infty$. Observing that $\mathbb{P}(X=0) = 0$ and introducing a new random variable B with $\mathbb{P}(X=k) = \mathbb{P}(B=k-1)$ we see that B is a negative binomial distributed random variable with parameters $r = 2$ and $p = \frac{3}{4}$, which finishes the proof. $\square$

5.2 Returns

Theorem 5.3. *Let the random variable Y_n counting the number of returns be defined as in Theorem 3.3. Then $Y := \lim_{n\to\infty} Y_n$ is distributed as follows*

$$\mathbb{P}(Y=0) = 0$$

and

$$\mathbb{P}(Y=k) = \mathbb{P}(\tilde{B} = k-1) \quad \textit{for } k \geq 1$$

where $\tilde{B}$ is a negative binomial distributed random variable with parameters $r = 2$ and $p = \frac{1}{2}$.

Proof: The proof of this theorem is similar to the proof of Theorem 5.2. Now we look at $[z^n x^k]F(z,x)$ for $n \to \infty$ and k fixed. The function

$$F(z,x) = \frac{(1-2z-\sqrt{1-4z})^k}{2^k(1-2z)^k}$$

has singularities at $z = \frac{1}{4}$ and $z = \frac{1}{2}$. The singularity at $z = \frac{1}{4}$ is the dominant one, the other singularity at $z = \frac{1}{2}$ lies outside of every Δ-region around $z = \frac{1}{4}$. By expanding the denominator with the binomial series and by using Theorem 5.1 we obtain

$$[z^n x^k]F(z,x) \sim -\frac{k}{2^k} \cdot \frac{(\frac{1}{2})^k}{(\frac{1}{2})^{k+1}} \cdot \frac{n^{-3/2}4^n}{\Gamma(-\frac{1}{2})} = \frac{k4^n}{2^{k-1}\sqrt{n^3\pi}}$$

for $n \to \infty$. We obtain

$$\mathbb{P}(X_n = k) = \frac{[z^n x^k]F(z,x)}{C_{n+1}} \sim \frac{k}{2^{k+1}}\left(1 + O\left(\frac{1}{n}\right)\right)$$

for $n \to \infty$. Introducing the random variable $\tilde{B}$ as in the theorem and observing that it indeed is negative binomial distributed with parameters $r = 2$ and $p = \frac{1}{2}$ concludes the proof. $\square$

6 Conclusion

We have studied the distribution of contacts and returns in 2-watermelons without wall and arbitrary deviation via a bijection with weighted Motzkin paths and derived both exact and asymptotic results. We discovered that the expected value of these parameters tends to a finite limit and that the asymptotic distribution behaves like a shifted negative binomial distribution.

The deviation, i.e. the height of the endpoints of our watermelons was arbitrary. A natural question would be to ask what happens for watermelons with a fixed deviation. This can be encoded in the following way: let z mark the length of the watermelon (or Motzkin path) and y the deviation. The deviation is then given by the number of level steps marked with u minus the number of level steps marked with d. Taking this into account when constructing the functional equation we obtain

$$F(z,y) = \frac{1}{1 - (z^2F + (y + y^{-1})z)}.$$

This then could be used to derive similar results for watermelons with a given deviation.

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