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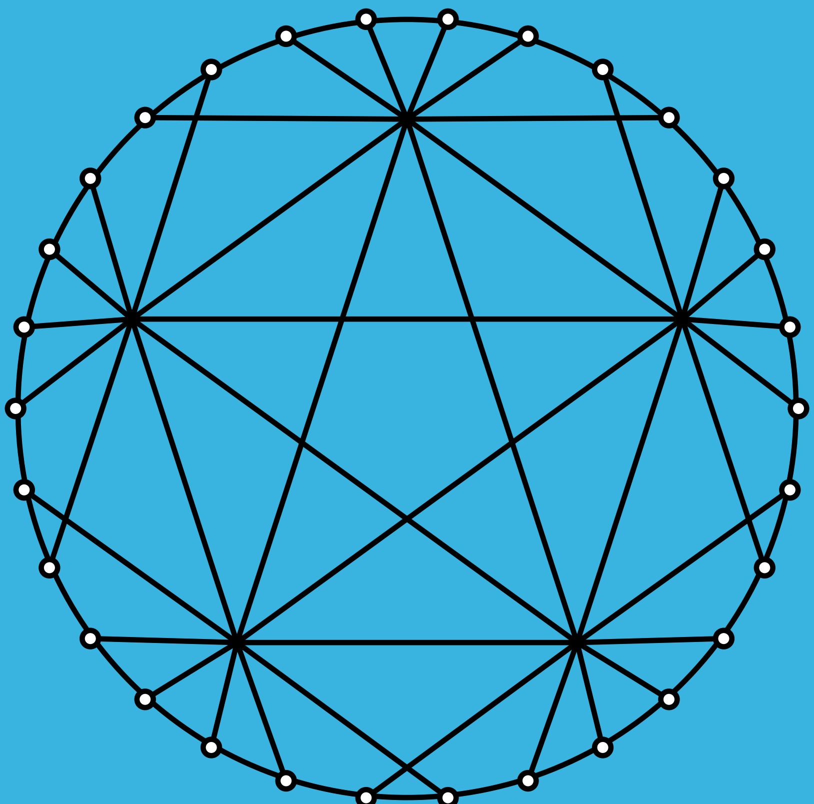

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# Two related (?) 2-edge-Hamiltonian bigraph conjectures

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**Abstract:** It is conjectured that the polygonal bigraphs, denoted $\mathrm{P}_{m,k}$, are all 2-edge-Hamiltonian and that a 3-connected bicubic graph is Hamilton-laceable if and only if it is 2-edge-Hamiltonian. There is persuasive evidence for the first conjecture based on proofs for several infinite families of the $\mathrm{P}_{m,k}$ as well as exhaustive computer searches for 56 or fewer vertices and suggestive evidence for the second. However, none of the case proofs suggest a general technique which is why the conjectures are put forward.

## 1 Introduction

There are four Hamiltonian graph properties listed below in roughly increasing order of restrictiveness (strength).

1. A graph is *Hamiltonian* if there exists a Hamilton cycle on the vertices.
2. A graph is *edge-Hamiltonian* if there exists a Hamilton cycle on every edge.
3. A graph is 2-*edge-Hamiltonian* if there exists a Hamilton cycle on every pair of edges.
4. A graph is *Hamilton-connected* if there exists a Hamilton path between every pair of vertices.

**Key words and phrases:** Hamilton-laceable,2-edge-Hamiltonian,polygonal bigraphs,Sullivan graphs

**AMS (MOS) Subject Classifications:** 05C45, 05C38, 05C30, 05C62

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Hamiltonian and edge-Hamiltonian are each implied by any stronger property, i.e. by a property below it in the list, and neither implies a stronger property. The graph in Figure 1 is edge-Hamiltonian since every edge is ei-

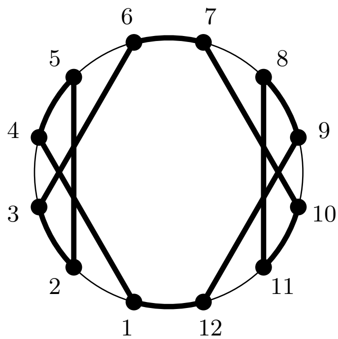



Figure 1

ther in the outer cycle or in the bold cycle, but can't be 2-edge-Hamiltonian since no Hamilton cycle is possible on the edge pair $2\ \bullet\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\bullet\ 4$ , nor can it be Hamilton-connected since no Hamilton path exists between vertices 1 and 6.

A *bigraph* (bipartite graph) can only host a Hamilton cycle if it is equitable, i.e. if the two parts have the same cardinality. An equitable bigraph can be Hamiltonian, edge-Hamiltonian or 2-edge-Hamiltonian but cannot be Hamilton-connected since there can't be a Hamilton path between two vertices in the same part. An equitable bipartite graph is *Hamilton-laceable* if there exists a Hamilton path between every pair of vertices in different parts. Hamilton-laceable is the analogous property for bipartite graphs to Hamilton-connected for general graphs.

Hamilton cycles include exactly two edges incident to each vertex. Hence a $k$-edge-Hamiltonian graph with $k > 2$ cannot have a vertex of degree exceeding 2 and thus when $k > 2$ the only graph of order $n \geq k$ that is $k$-edge-Hamiltonian is trivially the cycle $C_n$. Therefore edge-Hamiltonian and 2-edge-Hamiltonian exhaust the non-trivial possibilities for $k$-edge-Hamiltonicity. Furthermore a 2-edge-Hamiltonian graph $G$ cannot have a vertex $x$ of degree 2 adjacent to a vertex $y$ of degree exceeding 2, because no cycle on a pair of edges on $y$ and not on $x$ can include vertex $x$. Therefore all the vertices in a non-trivial 2-edge-Hamiltonian graph have degree at least 3. The interesting graphs to consider are the edge-minimal cubics and *bicubics* (cubic bigraphs), the latter of which are necessarily equitable.

It is easy to show that 2-edge-Hamiltonian bicubic graphs are 3-connected. Let $G$ be a 2-edge-Hamiltonian bicubic graph. Suppose $G$ has a cutset with two vertices $u$ and $v$. Since $G$ is Hamiltonian, $G \setminus \{u, v\}$ has two components $A$ and $B$. If $u \bullet\!\!-\!\!\bullet v$ is an edge, then it is easy to see it cannot be in a Hamilton cycle so that $G$ is not even edge-Hamiltonian. We may assume $u$ and $v$ are not adjacent. By the pigeonhole principle, $u$ has at least two neighbors $x$, $y$ in one part, say $A$. The edges $u \bullet\!\!-\!\!\bullet x$ and $u \bullet\!\!-\!\!\bullet y$ cannot be in a Hamilton cycle $H$ because $H$ then must pass between $A$ and $B$ more than once and $v$ is the only available vertex to facilitate this. Therefore, $G$ is 3-connected.

Virtually the same argument can be used to show that Hamilton-laceable bicubic graphs are also 3-connected. Let $G$ be a Hamilton-laceable bicubic graph with bipartition $A, B$. If $u$, $v$ are in different parts, then they are the ends of a Hamilton path. This implies that $G \setminus \{u, v\}$ is connected so that if $\{u, v\}$ is a cutset, then $u$, $v$ belong to the same part, say $A$, and are not adjacent. Let $Z_1$, $Z_2$ be the components of $G \setminus \{u, v\}$. If $u$ has three neighbors $x_1$, $x_2$, $x_3$ in the same component, say $Z_1$, then there is no Hamilton path from $u$ to $x_1$ in $G$ because the path must go through $v$ to reach $Z_2$ and there is no way to return to $Z_1$. Thus, neither $u$ nor $v$ have three neighbors in the same component. If both $u$ and $v$ have two neighbors in the same component, say $Z_1$, then there is no Hamilton path from $u$ to $x$, where $x$ is the neighbor of $v$ in $Z_2$. Finally, if $u$ has one neighbor $y_1$ in $Z_1$ and $v$ has two neighbors in $x_1$, $x_2$ in $Z_1$, then there is no Hamilton path from $v$ to $y_1$. Therefore, $G$ is 3-connected.

Edge-Hamiltonian graphs, 2-edge-Hamiltonian graphs and Hamilton-laceable bigraphs are all Hamiltonian; the first two by definition and the latter by joining an edge to a Hamilton path between its endpoints. Any of the several constructions disproving Tutte's conjecture [4] that a 3-connected bicubic graph must be Hamiltonian therefore show that 3-connectedness is a necessary but not necessarily a sufficient condition for a bicubic graph to be 2-edge-Hamiltonian or Hamilton-laceable.

It was noted recently [3] that the Tutte 8-cage is Hamiltonian, edge-Hamiltonian, 2-edge-Hamiltonian and Hamilton-laceable; apparently a remarkable confluence of Hamilton related graph properties. However, since the first two properties follow from either of the latter two it is only remarkable that the graph is both Hamilton-laceable and 2-edge-Hamiltonian. Not mentioned though was the fact that the 8-cage is also edge-stable with respect to Hamilton-laceability. The 10-fold symmetry of the 8-cage greatly reduced the number of edges that had to be considered and the generation of Hamilton paths between the endpoints of an edge could be halted as soon as a cover for the remaining edges was found which is how it was first noticed that the 8-cage was 2-edge-Hamiltonian. These simplifications made it possible to deal with the 30 vertices and 45 edges in the 8-cage.

Besides the 8-cage, it was noted earlier the $m$-prisms ($m$ even) and the $m$-Möbius ladders ($m$ odd) are infinite families of bicubics which are also edge-stable with respect to Hamilton-laceability. This result was a serendipitous byproduct of enumerating Hamilton paths in the graphs [1] but can easily be exploited to show the graphs are also 2-edge-Hamiltonian. All that is required is that the considered graph be a Hamilton-laceable cubic bigraph that is also is edge-stable. Let $a \bullet\!\!-\!\!\bullet\, b$ and $c \bullet\!\!-\!\!\bullet\, d$ be an arbitrary pair of edges in a bicubic graph which is edge stable with respect to Hamilton-laceability. They are either disjoint or share a vertex, say $b = c$. In either case let the other two edges on $d$ be $d \bullet\!\!-\!\!\bullet\, e$ and $d \bullet\!\!-\!\!\bullet\, f$. If $d \bullet\!\!-\!\!\bullet\, e$ is deleted, by hypothesis there will still be a Hamilton path between vertices $a$ and $b$, which to include vertex $d$ must lie on edges $c \bullet\!\!-\!\!\overset{d}{\bullet}\!\!-\!\!\bullet\, f$. Irrespective of whether $b$ and $c$ are distinct or not appending edge $a \bullet\!\!-\!\!\bullet\, b$ to the Hamilton path between vertices $a$ and $b$ forms a Hamilton cycle on the pair of arbitrary edges.

These ladder-like graphs are the simplest members of a doubly infinite family of bicubics many of which (all?) share this dual property of being both Hamilton-laceable and 2-edge-Hamiltonian. To construct the graphs, extend the sides of a regular $m$-gon to define the $m\lfloor(m-1)/2\rfloor$ finite points of intersection (including the vertices of the $m$-gon) and draw centrally symmetric concentric circles such that each annulus includes just one set of the intersections. The *polygonal bigraph* $\mathrm{P}_{m,k}$ is defined on the $k^{\text{th}}$ circle, where vertices are the intersections of the circle with lines in the configuration and edges are the segments of the lines cut out by the circle and the arcs of the circle between the intersections. The innermost circle, $k = 1$, contains only the vertices of the $m$-gon itself and defines $\mathrm{P}_{m,1}$ which are just the $m$-prisms ($m$ even) and the $m$-Möbius ladders ($m$ odd). Figure 2 shows the three $\mathrm{P}_{7,k}$ defined by a heptagon.

## 2 Conjecture 1

*The polygonal bigraphs, $P_{m,k}$, are 2-edge-Hamiltonian.*

An alternative description of $\mathrm{P}_{m,k}$, which obscures the relationship to the defining $m$-gon though, is: in a clockwise numbered cycle $\mathrm{C}_{2m}$ connect

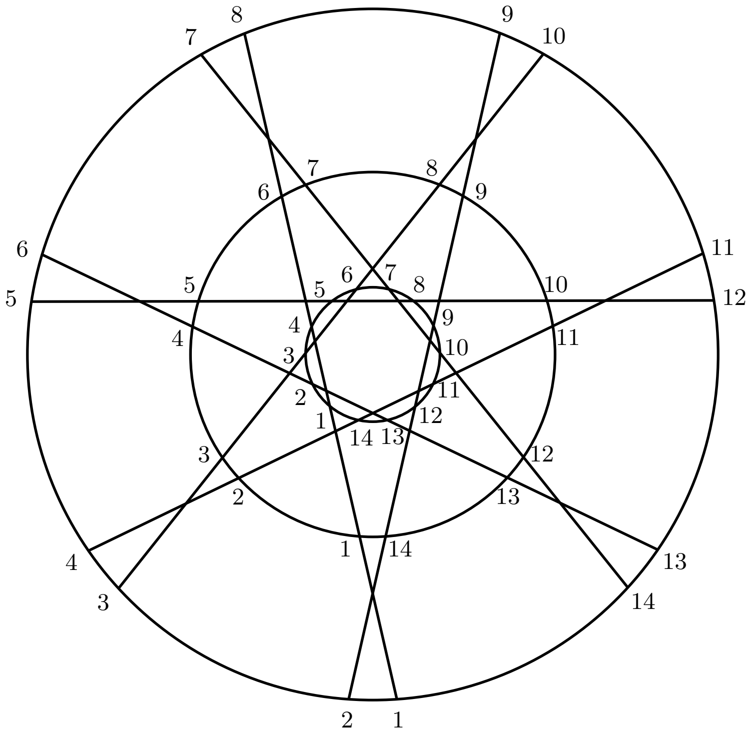


Figure 2: $\mathrm{P}_{7,k}$

each odd vertex to the even vertex 2k+1 clockwise from it in the cycle. It is worth noting that the three polygonal bigraphs generated by a heptagon are each a member of an infinite family, $\mathrm{P}_{m,1}$, $\mathrm{P}_{m,2}$ or $\mathrm{P}_{m,3}$, all of whose members have been proven to be Hamilton-laceable [2]. This description of $\mathrm{P}_{m,k}$ however makes obvious the LCF notation: $\left[(2k+1),(2k+1)^{-1}\right]^{m}$, $k \leq \lfloor (m-1)/2m \rfloor$.

**Theorem 2.1.** *The $P_{m,k}$ are* 3*-connected.*

*Proof.* The $\mathrm{P}_{m,k}$ are vertex-transitive because both rotation by two positions and reflection about the axis joining the midpoints of the arcs between 1 and 2, and $m+1$ and $m+2$ are automorphisms. Watkins [5] proved that the connectivity of a connected vertex-transitive graph of degree $d$ is strictly greater than two-thirds $d$ from which the result follows. $\square$

A Sullivan subgraph $S_{m,k}$ is a unique graph associated with each chord in a polygonal bigraph $P_{m,k}$. $S_{m,k}$ contains the two pairs of edges in the outer cycle lying on the endpoints of the defining chord with no other pair of edges in the outer cycle having a vertex in common in $S_{m,k}$. The $S_{12,3}$ defined by chord $1 \bullet\!\!-\!\!\bullet\ 8$ in $P_{12,3}$ is shown by the bold lines in Figure 3. If

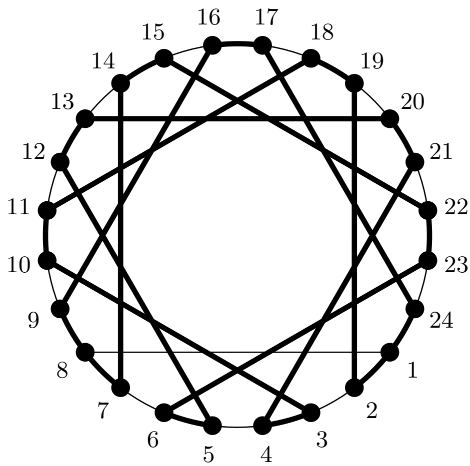



Figure 3: $S_{12,3}$

$m = 2k+1$, $P_{m,k}$ is just the cycle $C_{2m}$ and the $m$ diameters, in which case $S_{m,k}$ is degenerate and consists of a $C_6$ cycle on the two pairs of edges on the endpoints of the defining chord and $k-1$ $C_4$ cycles on the remaining pairs of opposite edges in $C_{2m}$. It will be shown that $P_{m,k}$ is both Hamilton-laceable and 2-edge-Hamiltonian in this case so it suffices to consider cases $m \geq 2k+2$.

**Theorem 2.2.** *Every pair of edges in $P_{m,k}$ is covered by some rotation of an $S_{m,k}$.*

*Proof.* The edges in $P_{m,k}$ have a natural partition into three perfect matchings: the chords, the plus edges and the minus edges in the outer cycle; where an edge is said to be "plus" if the chords on its endpoints do not cross and "minus" if they do. It is easy to see that every pair of edges lies in some rotation of an $S_{m,k}$. If the pair of edges to be covered in $P_{m,k}$ are either both in the outer cycle or are both chords the statement is obviously true. The only other possibility is that one of the edges is a chord and the other an edge in the outer cycle. It is easier to treat plus and minus edges separately. The $k+1$ outer edges in $\mathrm{S}_{m,k}$ in the circular segment cut off by the defining chord are all minus while the $m-k$ in its complement are all plus. There are at least two plus edges in the complement that are not in either of the pairs of edges on the ends of the defining chord. Any plus edge paired with the defining chord in $\mathrm{P}_{m,k}$ will be covered by the rotation of one of these two edges to lie on the edge in question, so every pairing of a plus edge with a chord is covered by a rotation of $\mathrm{S}_{m,k}$. For all $\mathrm{P}_{m,k}$ the minus edges $1 \bullet\!\!-\!\!\bullet\ 2$ and $(2k+1) \bullet\!\!-\!\!\bullet\ (2k+2)$ are paired with all chords except for the defining chord $1 \bullet\!\!-\!\!\bullet\ (2k+2)$ on vertex 1 in the first case and on $2k+2$ in the second. Rotating $\mathrm{S}_{m,k}$ $2k$ positions counterclockwise in the first case or clockwise in the second will cause one of the minus edges to lie on the other and bring the chord on 2 or on $2k+2$ to the position originally occupied by the defining chord. □

**Corollary 2.3.** *If $S_{m,k}$ is a Hamilton cycle, $P_{m,k}$ is 2-edge-Hamiltonian.*

It was this corollary which motivated the introduction of the Sullivan graphs since the conjecture that polygonal bigraphs are 2-edge-Hamiltonian would be proven if the Sullivan graphs were all Hamilton cycles. Obviously, since the conjecture is still open, they are not, but most are as shown in Table 1 where '-' indicates $\mathrm{S}_{m,k}$ is a Hamilton cycle and 'N' that it is not. Several patterns are evident in the entries, all of which are accounted for by Theorem 2.4.

**Theorem 2.4.** *The Sullivan subgraph $S_{m,k}$ is a Hamilton cycle in $P_{m,k}$ if and only if $gcd(m+1,k+1) \leq 2$.*

*Proof.* The basic idea is simple: show that if $gcd(m+1,k+1) > 2$, a path originating on vertex 4 can not connect to either end of the edge pair $(2m-1) \bullet\!\!-\!\!\bullet\ 2$ in the $\mathrm{S}_{m,k}$ defined by edge $1 \bullet\!\!-\!\!\bullet\ (2k+2)$.

Since the vertices in $\mathrm{S}_{m,k}$ are all of degree 2, $\mathrm{S}_{m,k}$ is either a Hamilton cycle or a union of two or more cycles. By construction the path reverses direction from clockwise to counterclockwise or vice versa only on the edge pairs on the ends of the defining chord. This forces the two edge pairs to be in the same cycle.

To show the necessity of the condition that $gcd(m+1,k+1) \leq 2$ if $\mathrm{S}_{m,k}$ is to be a Hamilton cycle, consider the path originating on vertex 4 and continuing 3, $2k+4$ etc. The next even vertex will be $4k+6$ etc., better written $2(k+1)+2$, $4(k+1)+2$. If this sequence doesn't land on one of $2m$ or 2 before, it will eventually land again on a minus edge in the circular segment as a result of the formal sum exceeding $2m$. When this happens $2m$ must be subtracted to get the actual vertex label. Formally the vertex labels in the path are of the form $2\alpha(k+1) - 2\beta(m+1) + \delta$, where $\alpha$ is the number of vertices in the path thus far, $\beta$ is the number of times the path has wrapped around the graph (also the number of minus edges in the path thus far) and $\delta$ is 4 if the current vertex is in the circular segment or 2 if it is not. The explanation for the difference in the value of $\delta$ is that the path moves counterclockwise on a minus edge and clockwise on a plus edge. If the path is to be in a Hamilton cycle, $2\alpha(k+1) - 2\beta(m+1) + 2$ must equal $2m$ or 2 for some set of path defined $\alpha$ and $\beta$. Then $\delta = 2$ since the vertex preceding either $2m$ or 2 will be in the complement.

**Case 1:** $2\alpha(k+1) - 2\beta(m+1) + 2 = 2m$ can be rewritten in the form $\alpha(k+1) - (\beta-1)(m+1) = -2$, which is impossible when $gcd(m+1, k+1) = j > 2$ since $j$ divides each of the terms on the left, but doesn't divide 2.

**Case 2:** $2\alpha(k+1) - 2\beta(m+1) + 2 = 2m+2$ can be rewritten in the form $\alpha(k+1) - (\beta)(m+1) = m$ which is also impossible since $j$ divides each of the terms on the left but not the one on the right since $gcd(m+1, m) = 1$.

But the path on vertex 4 must close to form a cycle irrespective of the value of $j$. The preceding argument shows this cycle will be disjoint from the one including the edge pairs on the ends of the defining chord when $j > 2$, hence $\mathrm{S}_{m,k}$ cannot be a Hamilton cycle in this case.

It isn't just that the path originating on vertex 4 can't land on either of vertices $2m$ or 2 if $j > 2$, but that it must if $j > 2$. Depending on the values of $m$ and $k$ it can be either; for example when $m = 15$ and $k = 4$ it lands on vertex 2 and when $m = 15$ and $k = 5$ it lands on vertex $30 = 2m$. There are two paths originating on vertex $2(k+1)$, one starting with the plus edge $(2k+2) \bullet\!\!-\!\!\bullet (2k+3)$, the other with the minus edge $(2k+2) \bullet\!\!-\!\!\bullet (2k+1)$. The corresponding formal vertex sequences are:

$$2\alpha(k+1) - 2\beta(m+1) \tag{A}$$

and

$$2\alpha'(k+1) - 2\beta'(m+1) - 2 \tag{B}$$

where $\alpha$, $\beta$, $\alpha'$ and $\beta'$ have the same interpretation as before. By Bézout's identity the integers of the form $a(k+1) + b(m+1)$ are multiples of $gcd(k+1, m+1)$. Hence when $gcd(k+1, m+1) \leq 2$, sequences (A) and (B)

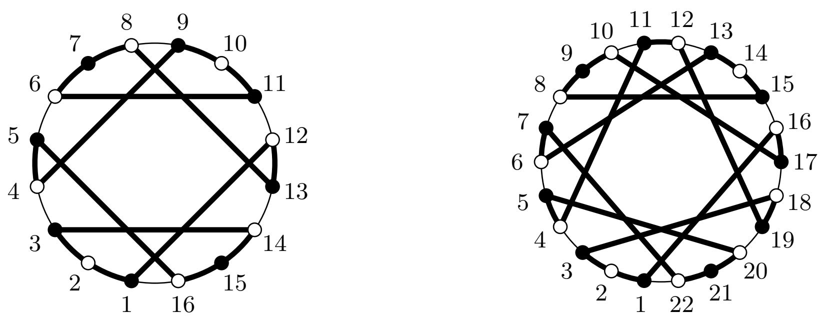
Table 1

| $m \backslash k$ | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **5** | N | | | | | | | | | | | |
| **6** | - | | | | | | | | | | | |
| **7** | - | N | | | | | | | | | | |
| **8** | N | - | | | | | | | | | | |
| **9** | - | - | N | | | | | | | | | |
| **10** | - | - | - | | | | | | | | | |
| **11** | N | N | - | N | | | | | | | | |
| **12** | - | - | - | - | | | | | | | | |
| **13** | - | - | - | - | N | | | | | | | |
| **14** | N | - | N | N | - | | | | | | | |
| **15** | - | N | - | - | - | N | | | | | | |
| **16** | - | - | - | - | - | - | | | | | | |
| **17** | N | - | - | N | - | - | N | | | | | |
| **18** | - | - | - | - | - | - | - | | | | | |
| **19** | - | N | N | - | - | N | - | N | | | | |
| **20** | N | - | - | N | N | - | N | - | | | | |
| **21** | - | - | - | - | - | - | - | - | N | | | |
| **22** | - | - | - | - | - | - | - | - | - | | | |
| **23** | N | N | - | N | - | N | N | - | - | N | | |
| **24** | - | - | N | - | - | - | - | N | - | - | | |
| **25** | - | - | - | - | - | - | - | - | - | - | N | |
| **26** | N | - | - | N | - | - | N | - | - | N | - | |
| **27** | - | N | - | - | N | N | - | - | - | N | - | N |
| **28** | - | - | - | - | - | - | - | - | - | - | - | - |

together generate all the even integers between 2 and $2m$ inclusive if each formal sequence is stopped as soon as 2 or $2m$ is reached: in other words, $\mathrm{S}_{m,k}$ is a Hamilton cycle when $gcd(m+1,k+1) \leq 2$. $\square$

The probability the greatest common divisor of a pair of randomly drawn integers $x$ and $y$, $x=m+1$ and $y=k+1$, will be no greater than 2 is just the probability they are relatively prime plus the probability their GCD is 2, i.e. $15/2\pi^2$ or approximately 0.76. In other words the Sullivan graphs show that at least three out of four of the $\mathrm{P}_{m,k}$ are 2-edge-Hamiltonian. Of course the conjecture is that they all are.

Since the Sullivan graphs were so effective in constructing edge pair covering Hamilton cycles in $\mathrm{P}_{m,k}$ when $gcd(m+1,k+1) \leq 2$ it is natural to look for another family of graphs with similar properties for the cases in which $gcd(m+1,k+1) > 2$. An anti-Sullivan subgraph $\overline{\mathrm{S}}_{m,k}$ in a polygonal bigraph $\mathrm{P}_{m,k}$ is a unique graph associated with each minus edge in the outer cycle in $\mathrm{P}_{m,k}$. $\overline{\mathrm{S}}_{m,k}$ contains the two pairs of contiguous edges in the outer cycle lying on either side of the defining minus edge and the $m-2$ chords not incident on the shared vertex in either edge pair. Figure 4 shows in bold edges the first two anti-Sullivan graphs, $\overline{\mathrm{S}}_{8,2}$ and $\overline{\mathrm{S}}_{11,3}$ defined by the minus edge $1 \bullet\!\!-\!\!\bullet\, 2m$ in $\mathrm{P}_{8,2}$ and $\mathrm{P}_{11,3}$ respectively, in the familiar representation as a cycle on $2m$ vertices with, in this case, $m-2$ chords.




$\overline{\mathrm{S}}_{8,2}$ $\qquad$ $\overline{\mathrm{S}}_{11,3}$

Figure 4

The Sullivan and anti-Sullivan graphs differ in virtually every respect. The one property they share is that the rotations of either covers all the edge pairs in $\mathrm{P}_{m,k}$. The anti-Sullivan graphs are only defined for $m = n(k+1)-1$, $n \geq 3$; a forced consequence of the definition. $\mathrm{S}_{m,k}$ is defined for all parameter pairs $m$ and $k$, $k \leq \lfloor (m-1)/2 \rfloor$, where the bound on $k$ is simply to avoid repetition. $\overline{\mathrm{S}}_{n(k+1)-1,k}$ can be generated by simple repetitive extensions of $\overline{\mathrm{S}}_{3(k+1)-1,k}$ for all $n > 3$. There is no comparable operation for the $\mathrm{S}_{m,k}$. The biggest difference though is that whereas $\mathrm{S}_{m,k}$ is only a Hamilton cycle roughly three times out of four and then subject to less than obvious conditions, it is almost trivial to show that $\overline{\mathrm{S}}_{m,k}$ is always a Hamilton cycle. The remarkable thing about the two families of graphs though is that the $\overline{\mathrm{S}}_{m,k}$ are Hamilton cycles in precisely those cases in which the $\mathrm{S}_{m,k}$ are not. Unfortunately there are parameter pairs which do not define a Hamilton cycle in either construction; when $m+1$ and $k+1$ share a proper divisor greater than 2. Just as there is a Sullivan graph associated with every chord in $\mathrm{P}_{m,k}$ there is an anti-Sullivan graph associated with every minus edge in the outer cycle.

**Theorem 2.5.** *Every pair of edges in $P_{m,k}$ is covered by some rotation of an $\overline{S}_{m,k}$.*

*Proof.* Identical to the proof of Theorem 2.2. □

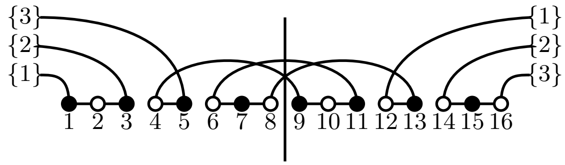
**Theorem 2.6.** *$\overline{S}_{n(k+1)-1,k}$ , $n \geq 3$ is a Hamilton cycle.*

*Proof.* Unlike the case for $\mathrm{S}_{m,k}$ where it was difficult to determine whether a particular graph was a Hamilton cycle or not, crucial to showing $\mathrm{P}_{m,k}$ is 2-edge-Hamiltonian, it becomes little more than a remark for $\overline{\mathrm{S}}_{n(k+1)-1,k}$ given the representation in Figure 5. For the moment consider only the minimal case in which $m = 3(k+1)-1$. The structure is forced by the definition; there will be $k-2$ plus edges between the two central pair of edge pairs and $k-1$ minus edges between the edge pairs on either end and the central ones. $\overline{\mathrm{S}}_{11,3}$ suffices to illustrate how easy it is to show $\overline{\mathrm{S}}_{m,k}$ is always a Hamilton cycle. Notice that the path from endpoint labeled $\{3\}$ on the left connects to a minus edge on the left, then to a plus edge in the middle and finally to a minus edge on the right to exit at an endpoint labeled $\{2\}$ on the right. For an arbitrary $k > 3$ there will be $k-2$ such paths which daisy chain to each other to connect endpoint labeled $\{k\}$ on the left to the endpoint labeled $\{2\}$ on the right. Irrespective of how large $k$ may be, the endpoint labeled $\{1\}$ on the right connects to the endpoint labeled $\{k+1\}$ on the left. These two partial paths are linked by the path reversals on edges $1 \; \overset{2}{\bullet\!-\!\bullet\!-\!\bullet} \; 3$ and $(2m-2) \; \overset{(2m-1)}{\bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet} \; (2m)$ to form a Hamilton cycle. Therefore $\overline{\mathrm{S}}_{3(k+1)-1,k}$ is a Hamilton cycle for $k \geq 2$.

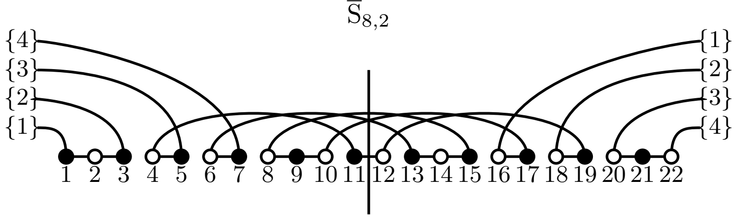
Next we show that $\overline{\mathrm{S}}_{n(k+1)-1,k}$ is a Hamilton cycle for all $n \geq 3$. Figure 6 shows examples for $k = 2$ and $k = 3$ of links which can be spliced into $\overline{\mathrm{S}}_{8,2}$

or $\overline{\mathrm{S}}_{11,3}$ at the indicated vertical bars. Since the paths described above are merely extended by such an operation, the end labels will be unchanged and the extended graphs will still be Hamilton cycles. Arbitrarily many such extensions can be made to form $\overline{\mathrm{S}}_{n(k+1)-1,2}$ or $\overline{\mathrm{S}}_{n(k+1)-1,3}$. For other values of $k$ the links will have $k+1$ plus edges arranged in the same manner as shown in the two links in Figure 6 depending on whether k is even or odd. Therefore $\overline{\mathrm{S}}_{n(k+1)-1,k}$ is a Hamilton cycle for $k \geq 2$. □

**Corollary 2.7.** *$P_{m,k}$ is 2-edge-Hamiltonian when* $\gcd(m+1, k+1) = k+1$.

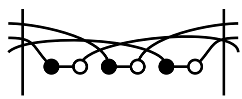



$\overline{\mathrm{S}}_{8,2}$

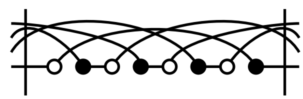



$\overline{\mathrm{S}}_{11,2}$

Figure 5



$k = 2$ link



$k = 3$ link

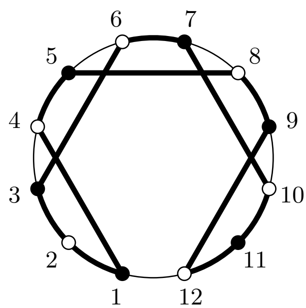
Figure 6

To summarize, $\mathrm{P}_{m,k}$ has been shown to be 2-edge-Hamiltonian except when $2 < (m+1, k+1) < k+1$. Several infinite families in this range can also be shown to be 2-edge-Hamiltonian, but since these special cases don't affect the asymptotic result only a couple of cases that are both 2-edge-Hamiltonian and Hamilton-laceable will be mentioned since they support Conjecture 2. $\mathrm{P}_{m,2}$ and $\mathrm{P}_{m,3}$, the entries in the first two columns in Table

1, and their isomorphs are obvious examples: 2-edge-Hamiltonian in consequence of Theorems 2.4 and 2.6 here and proven to be Hamilton-laceable in [2]. The entries on the diagonal $k = \lfloor (m-1)/2 \rfloor$ in Table 1 are also both 2-edge-Hamiltonian and Hamilton-laceable as shown in Theorems 2.8 and 2.9. $\mathrm{P}_{m,1}$ was shown to be Hamilton-laceable in [1] but there is such a simple proof using the Sullivan subgraphs that the new proof is included here. $\mathrm{P}_{m,k}$ is vertex-transitive so all that is required is to show there exists a Hamilton path between vertex 1 and every even vertex.

**Theorem 2.8.** *$P_{m,1}$ is 2-edge-Hamiltonian and Hamilton-laceable.*

*Proof.* Let $\mathrm{S}_{m,1}$ be the Sullivan subgraph defined in $\mathrm{P}_{m,1}$ by edge $2 \bullet\!\!-\!\!\bullet (2m-1)$ , i.e. the $\mathrm{S}_{m,1}$ in which the path reverses direction on vertices 1 and $2m$; See Figure 7. Clearly $\mathrm{S}_{m,1}$ is a Hamilton cycle so $\mathrm{P}_{m,1}$ is 2-edge-Hamiltonian.



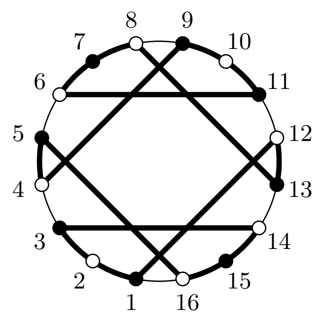




Figure 7

Define a Hamilton path between 1 and an arbitrary even vertex $y$ by following the outer cycle in a clockwise direction from vertex 1 to the odd vertex just before $y$ and then following $\mathrm{S}_{m,1}$ until it returns to $y$. This construction is well defined for all even $y$: if $y = 2$, 1 is the vertex preceding $y$ so $\mathrm{S}_{m,1}$ is followed from the beginning and if $y = 2m$, $2m - 1$ is the vertex preceding $y$ so only the final edge $(2m-1) \bullet\!\!-\!\!\bullet 2m$ is followed. □

**Theorem 2.9.** *$P_{m,\lfloor (m-1)/2 \rfloor}$ is 2-edge-Hamiltonian and Hamilton-laceable.*

*Proof.* If $m$ is odd, $\mathrm{P}_{m,\lfloor (m-1)/2 \rfloor} \cong \mathrm{P}_{m,1}$ which was just shown to be 2-edge-Hamiltonian and Hamilton-laceable. If $m$ is even let $2d$ be the minimum number of vertices between endpoints $x$ and $y$ on $C_{2m}$. For $k = \lfloor (m-1)/2 \rfloor$ the chords are parallel and on adjacent vertices. Form a cross of d pairs of the parallel edges as shown in Figure 8 for $d = 1$ and 2 and connect the open ends to endpoints $x$ and $y$ as shown to form a Hamilton path between $x$ and $y$. Since $S_{2n+2,n}$ is a Hamilton cycle for all $n \geq 1$, $P_{2n+2,n}$ is 2-edge-Hamiltonian. $\square$

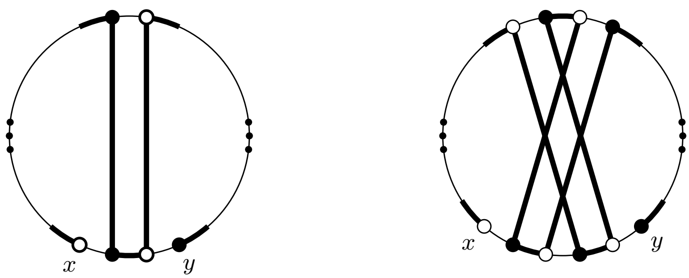



Figure 8

Of the 156 $m, k$ parameter pairs in Table 1 all but 10 define $P_{m,k}$ which are 2-edge-Hamiltonian as a consequence of Theorems 2.4, 2.6 and 2.9. The smallest case in doubt is $P_{14,5}$ which is isomorphic to $P_{14,3}$ [2] and hence both 2-edge-Hamiltonian and Hamilton-laceable. Obviously, if one member of an isomorphic pair of graphs possesses a property, the other member must also. This makes graph isomorphism a powerful tool in investigating how many members of a family of graphs possess a given property, and was in fact the primary tool used in studying the Hamilton laceability of polygonal bigraphs [2]. There, since the existence of Hamilton paths is the defining feature for Hamilton-laceability, all the Hamilton paths in $P_{m,k}$ were calculated using a backtracking program; approaching 10,000 paths on each vertex for several values of $k$ when $m = 21$. A striking feature of the tabulation was that for most $m$ there were several sets of two or three values of $k$ for which the path tally was the same, suggesting the associated graphs might be isomorphs. In fact, in every instance in which the number of Hamilton paths on a vertex in $P_{m,k}$ was the same as in $P_{m,k'}$ the bigraphs were one or the other of the following two simple isomorphs. In the case of polygonal bigraphs, there are two natural mappings: the Plus mapping in which plus edges on the outer cycle alternately join chords in $P_{m,k}$ and the Minus mapping in which the minus edges do. If either of these mappings defines a Hamilton cycle an isomorphism has been found, but as was pointed out in [2] either or both can be an auto-isomorphism which is of no assistance in identifying another Hamilton-laceable $P_{m,k'}$. An extreme example is $P_{13,3}$ in which both the Plus and Minus mappings are Hamilton cycles but in which both of the resulting graphs are isomorphic to the original $P_{13,3}$. $P_{13,3}$ does not appear in the table of isomorphs in [2] since neither isomorph leads to a new $P_{m,k}$. The fact that both the Plus and the Minus mappings are Hamilton cycles though suffices to prove $P_{13,3}$ is 2-edge-Hamiltonian: the outer cycle covers every edge pair in the cycle, the cycle defined by the Plus mapping every pair of chords and every pairing of a chord with a plus edge. Similarly for the Minus mapping, so even though the Plus and Minus mappings are auto-isomorphs to $P_{13,3}$, it is the fact that the three cycles cover all edge pairs that matters here since this shows $P_{13,3}$ is 2-edge-Hamiltonian. Of course this was already known for $P_{13,3}$ since $gcd(14, 4) = 2$. There are infinitely many $m$ though, $m$ a prime of form $6n + 1$ for example, which behave the same even though $2 < (m + 1, k + 1) < k + 1$: $P_{79,23}$ is one example.

Similar appeals to isomorphism show that all but four of the remaining cases are 2-edge-Hamiltonian but doesn't say whether they are Hamilton-laceable or not since $k > 3$ for them. The four undecided cases are $P_{19,7}$, $P_{20,5}$, $P_{20,8}$ and $P_{24,9}$, all of which elude the Sullivan and anti-Sullivan constructions. $P_{19,7}$ is 2-edge-Hamiltonian by the argument in the previous paragraph since both the Plus and the Minus mappings define Hamilton cycles. In other words, all but 3 out of the 156 $m, k$ pairs, $m \leq 28$, i.e. over 98%, can be shown to represent 2-edge-Hamiltonian bigraphs without having to resort to direct computation. Because of their importance to supporting the conjectures all three of $P_{20,5}$, $P_{20,8}$ and $P_{24,9}$ were shown to be 2-edge-Hamiltonian by direct computation. It was barely possible for $P_{24,9}$: requiring twenty-three and a half hours dedicated computation with a 4GHz 16GB computer. The bottom line is that all of the 156 parameter pairs in Table 1 define 2-edge-Hamiltonian $P_{m,k}$.

The improvement from 76% to 98% depended on showing that some of the $P_{m,k}$ of unknown property were isomorphs to $P_{m,k}$ known to be 2-edge-Hamiltonian. Since the object here is to summarize the evidence for the conjectures, it is natural to try to estimate how much the existence of isomorphs improves the bound in general. Their existence is a very irregular function of $m$ and $k$, ranging from the extreme when m is a prime of the form $m = 6n - 1$ and every $P_{m,k}$ has two isomorphs, to the cases $m = 12$ or 24 where none do.

It is possible to estimate the probability a randomly chosen $P_{m,k}$ is 2-edge-Hamiltonian using only what has already been proven. If $P_{m,k}$ is in the 76% of cases covered by the Sullivan graphs there is nothing to prove. That is also true if it is one of the cases covered by the anti-Sullivan graphs or by Theorems 2.4, 2.6 and 2.9, but these cases do not affect the asymptotic likelihood, even though they figured in showing that 98% of the cases in Table 1 were 2-edge-Hamiltonian since recalcitrant cases could be considered individually. A case by case argument isn't feasible in general, but the existence of such special cases makes an estimate which doesn't invoke them conservative.

It is computationally easy to calculate whether a Plus or Minus mapping of $\mathrm{P}_{m,k}$ defines a distinct isomorph for arbitrarily large $m$ and $k$. Taken over all $m \leq M$, let $p_0$, $p_1$, and $p_2$ be the fraction of the $k$ for which $\mathrm{P}_{m,k}$ has no, one or two isomorphs respectively. Assume that in the limit isomorphs are uniformly distributed on $k$. From previously proven results the probability a randomly chosen $\mathrm{P}_{m,k}$, $m \leq M$, is 2-edge-Hamiltonian is then given by:

$$p = 0.760+0.240(0.760p_1+(1-(0.240)2)p_2) = 0.760+0.182p_1+0.226p_2 \quad (1)$$

The resulting probabilities are shown in Table 2 for selected values of $M$. The probability a $\mathrm{P}_{m,k}$ drawn from Table 1 is 2-edge-Hamiltonian is esti-

Table 2

| $M$ | 28 | 100 | 200 | 400 | 500 |
|---|---|---|---|---|---|
| $p$ | 0.893 | 0.919 | 0.926 | 0.931 | 0.932 |

mated by Equation (1) to be 0.893 when in fact it is 0.981. If the other estimates underestimate the true values comparably it is safe to conclude that for sufficiently large M almost all $\mathrm{P}_{m,k}$ are 2-edge-Hamiltonian. Of course the conjecture is that they all are.

# 3 Conjecture 2

*A 3-connected bicubic graph is Hamilton-laceable if and only if it is 2-edge-Hamiltonian.*

Conjecture 2 holds in the cases for which $\mathrm{P}_{m,k}$ was shown to be both 2-edge-Hamiltonian and Hamilton-laceable in the previous section, i.e. for $k = 1$, 2 and 3, and their isomorphs, the family of graphs in Theorem 2.8 and for the overlap between entries in Table 1 and a similar tabulation in [2] showing

$P_{k,m}$ is Hamilton-laceable for $m \le 21$. The $P_{m,k}$ are vertex-transitive by construction so each vertex hosts the same number of Hamilton paths. These were computed and tabulated for $m \le 21$, the limit of computational feasibility, in [2]. Roughly half of the tallies for a fixed value of $m$ appear more than once, suggesting some of the $P_{m,k}$ are isomorphs which was easily shown to be the case. The condition for $P_{m,k} \cong P_{m,k'}$ is that one of a pair of modular equations be satisfied [2] which is an easy computation for any pair of parameter values irrespective of the size of $m$ or $k$. The isomorphism classes contain one, two or three parameter pairs so the total number of classes is a slowly growing function of $m$; only 10 for $m = 59$. The fact that $P_{14,5}$ is isomorphic to $P_{14,3}$, which is known to be both

Table 3: Number of isomorphism classes of $P_{m,k}$

| m | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| | -- | -- | -- | 1 | 1 | 1 | 2 | 2 | 3 | 2 |
| 1- | 3 | 2 | 5 | 3 | 4 | 5 | 5 | 6 | 9 | 5 |
| 2- | 7 | 7 | 6 | 4 | 11 | 5 | 7 | 6 | 9 | 5 |
| 3- | 12 | 6 | 9 | 9 | 9 | 9 | 13 | 7 | 10 | 11 |
| 4- | 15 | 7 | 16 | 8 | 13 | 13 | 12 | 8 | 19 | 10 |
| 5- | 15 | 13 | 15 | 9 | 18 | 13 | 19 | 15 | 15 | 10 |

2-edge-Hamiltonian and Hamilton-laceable, was invoked in the proof of Theorem 2.6. Since the object here is to exhibit instances of bicubics which possess both properties, isomorphs of bicubics that do are an obvious source for other cases. Unfortunately, the only cases directly proven to have both properties are $k = 1$ [2], $k = 2$ and 3 [2] and $k = \lfloor (m-1)/2 \rfloor$ in Theorem 2.6, the latter of which includes the isomorphs of $P_{m,1}$ when m is odd. If $m \equiv 0 \pmod 6$ neither $k = 2$ nor $k = 3$ has any isomorphs, but each has at least one for all other values of m. The congruence conditions for the existence of an isomorphic pair of $P_{m,k}$ can be particularized for $k = 2$ and $k = 3$ as shown.

$k = 2$

$$P_{2n+1,2} \cong P_{2n+1,n-1}$$
$$P_{3n+1,2} \cong P_{3n+1,n}$$
$$P_{3n+2,2} \cong P_{3n+2,n}$$

$k = 3$

$$P_{3n+1,3} \cong P_{3n+1,n-1}$$
$$P_{3n+2,3} \cong P_{3n+2,n+1}$$
$$P_{4n+1,3} \cong P_{4n+1,n}$$
$$P_{4n+3,3} \cong P_{4n+3,n}$$

In each run of 6 values of m starting on a multiple of 6, there will therefore be 7 isomorphs for each of $k = 2$ and $k = 3$. Conjecture 2 holds for all the cases in which $\mathrm{P}_{m,k}$ has been shown to be both 2-edge-Hamiltonian and Hamilton-laceable, i.e. for $k = 1$, 2 and 3, and their isomorphs just described, the family of graphs in Theorem 2.6 and for the overlap between entries in Table 2.1 and a similar tabulation in [2] showing $\mathrm{P}_{k,m}$ is Hamilton-laceable for $m \leq 21$. It is possible to extend the later bound to $m \leq 23$, but not computationally feasible for $m > 23$. The $k$ isomorphism classes for $m = 22$ and 23 are [2]:

| | |
|---|---|
| $m = 22$ | $(2,7)(3,6)(4,8)(5,9)$ |
| $m = 23$ | $(1,11)(2,7,10)(3,5,8)(4,6,9)$ |
| $m = 24$ | No isomorphs |

Since the isolated cases $\mathrm{P}_{22,10}$ and $\mathrm{P}_{23,11}$ are known to be 2-edge-Hamiltonian and Hamilton-laceable from Theorem 2.6, to extend the bound to $m \leq 23$ it sufficed to show that $\mathrm{P}_{22,4}$, $\mathrm{P}_{22,5}$ and $\mathrm{P}_{23,4}$ are each Hamilton-laceable by direct computation. It is computationally hopeless to do the case $m = 24$ by direct computation, both because of the 48 vertices involved and by the complete absence of isomorphs to lessen the computation. It is disappointing to stop at 46 vertices, because the smallest bicubic known to be a counter example to Tuttes conjecture is the Georges graph on 50 vertices. It is likely that 50 is the least number of vertices a counter example to Conjecture 2 could have, but that is just over the horizon of computational feasibility.

# 4 Conclusion

Conjecture 1 has the strongest evidence for its validity. It was proven here that for all $m$, more than three out of four of the $\mathrm{P}_{m,k}$ are 2-edge-Hamiltonian and statistically, more than nine out of ten are. More compelling is the fact that all $\mathrm{P}_{m,k}$ on 56 or fewer vertices are 2-edge-Hamiltonian. It was these results that prompted an interest in 2-edge-Hamiltonicity in the first place since it appeared they might lead to either a proof or progress on an older conjecture that all $\mathrm{P}_{m,k}$ are Hamilton-laceable [2]. Both are global properties of the graphs and hence can be frustrated by local behavior arbitrarily distant from the location of the edge or vertex pairs defining a specific case. The number of $\mathrm{P}_{m,k}$ known to be Hamilton-laceable was approximately doubled here, but more significantly every case that is known to be Hamilton-laceable is also 2-edge-Hamiltonian; hence Conjecture 1.

There is no a priori reason to expect a connection between there being a Hamilton path between every permissible pair of endpoints in a 3-connected bicubic graph and there being a Hamilton cycle on every pair of edges. One of the most supportive results for the latter is the fact that if a bicubic graph is edge-stable with respect to Hamilton laceability it is also 2-edge-Hamiltonian. The proof makes essential use of edge stability. If Conjecture 2 is true this is just an artifact of the proof technique, but thus far no proof has been found that doesn't appeal to edge stability. Fortunately many of the case graphs considered here, even the Tutte 8-cage, are edge-stable with respect to Hamilton-laceability.

The direct evidence that there is a Hamilton cycle on every pair of edges in a 3-connected bicubic graph is meager. What has been proven mainly suggests the polygonal bigraphs possess this dual property: six infinite families of the $\mathrm{P}_{m,k}$, the cases $k = 1$, 2 or 3, the isomorphs to $k = 2$ and 3 and the case $\mathrm{P}_{m,\lfloor(m-1)/2\rfloor}$ in Theorem 2.9. A cautionary note though is that polygonal bigraphs are a vanishingly small fraction of the 3-connected bicubics since their number grows only linearly with m while the total grows factorially. For example, of the 13 connected bicubics on 14 vertices, two are not 3-connected, three are the $\mathrm{P}_{7,k}$ polygonal bigraphs shown in Figure 4, and the remaining eight satisfy Conjecture 2 but are not isomorphic to any of the polygonal bigraphs nor to each other. This is easily seen by the fact the polygonal bigraphs host 364 and 504 Hamilton paths on each vertex ($\mathrm{P}_{7,1} \cong \mathrm{P}_{7,3}$ so they each host 364) and each of the other eight host a unique number of Hamilton paths: 420, 436, 448, 464, 480, 488, 576 and 592. Still, the fact that in every case in which $\mathrm{P}_{m,k}$ has been shown to be Hamilton-laceable it is also 2-edge-Hamiltonian lends credence to Conjecture 2.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