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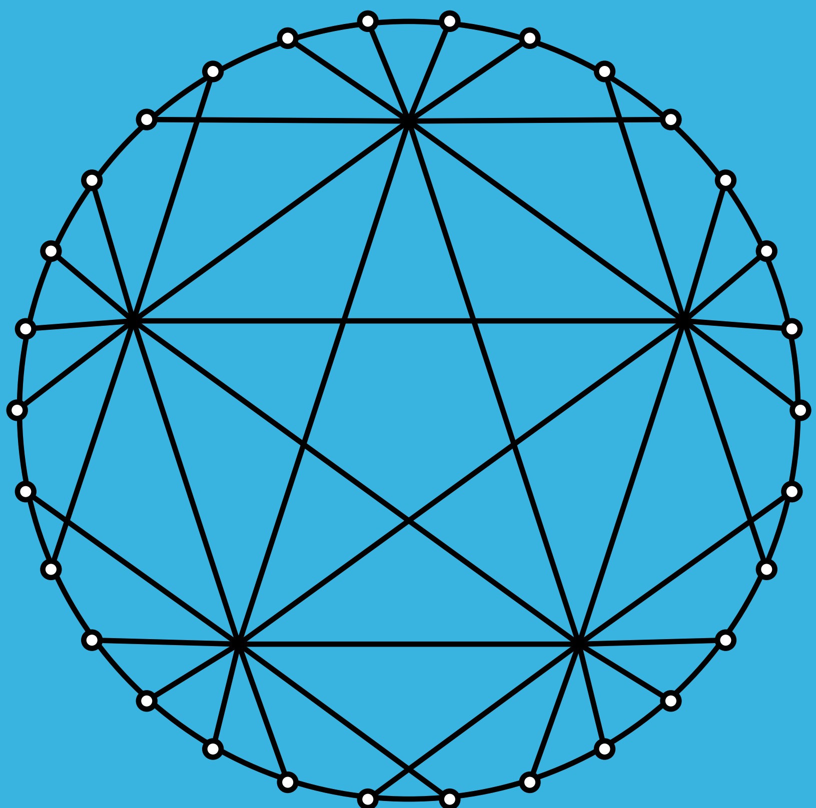

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Decomposition of complete graphs into bi-cyclic graphs with eight edges

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Abstract: We show that each of the 35 non-isomorphic bi-cyclic graphs with eight edges decomposes the complete graph K_n whenever the necessary conditions are satisfied.

1 Introduction

A *decomposition* of the complete graph K_n is a collection of mutually edge disjoint subgraphs $\mathcal{D} = \{G_1, G_2, \ldots, G_s\}$ such that every edge of K_n appears in exactly one graph $G_i \in \mathcal{D}$. If each subgraph G_r is isomorphic to a given graph G, then we say that the collection $\mathcal{D}$ forms a G-*decomposition* of K_n, or a G-*design*. When $s = n$, the decomposition is *cyclic* if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of K_n and an isomorphism $\phi : V(G_j) \to V(G_{j+1})$, such that for every $j = 1, 2, \ldots, n$, we have $\phi(x_i) = x_{i+1}$ for each $i = 1, 2, \ldots, n$. The subscripts are taken modulo n. Similarly, the decomposition is 1-*rotational* if there exists an ordering $(x_1, x_2, \ldots, x_n)$ of the vertices of K_n and an isomorphisms $\phi : V(G_j) \to V(G_{j+1})$ such that for every $j = 1, 2, \ldots, n-1$ we have $\phi(x_i) = x_{i+1}$ for each $i = 1, 2, \ldots, n-1$ where the subscripts here are taken modulo $n-1$ and $\phi(x_n) = x_n$.

A finite graph G with no loops or multiple edges is called *bi-cyclic* if it contains exactly two cycles. It can be also viewed as a forest with exactly

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two edges added such that once we add the first one and create a cycle, the second one shares at most one vertex with that cycle. In the following sections we present the necessary and sufficient conditions for decompositions of complete graphs into bi-cyclic graphs with eight edges and prove that each of them decomposes the complete graph K_n whenever the necessary conditions are satisfied.

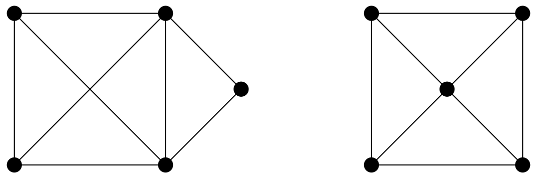
We will use standard decomposition methods based on ρ-labelings, introduced by Rosa [13] and later modified by other authors.

2 Related results

There has been a significant activity recently in the area of decompositions of complete graphs into graphs with eight edges.

This section will summarize what is known about classification of graphs where $|(E(G)| = 8$ that form decompositions of complete graphs.

Graphs with five vertices and eight edges were examined by Colbourn, Ge, and Ling [4]. There are only two non-isomorphic graphs with five vertices and eight edges, shown in Figure 1.



G_{20} G_{21}

Figure 1: Connected graphs with 8 edges and 5 vertices

Colbourn, Ge and Ling proved the following results for the graphs $G_{5,1}$ and $G_{5,2}$.

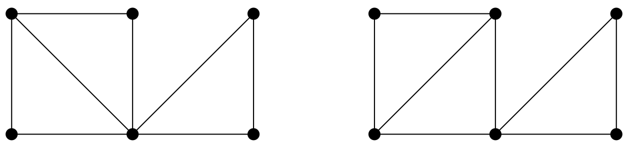
Theorem 2.1 (Colbourn, Ge, Ling 2008). *There exists a decomposition of K_n into G_{20} if and only if $n \equiv 0 \pmod{16}$ except possibly when $n = 32$ or $n = 48$.*

Theorem 2.2 (Colbourn, Ge, Ling 2008). *There exists a decomposition of K_n into G_{21} if and only if $n \equiv 0, 1 \pmod{16}$ except when $n = 16$ and possibly when $n = 48$.*

Kang, Yuan, and Liu researched graphs with six vertices and eight edges in 2005 [12]. There are 22 non-isomorphic graphs of this type, and they proved the following theorem with respect to decompositions of complete graphs.

Theorem 2.3 (Kang, Yuan, Liu 2005)**.** *Let G be a connected graph with six vertices and eight edges. Then G forms a decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$ with two possible exceptions for $n = 32$.*

The two missing cases were settled by Forbes and Griggs in 2018 [10]. The graphs are shown in Figure 2.



M_1 M_2

Figure 2: Graphs M_1 and M_2

Theorem 2.4 (Forbes, Griggs 2018)**.** *Graphs M_1 and M_2 shown in Figure 2 form a decomposition of K_{32}.*

Therefore, the following theorem holds.

Theorem 2.5 (Kang, Yuan, Liu 2005, Forbes, Griggs 2018)**.** *Let G be a connected graph with six vertices and eight edges. Then G forms a decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

We were unable to find any results on graphs with eight edges and seven vertices. Therefore, we study the connected ones along with other related graphs in the following sections.

For graphs with eight edges and eight vertices, Kang and Zhang [14] determined the spectrum completely for the four graphs shown in Figure 3.

Theorem 2.6 (Kang, Zhang 2015)**.** *Let G_i be a connected graph with eight vertices and eight edges shown in Figure 3. Then G_i forms a decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

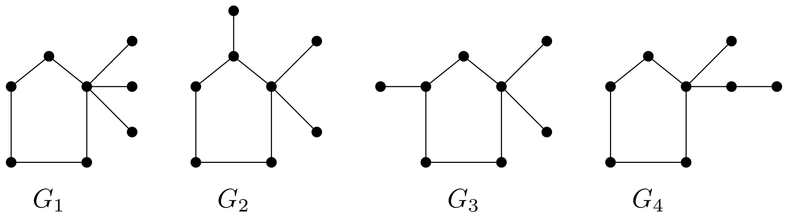


Figure 3: Graphs G_1, G_2, G_3, G_4 by Kang and Zhang

Bipartite connected unicyclic graphs with eight edges and eight vertices other than C_8 were studied by Fahnenstiel and Froncek in [7]; C_8 decompositions exist if and only if $n \equiv 1 \pmod{16}$ as proved by Rosa [13].

Theorem 2.7 (Fahnenstiel, Froncek 2019)**.** *Let G be a connected bipartite unicyclic graph with eight vertices and eight edges other than C_8. Then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

The disconnected case was recently completely settled by Freyberg and Tran [9].

Theorem 2.8 (Freyberg, Tran 2019)**.** *Let G be a bipartite disconnected unicyclic graph with eight edges. Then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

Connected unicyclic graphs with eight edges and eight vertices with pentagon were studied by Froncek and Kingston [11]; they have shown that each of the ten non-isomorphic connected unicyclic graphs with eight edges containing a pentagon decomposes the complete graph K_n whenever the necessary conditions are satisfied.

Theorem 2.9 (Froncek, Kingston 2019)**.** *Let G be a connected unicyclic graph with eight vertices and eight edges where the unique cycle is a pentagon. Then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

The disconnected case for unicyclic graphs with pentagon was completely solved by by Freyberg and Froncek [8]. They also fully solved the case of unicyclic graphs with a triangle or heptagon, both connected and disconnected.

Theorem 2.10 (Freyberg, Froncek 2019)**.** *Let G be a disconnected unicyclic graph with eight vertices and eight edges where the unique cycle is a pentagon. Then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

Theorem 2.11 (Freyberg, Froncek 2019)**.** *Let G be a unicyclic graph with eight vertices and eight edges where the unique cycle is a triangle or heptagon. Then there exists a G-decomposition of K_n if and only if $n \equiv 0, 1 \pmod{16}$ and $n \geq 16$.*

Therefore, the class of unicyclic graphs with eight edges is completely solved.

Theorem 2.12. *Let G be a unicyclic graph with eight edges. Then there exists a G-decomposition of K_n if and only if $G \ncong C_8$ and $n \equiv 0, 1 \pmod{16}$, or $G \cong C_8$ and $n \equiv 1 \pmod{16}$.*

As a natural next step, we further investigate graphs with exactly two cycles and eight edges.

3 Tools and methods

Our tools are graph labelings arising from the ρ-labeling, first defined by Rosa [13], who called it a *ρ-valuation.*

Definition 3.1. Let G be a graph with n edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, \ldots, 2n\}$ inducing the *length function* $\ell : E(G) \to \{1, 2, \ldots, n\}$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\}$$

with the property that

$$\{\ell(uv)|uv \in E(G)\} = \{1, 2, \ldots, n\}.$$

A more restrictive yet well studied is the *graceful labeling*, also introduced by Rosa in [13] and called originally a *β-valuation.*

Definition 3.2. Let G be a graph. A *graceful labeling* of G is a ρ-labeling such that $0 \leq f(u) \leq n$ for every vertex $u \in V(G)$ and $\ell(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$.

Rosa [13] proved that if a graph G with n edges has one of the above labelings, then a G-decomposition of the complete graph K_{2n+1} exists.

Theorem 3.3 (Rosa 1967). *A cyclic decomposition of the complete graph K_{2n+1} into subgraphs isomorphic to a given graph G with n edges exists if and only if there exists a ρ-labeling of the graph G.*

In some cases, more restrictive modifications of the above labelings allow decompositions of bigger complete graphs, in particular, K_{2nk+1} for any positive integer k. The following labeling was also introduced by Rosa [13].

Definition 3.4. An *α-labeling* is a graceful labeling with the additional property that there exists an integer λ such that for each edge uv either $f(u) \leq \lambda < f(v)$ or $f(v) \leq \lambda < f(u)$.

An α-labeled graph must be bipartite, and when V_1 and V_2 is a partition of the vertex set $V(G)$ of the graph G, then without loss of generality if $u \in V_1$, then $f(u) \leq \lambda$ and if $v \in V_2$, then $f(v) > \lambda$.

Rosa also proved the following [13]:

Theorem 3.5 (Rosa 1967). *If a graph G with n edges has an α-labeling, then there exists a decomposition of the complete graph K_{2nk+1} into subgraphs isomorphic to G for any positive integer k.*

Bunge, Chantasartrassmee, El-Zanati, and Vanden Eynden [3] found a more restrictive version of ρ-labeling, which is similar to α-labeling in the sense that it allows decompositions of bigger complete graphs into certain tripartite graphs.

Definition 3.6. Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A *ρ-tripartite labeling* of G is a one-to-one function $h : V(G) \to \{0, 1, 2, \ldots, 2n\}$ that satisfies

(r1) h is a ρ-labeling of G.

(r2) If $av \in E(G)$ with $a \in A$, then $h(a) < h(v)$.

(r3) If $e = bc \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = b'c'$ with $b' \in B$ and $c' \in C$ such that $|h(c)-h(b)|+|h(c')-h(b')| = 2n$.

(r4) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

Note that e and e' in (r3) need not be distinct.

They proved the following.

Theorem 3.7 (Bunge, Chantasartrassmee, El-Zanati, Vanden Eynden 2013)**.** *If a tripartite graph G with n edges has a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nk+1} for every positive integer k.*

The above labelings enable isomorphic decompositions of complete graphs of odd order, but similar methods exist for complete graphs of even order under certain circumstances. It is well known that certain ρ-labeled graphs can form isomorphic decompositions of K_{2nk} (see, e.g., [6]):

Theorem 3.8. *Let G be a graph with n edges and let v be a vertex of degree one in G. If $G - v$ has a ρ-labeling, then there exists a 1-rotational G-decomposition of K_{2n}.*

Theorem 3.8 is based on the following idea. We pick a vertex x_{2n} in K_{2n} and decompose $K_{2n} - x_{2n}$ cyclically into $2n - 1$ copies of the graph $G - v$. Then we identify v with x_{2n} to obtain a decomposition of K_{2n} by adding back the pendant edge uv, where u is the only neighbor of v in G.. Because the vertex u is each copy of $G - v$ projected onto a different vertex $x_i, i = 1, 2, \dots, 2n-1$, the edge uv is projected onto different edges $x_i x_{2n}$ in K_{2n}. The length of uv is denoted by ∞.

The labeling used in the above theorem can be formally defined as follows.

Definition 3.9. Let G be a graph with n edges and edge uv where $\deg(v) = 1$. A 1-*rotational ρ-labeling* of G consists of an injective function $f\colon V(G) \to \{0, 1, 2, \dots, 2n-2, \infty\}$ such that $f(w) = \infty$ that induces a *length function* $\ell\colon E(G) \to \{1, 2, \dots, n-1, \infty\}$ which is defined as

$$\ell(xy) = \min\{|f(x) - f(y)|, 2n - 1 - |f(x) - f(y)|\}$$

for $x, y \neq v$ and

$$\ell(uv) = \infty$$

with the property that

$$\{\ell(xv) : xv \in E(G)\} = \{1, 2, \dots, n-1, \infty\}.$$

A generalization of the above method for tripartite graphs was found by Bunge [2].

Definition 3.10. Let G be a tripartite graph with n edges, vertex tripartition $\{A, B, C\}$, and edge uw where $\deg(w) = 1$. A 1-*rotational* ρ-*tripartite labeling* of the graph G is a 1-rotational ρ-labeling that satisfies the following:

(t1) $f(w) = \infty$,

(t2) $f(a) < f(v)$ for all $av \in E(G) \setminus \{uw\}$ with $a \in A$, and

(t3) for every edge $bc \in E(G)$ with $b \in B$ and $c \in C$ there exists an edge $e' = b'c'$ with $b \in B$ and $c \in C$ such that $|h(c) - h(b)| + |h(c') - h(b')| = 2n$.

Bunge [2] proved the following theorem, which is another important tool in our decompositions.

Theorem 3.11 (Bunge 2018)**.** *If a tripartite graph G with n edges has a 1-rotational ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nk} for each positive integer k.*

4 Catalog

First we provide a catalog of all bi-cyclic graphs with eight edges. By $H_i(j, k; l)$ we denote the i-th type of a connected graph containing cycles C_j and C_k joined by a path with l edges, and by $D_i(j, k; l)$ a disconnected graph with the same parameters. When the cycles belong to different components, we deonte the graph by $D_i(j, k; -)$.

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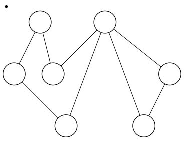
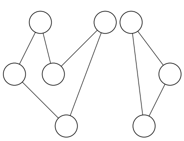
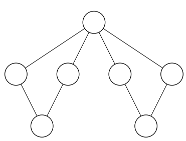


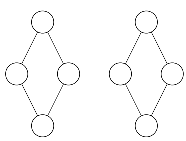
$H_1(5, 3; 0)$



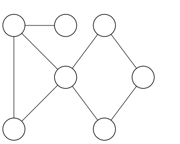
$D_1(5, 3; -)$



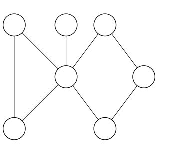
$H_1(4, 4; 0)$



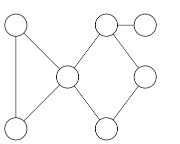
$D_1(4, 4; -)$



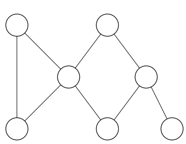
$H_1(4, 3; 0)$



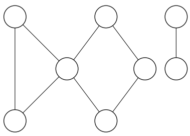
$H_2(4, 3; 0)$



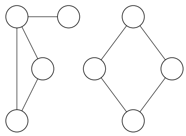
$H_3(4, 3; 0)$



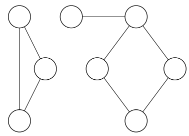
$H_4(4, 3; 0)$



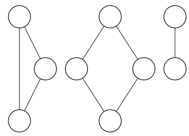
$D_1(4,3;0)$



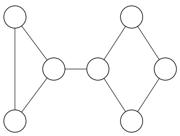
$D_1(4,3;-)$



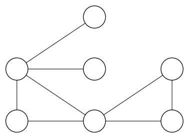
$D_2(4,3;-)$



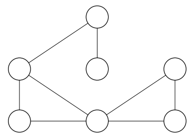
$D_3(4,3;-)$



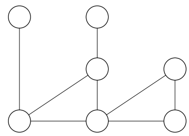
$H_1(4,3;1)$



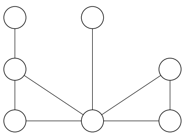
$H_1(3,3;0)$



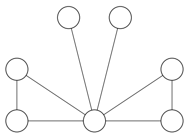
$H_2(3,3;0)$



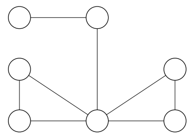
$H_3(3,3;0)$



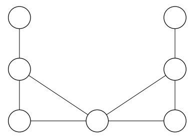
$H_4(3,3;0)$



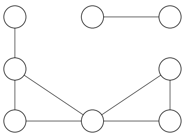
$H_5(3,3;0)$



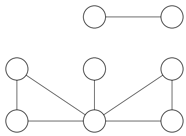
$H_6(3,3;0)$



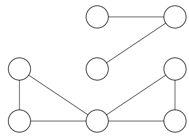
$H_7(3,3;0)$



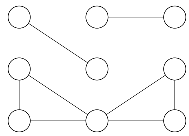
$D_1(3,3;0)$



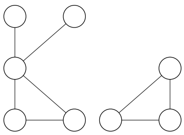
$D_2(3,3;0)$



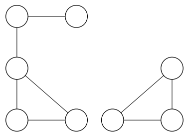
$D_3(3,3;0)$



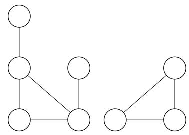
$D_4(3,3;0)$



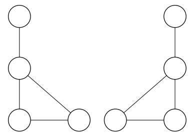
$D_1(3,3;-)$



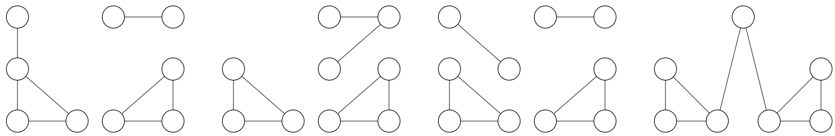
$D_2(3,3;-)$



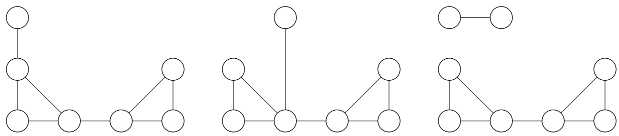
$D_3(3,3;-)$



$D_4(3,3;-)$



$D_5(3,3;-)$ $D_6(3,3;-)$ $D_7(3,3;-)$ $H_1(3,3;2)$



$H_1(3,3;1)$ $H_2(3,3;1)$ $D_1(3,3;1)$

5 Labelings and main result

We first observe that because the graphs $H_1(5,3;0), D_1(5,3;-)$, $H_1(4,4;0)$ and $D_1(4,4;-)$ have all vertices of even degrees, they cannot decompose K_{16n}, since its vertices have an odd degree.

Observation 5.1. *The graphs*

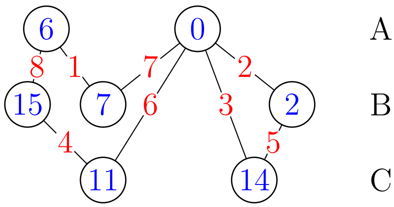
$$H_1(5,3;0), \quad D_1(5,3;-), \quad H_1(4,4;0) \quad \textit{and} \quad D_1(4,4;-)$$

do not decompose K_{16n} for any n.

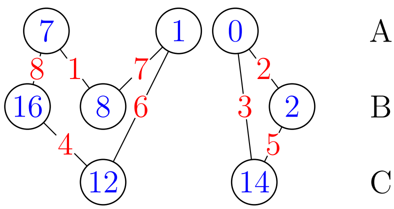
Although the graphs $H_1(4,3;1)$ and $H_1(3,3;2)$ do not admit a 1-rotational ρ-tripartite labeling, they still decompose K_{16n}, as shown by El-Zanati [5].

Theorem 5.2 (El-Zanati 2018)**.** *There exists a decomposition of K_{16n} into graphs $H_1(4,3;1)$ and $H_1(3,3;2)$ for every positive integer n.*

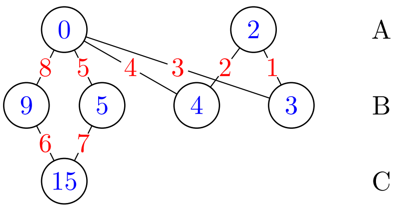
Now we present labelings for the graphs where the labelings exist. In the left column, we show a ρ-tripartite labeling for decompositions of K_{16n+1}, in the right one a 1-rotational ρ-tripartite labeling for decompositions of K_{16n}.



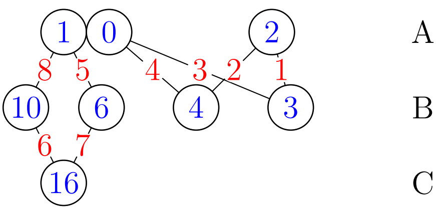
$H_1(5,3;0)$



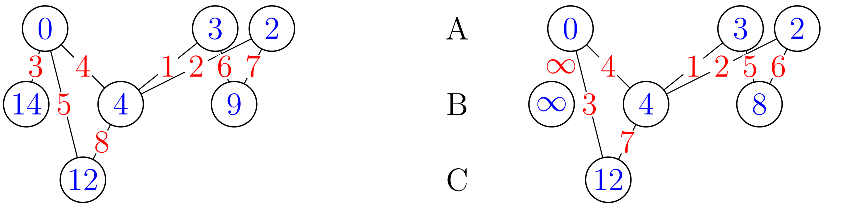
$D_1(5,3;-)$



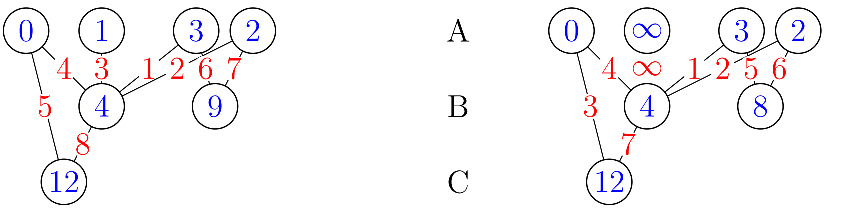
$H_1(4,4;-)$



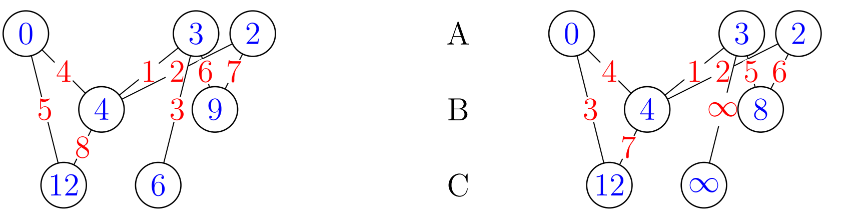
$D_1(4,4;0)$



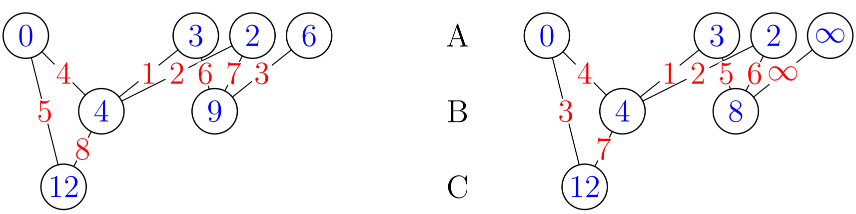
$H_1(4, 3; 0)$



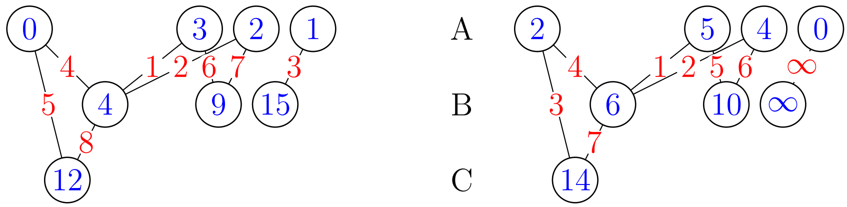
$H_2(4, 3; 0)$



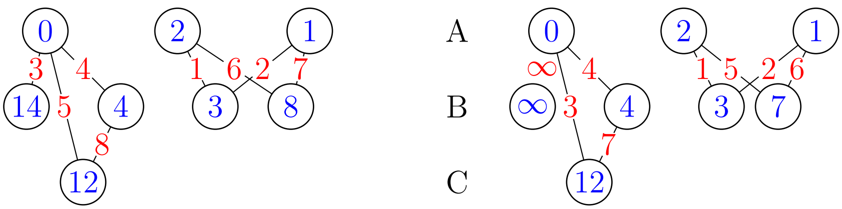
$H_3(4, 3; 0)$



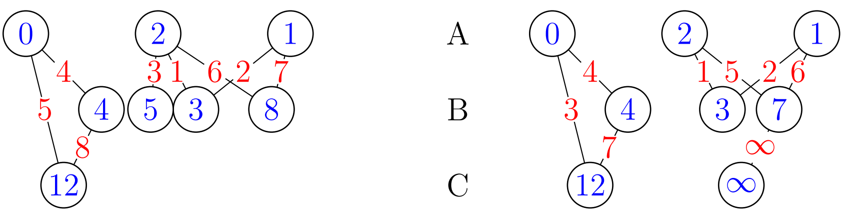
$H_4(4, 3; 0)$



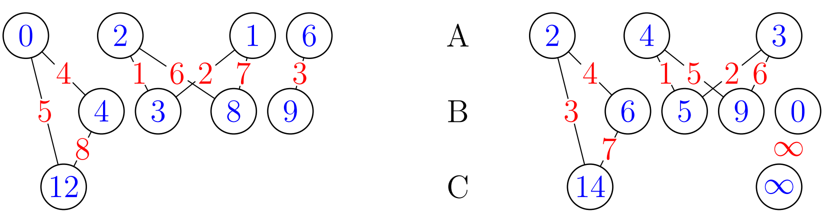
$D_1(4,3;0)$



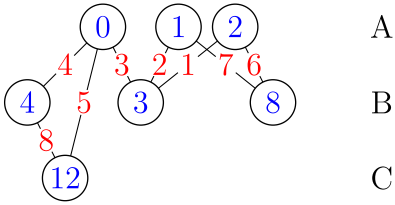
$D_1(4,3;-)$



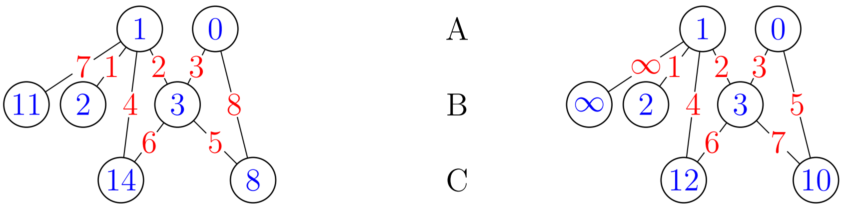
$D_2(4,3;-)$



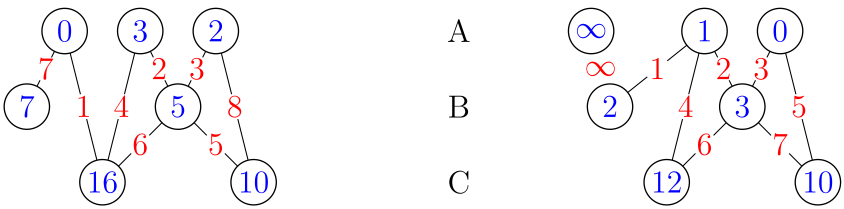
$D_3(4,3;-)$



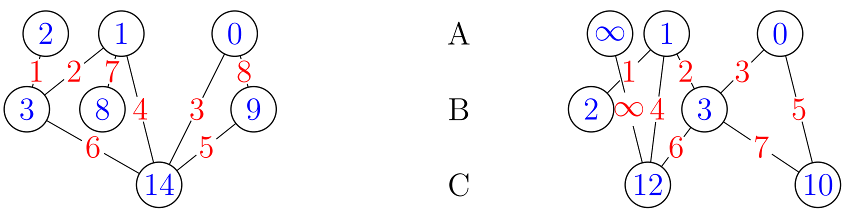
$H_1(4,3;1)$



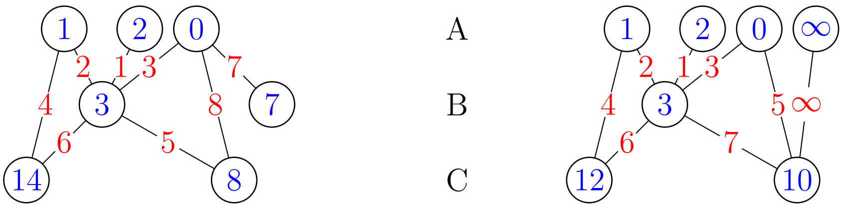
$H_1(3,3;0)$



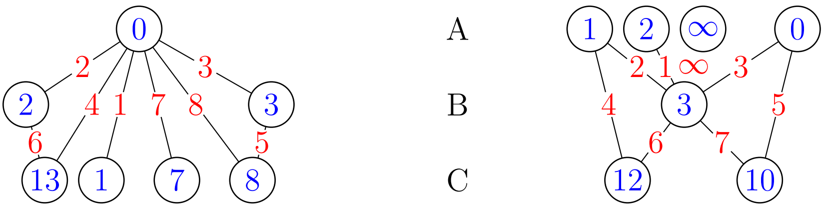
$H_2(3,3;0)$



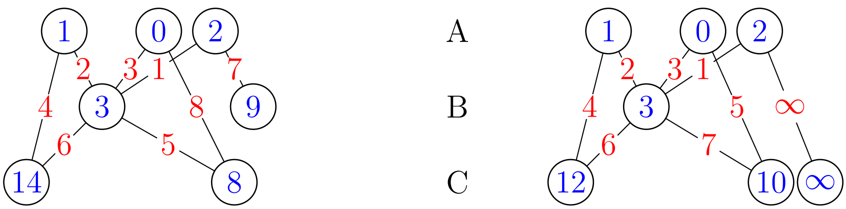
$H_3(3,3;0)$



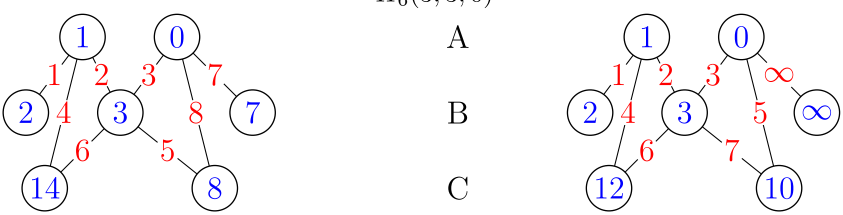
$H_4(3,3;0)$



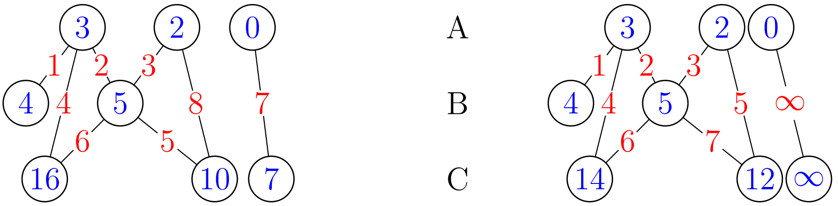
$H_5(3,3;0)$



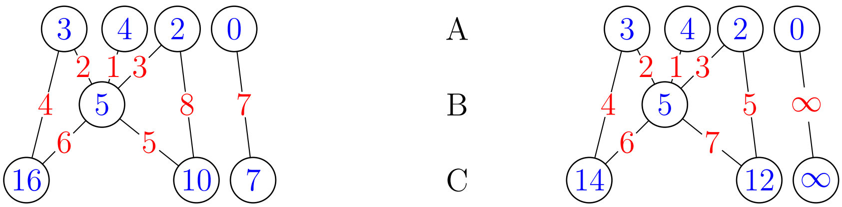
$H_6(3,3;0)$



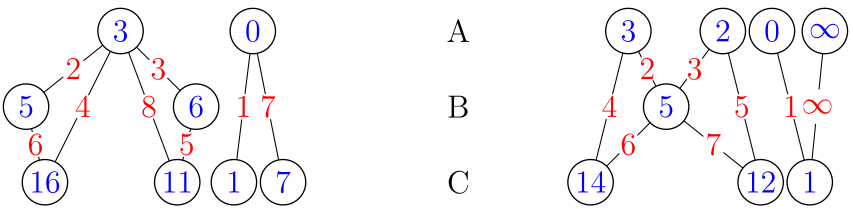
$H_7(3,3;0)$



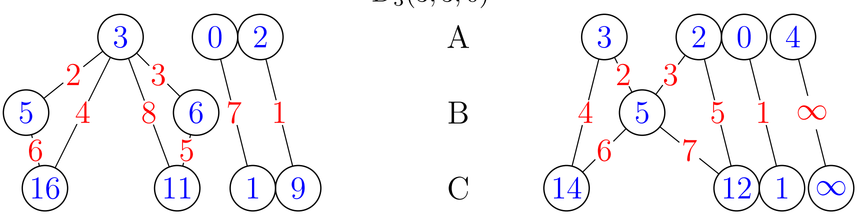
$D_1(3,3;0)$



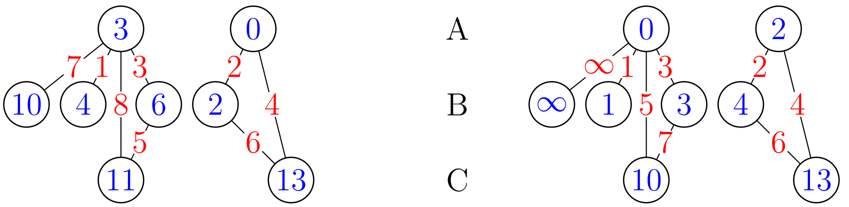
$D_2(3,3;0)$



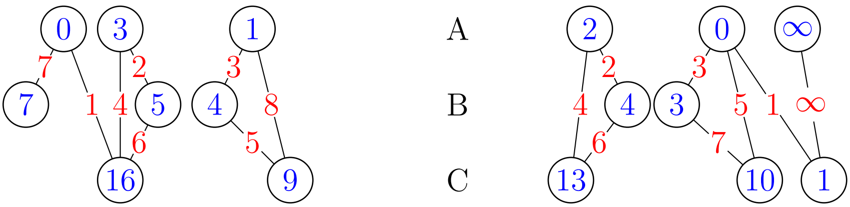
$D_3(3,3;0)$



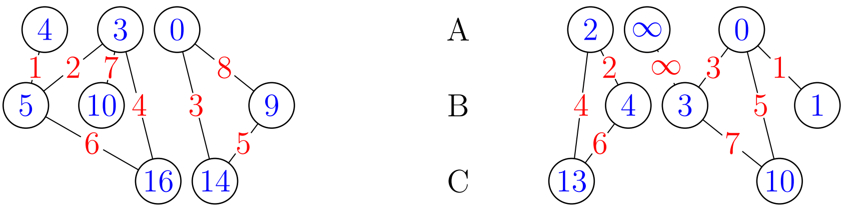
$D_4(3,3;0)$



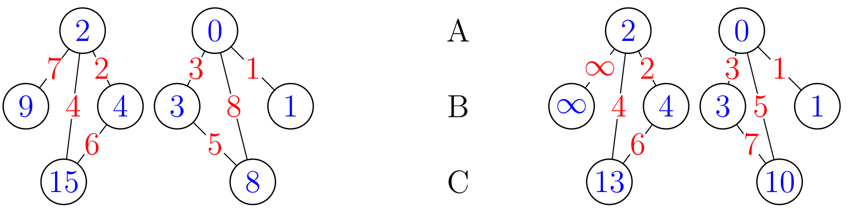
$D_1(3,3;-)$



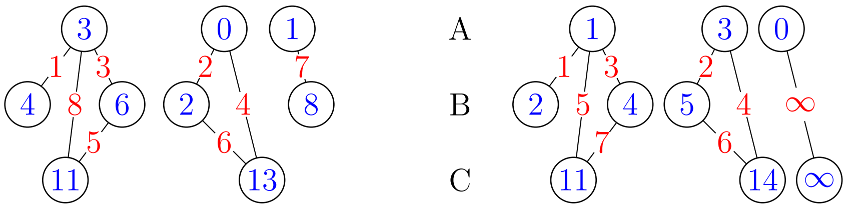
$D_2(3,3;-)$



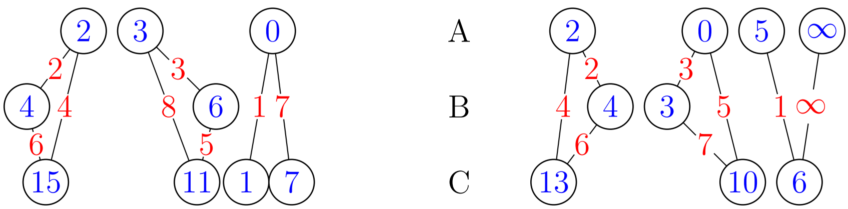
$D_3(3,3;-)$



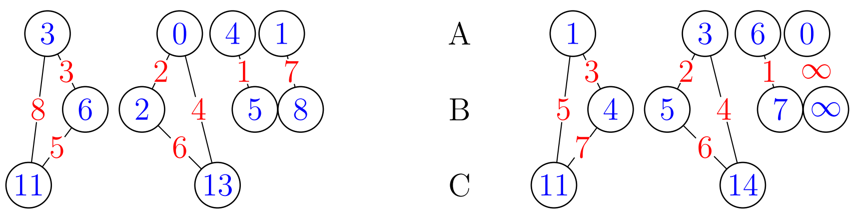
$D_4(3,3;-)$



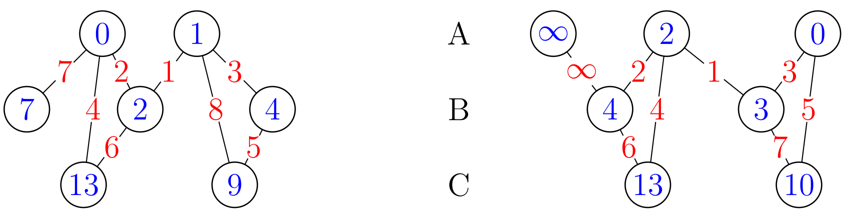
$D_5(3,3;-)$



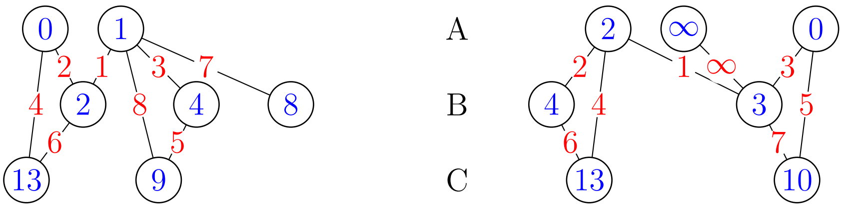
$D_6(3,3;-)$



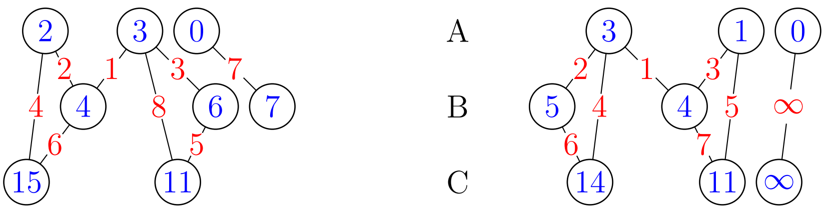
$D_7(3,3;-)$



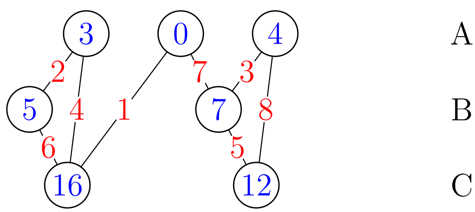
$H_1(3,3;1)$



$H_2(3,3;1)$



$D_1(3,3;1)$



$H_1(3,3;2)$

Our main result now follows directly from Observation 5.1, Theorem 5.2 and the above labelings.

Theorem 5.3. *A bi-cyclic graph G with eight edges decomposes the complete graph K_n if and only if*

- *there is a vertex of an odd degree and $n \equiv 0, 1 \pmod{16}$, or*
- *all vertices have even degrees and $n \equiv 1 \pmod{16}$.*

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