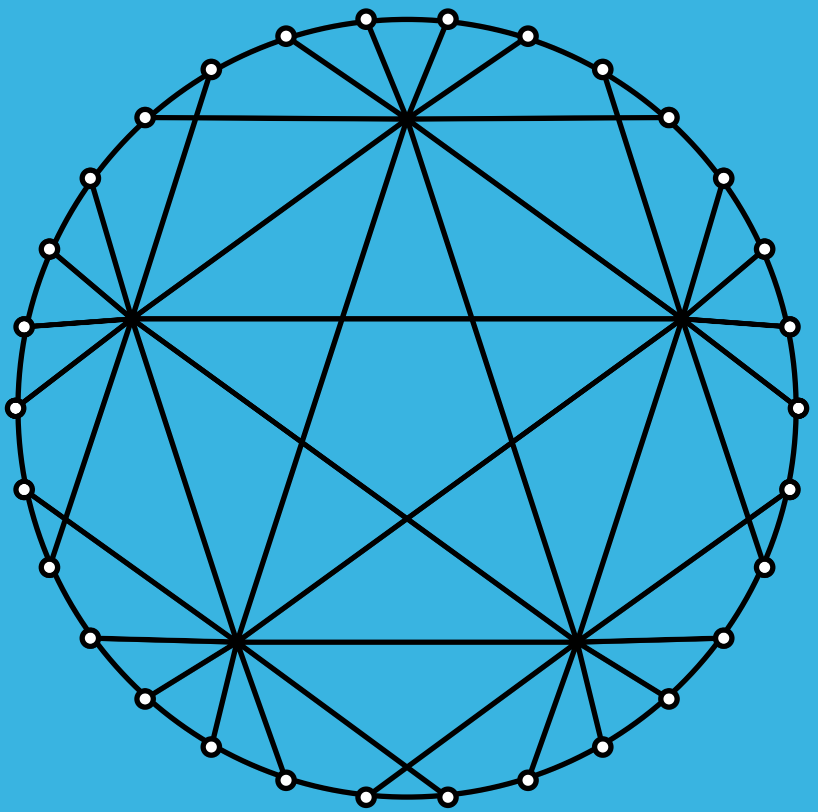
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 88
02 02 2020



Boca Raton, FL, U.S.A.

ISSN: 2689-0674 (Online)
ISSN: 1183-1278 (Print)

BULLETIN OF THE ICA
Volume **88** (2020), Pages 50–64

Distance-hereditary and strongly distance-hereditary graphs

Terry A. McKee

Wright State University, Dayton, Ohio USA
terry.mckee@wright.edu

Abstract: A graph G is distance-hereditary if the distance between vertices in connected induced subgraphs always equals the distance between them in G; equivalently, G contains no induced cycle of length 5 or more and no induced house, domino, or gem subgraph.

Define G to be strongly distance-hereditary if G is distance-hereditary and G^2 is strongly chordal. Although this definition seems completely unmotivated, it parallels a couple of ways in which strongly chordal graphs are the natural strengthening of chordal graphs; for instance, being distance-hereditary is characterized by the $k = 1$ case of a $\forall k \geq 1$ characterization of being strongly distance-hereditary. Moreover, there is an induced forbidden subgraph characterization of a distance-hereditary graph being strongly distance-hereditary.

1 Introduction and definitions

A graph G is *distance-hereditary* if, in every connected induced subgraph G' of G, the distance between vertices in G' equals their distance in G; see [2, 4, 8]. Proposition 1.1 will give three basic characterizations, where (1) is from [8], and (2) and (3) are from [2] (also see [1]). In characterization (1), chords ab and cd are *crossing chords* of a cycle C if the four vertices come

AMS (MOS) Subject Classifications: Primary 05C75; secondary 05C12
Key words and phrases: distance-hereditary graph; strongly chordal graph

Received: 8 June 2019
Accepted: 23 October 2019

in the order a, c, b, d around C. In (2), the *house*, *domino*, and *gem* graphs are defined in Figure 1. In (3), a *pendant vertex* is a vertex of degree 1, and *twin vertices* are vertices that have exactly the same neighbors (whether or not they are adjacent to each other).

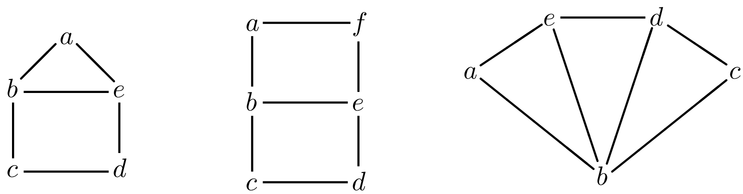


Figure 1: The forbidden subgraphs, along with C_n with $n \geq 5$, for being distance-hereditary: left to right, the *house*, *domino*, and *gem*.

Proposition 1.1. *Each of the following is equivalent to a graph G being distance-hereditary*:

(1) *Every cycle of length* 5 *or more in G has crossing chords.*

(2) *G has no induced subgraph isomorphic to C_n with $n \geq 5$ or to a house, domino, or gem graph.*

(3) *Every induced subgraph of G contains either a pendant vertex or twin vertices.*

The present paper's discussion of distance-hereditary and strongly distance-hereditary graphs requires a quick review of chordal and strongly chordal graphs. A graph is called *chordal* if every cycle with length 4 or more has a chord. A graph is *strongly chordal* if it is chordal and contains no induced n-sun subgraph with $n \geq 3$, where an *n-sun* is a chordal graph whose vertex set is partitioned into n vertices $w_1, \ldots, w_n$ that, in that order, form a cycle (possibly with chords) together with n independent vertices $u_1, \ldots, u_n$ that each have degree 2 such that each u_i is adjacent to w_i and w_{i-1} (calculating subscripts modulo n). See [4, 14] for other characterizations and the history of chordal and strongly chordal graphs. In spite of distance-hereditary graphs and chordal graphs being quite different from each other, they will play partially analogous roles several times in this paper.

Define a graph to be *strongly distance-hereditary* if it is distance-hereditary and its square is strongly chordal, where the *square* G^2 of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in G^2 if and only if they are distance at most 2 apart in G. Although this definition of strongly distance-hereditary seems totally unmotivated at this moment, one goal of this paper is to provide motivation. For now, note that every strongly distance-hereditary graph is automatically distance-hereditary, and that every ptolemaic graph is strongly distance-hereditary (where *ptolemaic graphs* are the graphs that are both chordal and distance-hereditary; ptolemaic graphs are strongly chordal, and so their squares are also strongly chordal, see [4]). Finally, the square of every distance-hereditary graph is chordal by [1] (but the squares of C_5 and the house and gem graphs are chordal without those graphs being distance-hereditary). Other ways to strengthen distance-hereditary graphs are discussed in [13].

As in [4], define $\mathcal{C}(G)$ to be the set of all inclusion-maximal complete induced subgraphs of G, and define $\mathcal{CC}(G)$ to be the set of all inclusion-maximal connected induced subgraphs of G that have no induced P_4 subgraphs. Note that P_n denotes the graph that consists of a chordless path of order n, and so has length $n-1$. Thus $\mathcal{C}(G)$ is, equivalently, the set of all inclusion-maximal connected induced subgraphs of G that have no induced P_3 subgraphs. The $\mathcal{CC}(G)$ notation comes from the graphs without induced P_4 subgraphs being commonly called *cographs*, see [6], and so $\mathcal{CC}(G)$ is defined using connected cographs in the same way that $\mathcal{C}(G)$ uses complete graphs.

2 Intersection graph motivation

This section sketches the original motivation for the notion of strongly distance-hereditary graphs from [10]; details will not be needed in the remainder of the present paper.

For any graph G and family $\mathcal{F}(G) = \{S_1, \ldots, S_n\}$ of subsets of $V(G)$, the *intersection graph* $\Omega(\mathcal{F}(G))$ has vertex set $\mathcal{F}(G)$ with two vertices adjacent in $\Omega(\mathcal{F}(G))$ if and only if the corresponding subsets $S_i, S_j \in \mathcal{F}$ have $S_i \cap S_j \neq \emptyset$. An $\mathcal{F}(G)$*-tree* for G is a spanning subtree T of $\Omega(\mathcal{F}(G))$ such that, for each $v \in V(G)$, the vertices of T that contain v induce a subtree of T. One of the best known characterizations of being chordal is that G is chordal if and only if G has a $\mathcal{C}(G)$ tree; see [4, 14]. Proposition 2.1 is from [15] (also see [3]).

Proposition 2.1. *A graph G is distance-hereditary if and only if G has a $\mathcal{CC}(G)$ tree.*

If G has an $\mathcal{F}(G)$ tree T, then the intersection graph $\Omega(\mathcal{F}'(G))$ makes sense where $\mathcal{F}'(G) = \{S_i \cap S_j : S_iS_j \in E(T)\}$. If G also has an $\mathcal{F}'(G)$ tree T', then repeat this to define $\mathcal{F}''(G) = \{S_i \cap S_j : S_iS_j \in E(T')\}$ with the possible existence of an $\mathcal{F}''(G)$ tree T'', and so on. If all such $T, T', T'', \ldots, T^{(i)}$ exist until $E(T^{(i)}) = \emptyset$, then call T a *strong* $\mathcal{F}(G)$ *tree* for G. (See [14] for an alternative, but equivalent definition.) One result in [11] is that a chordal graph is strongly chordal if and only if it has a strong $\mathcal{C}(G)$ tree. The strongly distance-hereditary analog in Theorem 2.2 is from [10].

Theorem 2.2. *A graph G is strongly distance-hereditary if and only if G has a strong $\mathcal{CC}(G)$ tree.*

3 Forbidden subgraph characterization

Theorem 3.1. *A graph G is strongly distance-hereditary if and only if G is distance-hereditary and G^2 contains no induced 3-sun subgraph.*

Proof. Suppose G is distance-hereditary, and so by [1] all even powers of G are chordal. The "only if" direction follows directly from the definitions of strongly distance-hereditary and strongly chordal. The "if" direction will also follow from those definitions, after the proof in the following paragraph that G^2 cannot contain an induced n-sun with $n \geq 4$ (this is also proved in [1]).

Suppose G^2 contains an induced n-sun with $n \geq 4$ (arguing by contradiction), with vertices $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ as in the definition of n-sun. Thus the distance between u_i and u_j in G^2 is 2 if $|i-j| = 1$ and is 3 or more if $|i-j| > 1$. Therefore, the distance between u_i and u_j in G^4 is 1 if $|i-j| = 1$ and is 2 or more if $|i-j| > 1$. But means that $u_1, \ldots, u_n$, in that order, form a chordless n-cycle in G^4, contradicting that G^4 is chordal by [1]. □

Because squares of distance-hereditary graphs are chordal and because chordal graphs without induced 3-sun subgraphs have been characterized in various ways, there are several alternative statements for Theorem 3.1. One of those, using the *minimum radius property* from [5], is that G is strongly distance-hereditary if and only if G is distance-hereditary and G^2 satisfies $\mathrm{rad}(H) = \lceil \mathrm{diam}(H)/2 \rceil$ for all of its connected induced subgraphs H. Similarly, one could use G being distance-hereditary and G^2 being "hereditarily clique-Helly" as in [16] (also see [12]). But we really want a forbidden subgraph characterization in terms of G, not G^2, extending condition (2) of Proposition 1.1. Theorem 3.2 will do this.

Reference [10] wrongly claimed to characterize strongly distance-hereditary graphs by being distance-hereditary and forbidding a single order-7 induced subgraph (namely, H_7 in Figure 2, which is also the unique forbidden subgraph for certain distance-hereditary graphs studied in [7]). The proof in [10] was wrong, and [10, Thm. 3] and [10, Cor. 2] are false (but in spite of that, [10, Cor. 1] remains true; indeed, it is the conjunction of Theorems 2.2 and 3.1 above). Theorem 3.2 below corrects the characterization, now using the two forbidden subgraphs shown in Figure 2. Theorem 3.2 will be used in section 4 and will be proved in section 5.

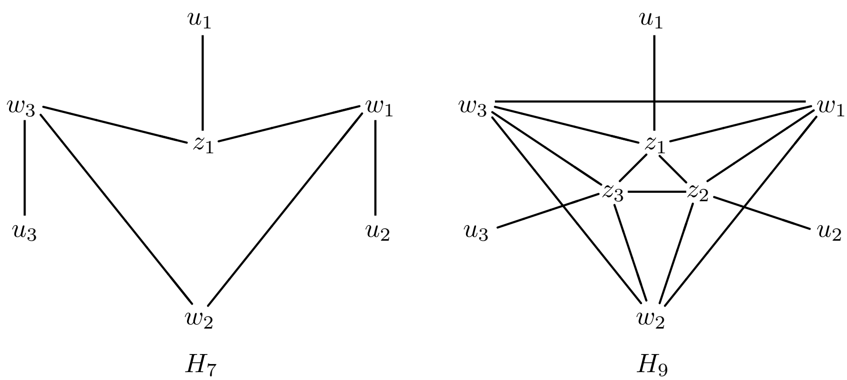


Figure 2: The two forbidden induced subgraphs for a distance-hereditary graph to be strongly distance-hereditary.

Theorem 3.2. *A graph is strongly distance-hereditary if and only if it is distance-hereditary with neither of the graphs H_7 and H_9 in Figure 2 as an induced subgraph.*

4 Cycle-based motivation

For any subgraph H of G—typically H will be an edge, a path of length 2, or a triangle of G—define $\mathrm{str}_{\mathcal{C}}(H)$ to be the number of members of $\mathcal{C}(G)$ that contain H. In Figure 2, for example, $\mathcal{C}(H_7) = E(H_7)$, and $\mathcal{C}(H_9)$ consists of the eight triangles and the three pendant edges; thus $\mathrm{str}_{\mathcal{C}}(H) = 1$ in H_9 when H is a pendant edge, while $\mathrm{str}_{\mathcal{C}}(H) = 2$ for the other edges. Note that $\mathrm{str}_{\mathcal{C}}(H) \geq 1$ always holds automatically when H is an edge or a triangle.

For any subgraph H of G, define $\mathrm{str}_{\mathcal{CC}}(H)$ to be the number of members of $\mathcal{CC}(G)$ that contain H. In Figure 2, for example, $\mathcal{CC}(H_7)$ consists of the three $K_{1,3}$ subgraphs induced by the closed neighborhoods $N[w_1]$, $N[w_3]$, $N[z_1]$, along with the $K_{2,2}$ 4-cycle induced by the non-pendant vertices; thus each pendant edge xy has $\mathrm{str}_{\mathcal{CC}}(xy) = 1$, $\mathrm{str}_{\mathcal{CC}}(w_1w_2) = \mathrm{str}_{\mathcal{CC}}(w_2w_3) = 2$, and $\mathrm{str}_{\mathcal{CC}}(w_1z_1) = \mathrm{str}_{\mathcal{CC}}(w_3z_1) = 3$ in H_7. Similarly, $\mathcal{CC}(H_9)$ consists of the three subgraphs induced by the closed neighborhoods $N[z_i]$, along with the $K_{2,2,2}$ octahedron induced by the non-pendant vertices; thus each pendant edge has $\mathrm{str}_{\mathcal{CC}}(u_iz_i) = 1$, each $\mathrm{str}_{\mathcal{CC}}(w_iw_j) = 2$, each $\mathrm{str}_{\mathcal{CC}}(w_iz_j) = 3$, and each $\mathrm{str}_{\mathcal{CC}}(z_iz_j) = 4$ in H_9. Note that $\mathrm{str}_{\mathcal{CC}}(H) \geq 1$ always holds automatically when H is an edge, a path of length 2, or a triangle.

Recall that G is chordal if and only if every cycle either has a chord or is a triangle. This corresponds to the $k = 1$ instance of the characterization of strongly chordal in [9], which states that G is strongly chordal if and only if, for every $k \geq 1$ and every cycle C all of whose edges x_hx_{h+1} have $\mathrm{str}_{\mathcal{C}}(x_hx_{h+1}) \geq k$, either C has a chord x_ix_j with $\mathrm{str}_{\mathcal{C}}(x_ix_j) \geq k$ or C is a triangle with $\mathrm{str}_{\mathcal{C}}(C) \geq k$. As an added observation, this characterization is equivalent, see [12], to G being chordal and, for every induced subgraph H of G and every cycle C of H all of whose edges x_hx_{h+1} have $\mathrm{str}_{\mathcal{C}}(x_hx_{h+1}) \geq 2$ in H, either C has a chord x_ix_j with $\mathrm{str}_{\mathcal{C}}(x_ix_j) \geq 2$ in H or is a triangle with $\mathrm{str}_{\mathcal{C}}(C) \geq 2$ in H.

Theorem 4.1 [respectively, Theorem 4.3] below will be a characterization of [strongly] distance-hereditary graphs that mimics the chord-and-triangle characterization of [strongly] chordal graphs stated in the preceding paragraph. They will illustrate a simple sense in which being a strongly distance-hereditary graph strengthens being a distance-hereditary graph: namely, the characterization of distance-hereditary in Theorem 4.1 will correspond to the $k = 1$ instance of the $\forall k \geq 1$ characterization of strongly distance-hereditary in Theorem 4.3 (the $\mathrm{str}_{\mathcal{CC}}$ statements there hold automatically when $k = 1$). The choice of a suitable distance-hereditary characterization for Theorem 4.1 is not obvious, however, and it seemingly cannot be simply in terms of cycles and chords. (This is in spite of condition (2) of Proposition 1.1, since H_7 in Figure 2 is not strongly distance-hereditary and has only one cycle, a 4-cycle.) The characterization chosen for Theorem 4.1 involves, instead, adjacent edges of a cycle, reflecting that such edges of G lead to edges of G^2.

Theorem 4.1. *A graph is distance-hereditary if and only if every two adjacent edges that are in a cycle are in a cycle of length* 3 *or* 4*.*

Proof. For the "only if" direction, suppose G is a distance-hereditary graph, and suppose cycle C of G contains edges ab and bc where C has minimum length $n \geq 5$ among all such cycles; thus ab and bc are not in a length 3 or 4 cycle (arguing by contradiction). But then a and c are a distance 2 apart in G and distance at least 3 apart in $G - b$ (by the minimality of $n \geq 5$), contradicting that G is distance-hereditary.

For the "if" direction, if G is not distance-hereditary, then by Proposition 1.1, G contains an induced subgraph H that is either C_n with $n \geq 5$ or the house, domino, or gem graph. Let a, b, c be three consecutive vertices around the cycle if $H = C_n$ or be those vertices as labeled in Figure 1. But then the adjacent edges ab and bc are in a unique cycle, and that cycle has length 5 or more. □

Lemma 4.2. *If H is a subgraph of a distance-hereditary graph G, then* $\mathrm{str}_{\mathcal{CC}}(H)$ *in G equals* $\mathrm{str}_{\mathcal{C}}(H)$ *in G^2.*

Proof. If H is a distance-hereditary graph, then H has no induced P_4 subgraph if and only if every connected subgraph of H has diameter at most 2; see [6, 17]. Therefore, H is a connected induced subgraph of G that has no induced P_4 subgraph if and only if $V(H)$ induces a complete subgraph of G^2. The lemma then follows from the definitions of $\mathcal{C}(G)$ and $\mathcal{CC}(G)$ and of $\mathrm{str}_{\mathcal{C}}(H)$ and $\mathrm{str}_{\mathcal{CC}}(H)$. □

Theorem 4.3. *A graph is strongly distance-hereditary if and only if, for every $k \geq 1$, every two adjacent edges ab and bc that are in a cycle are in a cycle C of length* 3 *or* 4 *such that:*

- **(1)** *If $E(C) = \{ab, bc, ac\}$ and each $xy \in E(C)$ has $\mathrm{str}_{\mathcal{CC}}(xy) \geq k$, then $\mathrm{str}_{\mathcal{CC}}(C) \geq k$.*
- **(2)** *If $E(C) = \{ab, bc, cd, ad\}$, each $xy \in E(C)$ has $\mathrm{str}_{\mathcal{CC}}(xy) \geq k$, and the subpath $\pi = a, b, c$ of C has $\mathrm{str}_{\mathcal{CC}}(\pi) \geq k$, then the subpath $\pi' = a, d, c$ of C has $\mathrm{str}_{\mathcal{CC}}(\pi') \geq k$.*

Proof. For the "only if" direction, suppose that G is strongly distance-hereditary, $k \geq 1$, and adjacent edges ab and bc are in a cycle. Theorem 4.1 ensures that ab and bc are in a cycle C of length 3 or 4.

For (1), suppose $E(C) = \{ab, bc, ac\}$ and each $xy \in E(C)$ has $\text{str}_{\mathcal{CC}}(xy) \geq k$ in G. By Lemma 4.2, each $xy \in E(C)$ has $\text{str}_{\mathcal{C}}(xy) \geq k$ in G^2, so $\text{str}_{\mathcal{C}}(C) \geq k$ in G^2 (since G^2 is strongly chordal, using the characterization strongly chordal from [9] stated in the third paragraph of this section). By Lemma 4.2, $\text{str}_{\mathcal{CC}}(C) \geq k$ in G, and so (1) holds.

For (2), suppose $E(C) = \{ab, bc, cd, ad\}$, each $xy \in E(C)$ has $\text{str}_{\mathcal{CC}}(xy) \geq k$ in G, and $\pi = a, b, c$ has $\text{str}_{\mathcal{CC}}(\pi) \geq k$ in G. By Lemma 4.2, each $xy \in E(C)$ has $\text{str}_{\mathcal{C}}(xy) \geq k$ in G^2 and $\text{str}_{\mathcal{C}}(\pi) \geq k$ in G^2. Since $ac \in E(G^2)$, the triangle abc of G^2 also has $\text{str}_{\mathcal{C}}(abc) \geq k$ in G^2. Thus the edge ac of G^2 has $\text{str}_{\mathcal{C}}(ac) \geq k$ in G^2, so the triangle adc of G^2 has $\text{str}_{\mathcal{C}}(adc) \geq k$ in G^2 (again since G^2 is strongly chordal, using [9]), and so $\text{str}_{\mathcal{C}}(\pi') \geq k$ in G^2. By Lemma 4.2, $\text{str}_{\mathcal{CC}}(\pi') \geq k$ in G, and so (2) holds.

For the "if" direction, suppose graph G is not strongly distance-hereditary. If G is not even distance-hereditary, then by Theorem 4.1 there will be adjacent edges that are in a cycle, but not a cycle of length 3 or 4. Hence suppose G is distance-hereditary and so, by Theorem 3.2, either H_7 or H_9 in Figure 2 is an induced subgraph of G. Every two edges that are in a cycle of H_7 or H_9 are in a cycle of length 3 or 4. For H_7, if $a = w_1$, $b = z_1$, $c = w_3$, $d = w_2$, $\pi = w_1, z_1, w_3$, and C is the 4-cycle induced by the non-pendant vertices, then each $xy \in E(C)$ has $\text{str}_{\mathcal{CC}}(xy) \geq k = 2$ and $\text{str}_{\mathcal{CC}}(\pi) = 2$, but $\text{str}_{\mathcal{CC}}(\pi') = 1$; thus (2) would fail. For H_9, if $a = w_1$, $b = w_2$, $c = w_3$, and C is the triangle $w_1w_2w_3$, then each $xy \in E(C)$ has $\text{str}_{\mathcal{CC}}(xy) = k = 2$, but $\text{str}_{\mathcal{CC}}(C) = 1$; thus (1) would fail. □

As an added observation, since conditions (1) and (2) fail when $k = 2$ in the "if" direction, the proof of Theorem 4.3 also shows that G is strongly distance-hereditary if and only if G is distance-hereditary and, for every induced subgraph H of G, every two adjacent edges ab and bc that are in a cycle of H are in a cycle C of H of length 3 or 4 such that the $k = 2$ instances of conditions (1) and (2) of Theorem 4.3 hold in H.

5 Proving theorem 3.2

For every graph G and $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G that is induced by S. Thus $G[S]^2$ is the square of the subgraph of G induced by S, and $G^2[S]$ is the subgraph of G^2 induced by S. Taking $G = C_5$ and $S \subset V(G)$ with $|S| = 4$ gives an example with $G[S]^2 \not\cong G^2[S]$.

Lemma 5.1. *If G is a distance-hereditary graph, $S \subseteq V(G)$, and $G[S]$ is connected, then $G[S]^2 \cong G^2[S]$.*

Proof. Suppose G is distance-hereditary, $a, b \in S \subseteq V(G)$, and $H = G[S]$ is connected. The definition of H^2 ensures that $d_{H^2}(a,b) = 1$ if and only if $1 \leq d_H(a,b) \leq 2$. Since H is a connected induced subgraph of G and G is distance-hereditary, $d_H(a,b) = d_G(a,b)$. That equality and the definition of G^2 ensure that $1 \leq d_H(a,b) \leq 2$ if and only if $d_{G^2}(a,b) = 1$. Therefore, $G[S]^2 = H^2 \cong G^2[S]$. □

We now prove Theorem 3.2: *A graph is strongly distance-hereditary if and only if it is distance-hereditary with neither of the graphs H_7 and H_9 in Figure 2 as an induced subgraph.*

The "only if" direction follows immediately from H_7^2 and H_9^2 both being distance-hereditary, having the form shown in Figure 3, and not being strongly chordal (since the non-z_i vertices induce a 3-sun). Therefore, no graph that contains an induced H_7 or H_9 subgraph can be strongly distance-hereditary.

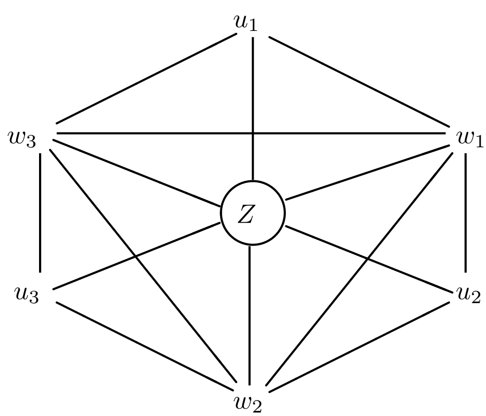


Figure 3: The subgraph G^* of G^2, where Z represents the set $\{z_1, \ldots, z_t\}$ and each z_i is adjacent to all the other $t + 5$ vertices.

For the "if" direction, suppose G is distance-hereditary, so G^2 is chordal by [1], and yet G^2 is not strongly chordal (arguing that H_7 or H_9 is an induced subgraph of G). Moreover, suppose G is a minimal such graph in that, for every proper subgraph G' of G, the distance-hereditary graph G' has $(G')^2$ strongly chordal.

The chordal, but not strongly chordal, graph G^2 will contain an induced subgraph—call it G^*—by Theorem 3.8 of [1] that consists of a 3-sun together with $t \geq 1$ vertices $z_1, \ldots, z_t$ that are all the vertices that are adjacent in G^2 to all the vertices of the 3-sun. Say the 3-sun consists of independent degree-2 vertices u_1, u_1, u_2 and pairwise-adjacent degree-4 vertices w_1, w_2, w_3, with each u_i adjacent to w_i and w_{i-1} (calculating subscripts modulo 3 in this proof). Since G^* is chordal and $u_i w_{i+1}$ is not a chord of any 4-cycle induced by $\{u_i, w_{i+1}, z_j, z_k\}$, every two vertices z_j and z_k must be adjacent in G^*, and so $\{w_1, w_2, w_3, z_1, \ldots, z_t\}$ induces a K_{t+3} subgraph of G^*. The existence of z_1 ensures that $\text{diam}(G^*) = 2$.

Let $S = \{u_1, u_2, u_3, w_1, w_2, w_3, z_1, \ldots, z_t\}$, so $G^* = G^2[S]$, and let $H = G[S]$.

Claim 1: *H is connected.* Suppose instead that H is not connected (arguing by contradiction), with distinct components H' and H'' where (say) $z_1 \in V(H')$. By the definition of G^* in G^2 and since $H = G[S] = G[V(G^*)]$, there exist $r \geq 1$ vertices $x_1, \ldots, x_r \in V(G) - S$ such that each x_i is adjacent in G to at least one vertex in each of H' and H'' (in order to create H'-to-H'' edges, since G^* is connected). If $r = 1$, then $z_1 \in V(H')$ being adjacent in G^2 to every vertex in S implies that $x_1 z_1 \in E(G)$, and so x_1 is distance 1 or 2 in G from each vertex of H' and x_1 is adjacent in G to each vertex of H''; thus x_1 will be adjacent in G^2 to every u_i and w_i, which makes $x_1 = z_i$ for some i (contradicting $x_1 \notin S$). Thus, $r \neq 1$. Similarly, if every two $x_i, x_j \in \{x_1, \ldots, x_r\}$ have exactly the same neighbors in H', then each $x_i z_1 \in E(G)$, so each x_i is again adjacent in G^2 to all the vertices of G^*, and so each $x_i = z_j$ for some j (again contradicting $x_i, x_j \notin S$). Thus, without loss of generality, say x_1 and x_2 have different neighborhoods in H'.

Let $x_1', x_2' \in V(H')$ be neighbors in G of x_1 and x_2, respectively, but $x_1 x_2' \notin E(G)$ (so $x_1' \neq x_2'$) such that $d_{H'}(x_1', x_2')$ is minimum; also let $x_1'', x_2'' \in N(H'')$ be neighbors in G of x_1 and x_2, respectively, such that $d_{H''}(x_1'', x_2'')$ is minimum (possibly $x_1'' = x_2''$). The edges $x_1 x_1', x_1 x_1'', x_2 x_2', x_2 x_2''$ and minimum-length x_1'-to-x_2' and x_1''-to-x_2'' paths in, respectively, H' and H'' combine with the *possible* edges $x_1 x_2$ and $x_1' x_2$ to form either an induced cycle of length 5 or more or an induced house, domino, or gem of G. (For instance, if $x_1 x_2$, $x_1' x_2' \in E(G)$ and $x_1'' = x_2''$, then $\{x_1, x_1', x_1'', x_2, x_2'\}$ induces a gem or house in G, depending on whether or not $x_1' x_2 \in E(H)$.) But inducing such a subgraph would contradict G being distance-hereditary. Therefore, $H = G[S]$ is connected.

Claim 2: $G^* \cong H^2$. This follows from Claim 1 and Lemma 5.1.

Claim 3: *H has no twin vertices.* Since $G^* = G^2[S]$ has no twin vertices except for the adjacent twins in $\{z_1, \dots, z_t\}$, the graph $H = G[S]$ has no twins except possibly among $\{z_1, \dots, z_t\}$. But if z_i and z_j are twins in H, then $G^* - z_j \cong (H - z_j)^2$ and $G - z_j$ would contradict the minimality of G (from the first paragraph of the proof).

Claim 4: *No w_i or z_i is a pendant vertex of H.* The following are the only possibilities (up to permuting subscripts): If w_1 is on the unique edge u_1w_1 in H, then $u_2w_1 \in E(H^2) - E(H)$ would require $u_1u_2 \in E(H)$, contradicting that $u_1u_2 \notin E(H^2)$. If w_1 is on the unique edge w_1w_2 in H, then $u_1w_1 \in E(H^2) - E(H)$ would require $u_1w_2 \in E(H)$, contradicting that $u_1w_2 \notin E(H^2)$. If w_1 is on the unique edge w_1z_1 in H, then u_1w_1, $u_2w_1 \in E(H^2) - E(H)$ would require u_1z_1, $u_2z_1 \in E(H)$, contradicting that $u_1u_2 \notin E(H^2)$. If z_1 is on the unique edge u_1z_1 in H, then $u_2z_1 \in E(H^2) - E(H)$ would require $u_1u_2 \in E(H)$, contradicting that $u_1u_2 \notin E(H^2)$. If z_1 is on the unique edge w_1z_1 in H, then u_1z_1, $u_2z_1 \in E(H^2) - E(H)$ would require u_1w_1, $u_2w_1 \in E(H)$, contradicting that $u_1u_2 \notin E(H^2)$. Finally, if z_1 is on the unique edge z_1z_2 in H, then u_1z_1, $u_2z_1 \in E(H^2) - E(H)$ would require u_1z_2, $u_2z_2 \in E(H)$, contradicting that $u_1u_2 \notin E(H^2)$.

Claim 5: *Either each u_i is a pendant vertex of H or H contains a proper induced subgraph that is isomorphic to H_7 or H_9.* By Theorem 2.4 of [1], the distance-hereditary graph H, which has no twins by Claim 3, must therefore have at least two pendant vertices; moreover, the only possible pendant vertices are u_1, u_2, u_3 by Claim 4.

Suppose instead that (say) u_1 is not a pendant vertex of H (and so both u_2 and u_3 are pendant vertices of H). Since u_1u_2, u_1u_3, $u_1w_2 \notin E(H^2)$, vertex u_1 must be adjacent in H to at least two of w_1, w_3, $z_1, \dots, z_t$. Therefore, (up to permuting subscripts) there are only the following three possible cases for edges to be incident with u_1; each leads to either a contradiction or to H properly containing an induced H_7 or H_9 subgraph.

Case 1: $u_1w_1, u_1z_1 \in E(H)$. Thus u_2w_1, $u_2z_1 \notin E(H)$ since $u_1u_2 \notin E(H^2)$, and w_1w_2, $w_2z_1 \notin E(H)$ since $u_1w_2 \notin E(H^2)$, and $u_3z_1 \notin E(H)$ since $u_1u_3 \notin E(H^2)$. Since $u_2w_1 \in E(H^2) - E(H)$ is in only the triangles $u_2w_1w_2$, $u_2w_1z_1$, $u_2w_1z_i$ with $i > 1$ of G^* and w_1w_2, $u_2z_1 \notin E(H)$, there exists (say) z_2 such that w_1z_2, $u_2z_2 \in E(H)$ (to make $u_2w_1 \in E(H^2)$). Since $u_2z_1 \in E(H^2) - E(H)$ and u_2z_2 is the unique edge of H that is incident with the pendant vertex u_2, edge $z_1z_2 \in E(H)$. Similarly, $u_2w_2 \in E(H^2)$ $-E(H)$ and $w_2z_2 \in E(H)$. Thus, u_1z_2, u_3z_2, $w_3z_2 \notin E(H)$ since u_1w_2, u_3w_1, $u_2w_3 \notin E(H^2)$.

Since $u_3w_2 \in E(H^2)$ is in only the triangles $u_3w_2w_3$, $u_3w_2z_1$, $u_3w_2z_2$, $u_3w_3z_j$ with $j \geq 3$ of G^*, since $u_3w_3 \in E(H^2)$ is in only the triangles $u_3w_2w_3$, $u_3w_3z_1$, $u_3w_3z_2$, $u_3w_3z_j$ of G^*, since $u_3z_1 \in E(H^2) - E(H)$ is in only the triangles $u_3w_2z_1$, $u_3w_3z_1$, $u_3z_1z_2$, and $u_3z_1z_j$ of G^*, and since $u_3z_2 \in E(H^2) - E(H)$ is in only the triangles $u_3w_2z_2$, $u_3w_3z_2$, $u_3z_1z_2$, $u_3z_2z_j$ of G^*, therefore u_3 being a pendant vertex of H implies there exists (say) z_3 such that u_3z_3, w_2z_3, w_3z_3, z_1z_3, $z_2z_3 \in E(H)$. Thus u_1z_3, $w_1z_3 \notin E(H)$ since u_1u_3, $u_3w_1 \notin E(H^2)$. But then $\{u_1, w_1, z_1, z_2, z_3\}$ would induce either a gem or a house in H (depending on whether or not $w_1z_1 \in E(H)$), contradicting Proposition 1.1.

Case 2: $u_1w_1, u_1w_3 \in E(H)$, with each $u_1z_i \notin E(H)$. Thus w_1w_2, $w_2w_3 \notin E(H)$ since $u_1w_2 \notin E(H^2)$, and u_2w_1, $u_3w_3 \notin E(H)$ since u_1u_2, $u_1u_3 \notin E(H^2)$. Since $u_2w_1 \in E(H^2) - E(H)$ is in only the triangles $u_2w_1w_2$ and $u_2w_1z_i$ of G^* and $w_1w_2 \notin E(H)$, there exists (say) z_1 such that w_1z_1, $u_2z_1 \in E(H)$. Thus u_3z_1, $w_3z_1 \notin E(H)$ since u_3w_1, $u_2w_3 \notin E(H^2)$. By a symmetric argument, there exists (say) z_2 such that u_3z_2, $w_3z_2 \in E(H)$ (using that $z_1 \neq z_2$ since $u_3w_1 \notin E(H^2)$), and so u_2z_2, $w_1z_2 \notin E(H)$.

Since u_2z_1 is the unique edge of H that is incident with the pendant vertex u_2, edge $u_2w_2 \in E(H^2) - E(H)$ and $w_2z_1 \in E(H)$. A symmetric argument shows $w_2z_2 \in E(H)$. But then the 6-cycle C that has vertices $u_1, w_1, z_1, w_2, z_2, w_3$, in that order, could only possibly have chords w_1w_3 or z_1z_2; thus C would have no crossing chords, contradicting Proposition 1.1.

Case 3: u_1z_1, $u_1z_2 \in E(H)$, with u_1w_1, $u_1w_3 \notin E(H)$. Since u_1u_2, u_1u_3, $u_1w_2 \notin E(H^2)$, none of u_2, u_3, w_2 can be adjacent to z_1 or z_2 in H. Since u_2 and u_3 are pendant vertices of H, edges u_2w_1, u_2w_2 are not both in $E(H)$ and u_3w_2, u_3w_3 are not both in $E(H)$. Also, u_2w_2, u_3w_2 are not both in $E(H)$ since $u_2u_3 \notin E(H^2)$. Therefore, (up to permuting subscripts) there are only the following five possible subcases for the unique edges incident with u_2 and u_3.

Subcase 3.1: u_2w_1, $u_3w_2 \in E(H)$ and u_2w_2, $u_3w_3 \notin E(H)$. But $u_3z_1 \in E(H^2) - E(H)$ and $w_2z_1 \notin E(H)$ would then contradict u_3w_2 being the unique edge of H that is incident with u_3.

Subcase 3.2: u_3w_2, $u_2z_3 \in E(H)$ and u_2w_1, u_2w_2, $u_3w_3 \notin E(H)$. But $u_3z_1 \in E(H^2) - E(H)$ and $w_2z_1 \notin E(H)$ would again contradict u_3w_2 being the unique edge of H that is incident with u_3.

Subcase 3.3: u_2w_1, $u_3z_3 \in E(H)$ and u_2w_2, u_3w_2, $u_3w_3 \notin E(H)$. Thus $u_2z_3 \notin E(H)$ since $u_2u_3 \notin E(H^2)$. But then $u_2z_3 \in E(H^2) - E(H)$ and $w_1z_3 \notin E(H)$ would then contradict u_2w_1 being the unique edge of H that is incident with u_2.

Subcase 3.4: u_2w_1, $u_3w_3 \in E(H)$ and u_2w_2, $u_3w_2 \notin E(H)$. Since $u_3w_2 \in E(H^2) - E(H)$ is in only the triangles $u_3w_2w_3$ and $u_3w_2z_i$ with $i \geq 1$ of G^*, since $u_3z_1 \in E(H^2) - E(H)$ is in only the triangles $u_3w_2z_1$, $u_3w_3z_1$, $u_3z_1z_i$ with $i \geq 2$ in G^*, and since u_3w_3 is the unique edge of H that is incident with u_3, edges w_2w_3 and $w_3z_1 \in E(H)$. Similarly, w_1w_2, $w_1z_1 \in E(H)$. Moreover, $w_1w_3 \notin E(H)$, since $u_2w_1 \in E(H)$ and $u_2w_3 \notin E(H^2)$. Therefore, the subgraph of H that is induced by $\{u_1, u_2, u_3, w_1, w_2, w_3, z_1\}$ is H_7, contradicting the minimality of G (from the first paragraph of the proof).

Subcase 3.5: u_2z_3, $u_3z_4 \in E(H)$ and u_2w_1, u_2w_2, u_3w_2, $u_3w_3 \notin E(H)$ (and $z_3 \neq z_4$ since $u_2u_3 \notin E(H^2)$). Thus w_1z_4, $w_3z_3 \notin E(H)$ since u_3w_1, $u_2w_3 \notin E(H^2)$, and u_2z_4, $u_3z_3 \notin E(H)$ since $u_2u_3 \notin E(H^2)$. Since u_2w_1, u_2w_2, u_2z_1, u_2z_2, $u_2z_4, \in E(H^2) - E(H)$ and u_2z_3 is the unique edge of H incident with u_2, edges w_1z_3, w_2z_3, z_1z_3, z_2z_3, $z_3z_4 \in E(H)$. Similarly, w_2z_4, w_3z_4, z_1z_4, $z_2z_4 \in E(H)$. Since w_2z_3, $w_2z_4 \in E(H)$ and $u_1w_2 \notin E(H^2)$, edges u_1z_3, $u_1z_4 \notin E(H)$.

Since $u_1w_1 \in E(H^2) - E(H)$ is in only the triangles $u_1w_1w_3$, $u_1w_1z_i$ with $i \geq 1$ of G^* and $u_1w_3 \notin E(H)$, there must be some z_i adjacent to both u_1 and w_1 in H (thus $i \notin \{3, 4\}$). If $i \neq 1$, then the 5-cycle C that has vertices w_1, z_3, z_1, u_1, z_i, in that order, would by Proposition 1.1 have crossing chords w_1z_1 and z_3z_i (since u_1w_1, $u_1z_3 \notin E(H)$), and so z_1 is adjacent to both u_1 and w_1 in H. Thus, whether or not $i = 1$, edge $w_1z_1 \in E(H)$. Similarly, w_1z_2, w_3z_1, $w_3z_2 \in E(H)$.

The 5-cycle C' that has vertices w_1, z_3, z_4, w_3, z_1, in that order, has chords z_1z_3 and z_1z_4, but neither w_1z_4 nor w_3z_3, and C' has crossing chords by Proposition 1.1, so $w_1w_3 \in E(H)$. Moreover, $w_1w_2 \in E(H)$ (to prevent $\{w_1, w_2, z_2, z_3, z_4\}$ from inducing a gem) and $w_2w_3 \in E(H)$ (similarly). Therefore, the subgraph of H that is induced by $\{u_1, u_2, u_3, w_1, w_2, w_3, z_1, z_3, z_4\}$ is H_9, contradicting the minimality of G.

This concludes the (sub)case argument for Claim 5. In the remainder of the proof, suppose that all of u_1, u_2, u_3 are pendant vertices of H. Note that u_1u_2, u_2u_3, $u_1u_3 \notin E(H^2)$ implies no two of u_1, u_2, u_3 can be adjacent to the same vertex.

If u_iw_i, $u_{i+1}w_{i+1} \in E(H)$, then $u_{i+1}w_i \in E(H^2) - E(H)$ being in only the triangles $u_{i+1}w_iw_{i+1}$ and $u_{i+1}w_iz_j$ of G^* would contradict either that $w_iw_{i+1} \notin E(H)$ (since $u_iw_{i+1} \notin E(H^2)$) or that $u_{i+1}w_{i+1}$ is the unique edge of H that is incident with the pendant vertex u_{i+1} of H. If, instead, $u_iw_i, u_{i+1}z_1, u_{i+2}z_2 \in E(H)$ (where $z_1 \neq z_2$ since $u_{i+1}u_{i+2} \notin E(H^2)$), then $w_iz_2 \notin E(H)$ (since $u_{i+2}w_i \notin E(H^2)$) and u_iw_i being the unique edge of H incident with u_i would imply that the distance between u_i and z_2 in H is at least 3 (contradicting that $u_iz_2 \in E(H^2)$. Therefore (up to permuting subscripts), either u_1z_1, u_2w_1, $u_3w_3 \in E(H)$ or u_1z_1, u_2v_2, $u_3z_3 \in E(H)$.

First suppose u_1z_1, u_2w_1, $u_3w_3 \in E(H)$. Edges w_1w_3, $w_2z_1 \notin E(H)$ since u_3w_1, $u_1w_2 \notin E(H^2)$. Since $u_1w_1 \in E(H^2) - E(H)$ and u_1z_1 is the unique edge of H incident with u_1, edge $w_1z_1 \in E(H)$. Similarly, $w_3z_1 \in E(H)$. Since $u_2w_2 \in E(H^2) - E(H)$ and u_2w_1 is the unique edge of H incident with u_2, edge $w_1w_2 \in E(H)$. Similarly, $w_2w_3 \in E(H)$. Therefore, the induced subgraph $H[\{u_1, u_2, u_3, w_1, w_2, w_3, z_1\}] = H_7$.

For the remaining possibility, suppose u_1z_1, u_2z_2, $u_3z_3 \in E(H)$. Edges u_1z_2, u_1z_3, $w_2z_1 \notin E(H)$ since u_1u_2, u_1u_3, $u_1w_2 \notin E(H^2)$. Similarly, the edges u_2z_1, u_1z_3, w_3z_2, u_3z_1, u_3z_2, $w_1z_3 \notin E(H)$. Since $u_1w_1 \in E(H^2) - E(H)$ and u_1z_1 is the unique edge of H incident with u_1, edge $w_1z_1 \in E(H)$. Similarly, w_1z_2, w_2z_2, w_2z_3, w_3z_1, $w_3z_3 \in E(H)$. By Proposition 1.1, all the chords w_iw_j and z_iz_j (remembering that w_1z_3, w_2z_1, $w_3z_2 \notin E(H^2)$) of the 6-cycle that has vertices $z_1, w_1, z_2, w_2, z_3, w_3$, in that order, must exist. Therefore, the induced subgraph $H[\{u_1, u_2, u_3, w_1, w_2, w_3, z_1, z_2, z_3\}] = H_9$.

This concludes the proof of Theorem 3.2.

References

[1] H.-J. Bandelt, A. Henkmann and F. Nicolai, Powers of distance-hereditary graphs, *Discrete Math.*, **145** (1995), 37–60.

[2] H.-J. Bandelt and H.M. Mulder, Distance-hereditary graphs, *J. Combin. Theory, Ser. B*, **41** (1986), 182–208.

[3] A. Brandstädt, F.F. Dragan and F. Nicolai, Homogeneously orderable graphs, *Theoret. Comput. Sci.*, **172** (1997), 209–232.

[4] A. Brandstädt, V.B. Le and J.P. Spinrad, *Graph Classes: A Survey*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[5] G.J. Chang and G.L. Nemhauser, The k-domination and k-stability problems on sun-free chordal graphs, *SIAM J. Alg. Disc. Meth.*, **5** (1984), 332–345.

[6] D.G. Corneil, H. Lerchs and L.S. Burlingham, Complement reducible graphs, *Discrete Appl. Math.* **3** (1981), 163–174.

[7] F.F. Dragan and F. Nicolai, LexBFS-orderings and powers of distance hereditary graphs with application to the dimetral pair problem, *Discrete Appl. Math.*, **98** (2000), 191–207.

[8] E. Howorka, A characterization of distance-hereditary graphs, *Quart. J. Math. Oxford Ser. 2*, **28** (1977), 417–420.

[9] T.A. McKee, A new characterization of strongly chordal graphs, *Discrete Math.*, **205** (1999), 245–247.

[10] T.A. McKee, Maximal connected cographs in distance-hereditary graphs, *Util. Math.*, **57** (2000), 73–80.

[11] T.A. McKee, Subgraph trees in graph theory, *Discrete Math.*, **270** (2003), 3–12.

[12] T.A. McKee, Simplicial and nonsimplicial complete subgraphs, *Discuss. Math. Graph Theory*, **31** (2011), 577–586.

[13] T.A. McKee, Double-crossed chords and distance-hereditary graphs, *Australas. J. Combin.*, **65** (2016), 183–190.

[14] T.A. McKee and F.R. McMorris, *Topics in Intersection Graph Theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.

[15] F. Nicolai, A hypertree characterization of distance-hereditary graph. *Tech. Report Gerhard-Mercator-Universität-Gesamthochschule*, Duisburg SM-DU-255, 1994.

[16] E. Prisner, Hereditary clique-Helly graphs, *J. Combin. Math. Combin. Comput.*, **14** (1993), 216–220.

[17] D.P. Sumner, Dacey graphs, *J. Austral. Math. Soc.*, **18** (1974), 492–502.