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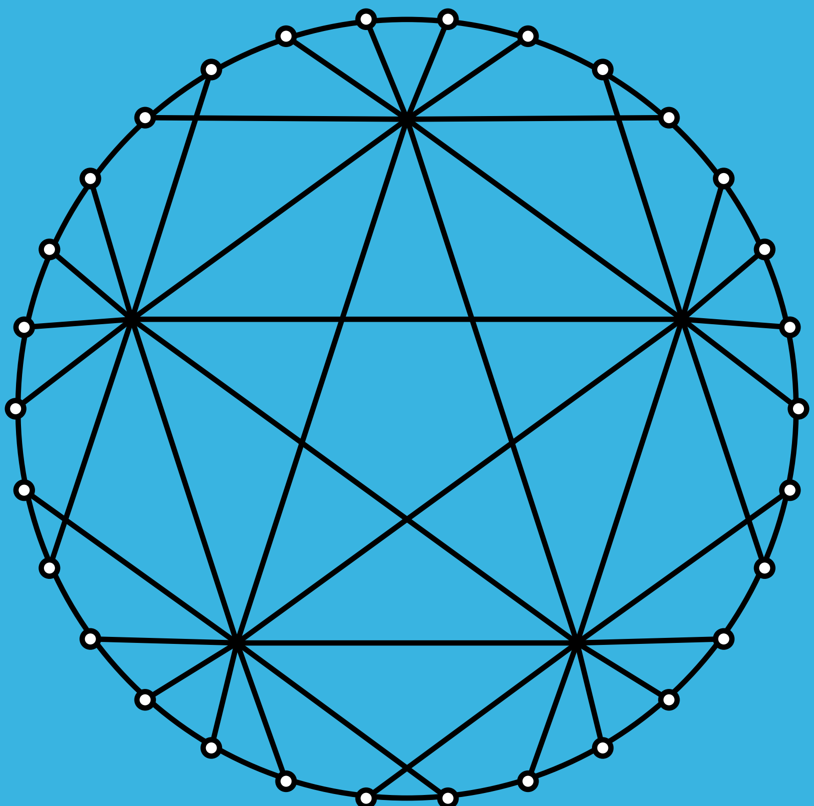

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On set graphs

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Abstract: If X is any nonempty set on $n \geq 2$ elements we define the set graph G_n to be the graph whose vertices are the $2^n - 2$ proper subsets of X with two vertices adjacent if and only if their underlying sets are disjoint. We discuss some properties of G_n. In particular we find its clique partition number and its product dimension. We also give bounds for its representation number.

We use standard graph theory terminology as given in [13].

A family of subsets $S_1, S_2, \ldots$ of a set S gives a graph in a natural way if we use these sets as vertices and let S_iS_j for $i \neq j$ be an edge if and only if the corresponding subsets have a nonempty intersection. In [12], Marczewski has established the converse, i.e. for any graph G there is a set S, such that a family of its subsets defines G according to the above description. Erdős, Goodman and Posa in [1] have remarked that one may replace the idea of a nonempty intersection with disjointness of the subsets since the same would then imply Marczewski's theorem for $\overline{G}$. They have then determined the minimum number of elements in the set S for an arbitrary G. The same problem for a given G can be studied from the viewpoint of intersection number of a graph (see [13]), or coprime index of a graph (see [5]). Hence the graph whose vertices are subsets of an n-set with two vertices adjacent

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if and only if they are disjoint is naturally of interest. We find it convenient to exclude the n-set and the empty set from the vertices of this graph. In the terminology of [7] this graph is the complement of the intersection graph on $2^{[n]} \setminus \{[n], \emptyset\}$.

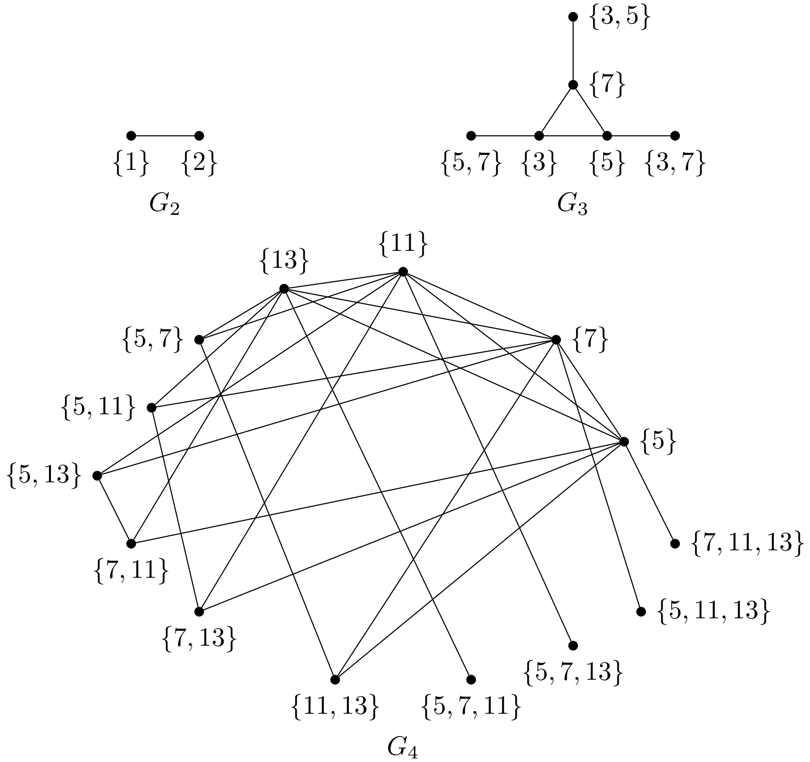
Definition 1. Let X be any nonempty set on $n \geq 2$ elements. We define the set graph $G(X) = G_n$ to be the graph whose vertices are the $2^n - 2$ proper subsets of X and whose two vertices are adjacent if and only if their underlying sets are disjoint.

If X_1 and X_2 are two sets with the same cardinality n, then any bijection between X_1 and X_2 induces a graph isomorphism between the corresponding set graphs $G(X_1)$ and $G(X_2)$. Hence we are justified in using the notation G_n (defined up to isomorphism). Figure 1 gives examples of G_2, G_3, and G_4. Here for G_2 we take $X = \{1, 2\}$, for G_3 we take $X = \{3, 5, 7\}$, and for G_4 we take $X = \{5, 7, 11, 13\}$. The labelling of the vertices for G_3 and G_4 can be used in coprime labelling mentioned below and also in connection with their representation numbers. G_n also arises in the following way. Recently Katre, Yahyaei and Arumugam have defined the concept of coprime index of a graph in [5]. They showed that for any simple graph G, one can label its vertices with distinct integers ≥ 2 in such a way that two vertices are adjacent if and only if the corresponding integers are relatively prime. For example, for G_3 and G_4, the products of the primes in the labellings in Figure 1 give a coprime labelling. The coprime index of G, denoted by $\mu(G)$, is then defined as the least number of primes using which such a labelling can be made. Given a graph the obvious problem is then to find its coprime index. Let us now consider the converse situation which we interpret as follows. Fix n primes $p_1, \ldots, p_n$. Consider the graph whose vertices are the proper divisors of $m = p_1 \cdots p_n$ and two vertices are joined by an edge precisely when the vertices are coprime. Clearly, this corresponds with the situation when $X = \{p_1, \ldots, p_n\}$ above and the graph in question is precisely G_n. We see easily that $\mu(G_n) = n$.

G_n is also studied in ring theory where it is isomorphic to the zero divisor graph of a finite Boolean ring [14]. A zero divisor graph of a finite Boolean ring has nonzero zero-divisors as its vertices, with two vertices u, v adjacent if and only if $u \cdot v = 0$. This connection also justifies our rejecting X and $\emptyset$ as vertices of G_n.

The graph G_n is related to the Kneser graph, which consists of k-subsets of an n-set as vertices, and where two vertices are adjacent if and only if they are disjoint. One way to generalize the notion of a Kneser graph would be to allow the subsets to be of a variable size taken from some finite set K.

Figure 1:



Under this scheme, the Kneser graph is obtained when $K = \{k\}$, and G_n is obtained when $K = \{1, \ldots, n-1\}$. Other types of well known graphs related to G_n are the Johnson graphs and Odd graphs. For $n = 3$, the graph G_n is also known as the net graph.

In view of the above, it is reasonable to study the graph G_n. In this paper we discuss properties of G_n such as its clique partition number, product dimension and representation number.

1 Some basic results

In this section we collect some basic results related to G_n which are implicitly available in the literature. For convenience, for subsets A, B of a set, we define $A - B = \{x \in A : x \notin B\}$.

Lemma 1.1. Let $|X| = n$ and $G_n = G(X)$. For any $\emptyset \neq v \subsetneq X$, $\deg(v) = 2^{n-|v|} - 1$. G_n has $\frac{1}{2}(3^n - 2^{n+1} + 1)$ edges and the degree sequence of G_n is

$$(\underbrace{1, \ldots, 1}_{\binom{n}{1}\text{times}}, \underbrace{3, \ldots, 3}_{\binom{n}{2}\text{times}}, \ldots, \underbrace{2^{n-1} - 1, \ldots, 2^{n-1} - 1}_{\binom{n}{n-1}\text{times}}).$$

Furthermore, G_n is connected and the diameter of G_n is 3 for $n \geq 3$.

Proof. If v is a vertex having k elements then the remaining $n-k$ elements form $2^{n-k} - 1$ nonempty subsets, each of which are adjacent to v. Since no other vertex is adjacent to v, $\deg(v) = 2^{n-k} - 1$. Also since k-element subsets are $\binom{n}{k}$ in number, for $1 \leq k \leq n-1$, so by the handshaking lemma the total number of edges is $\frac{1}{2}\sum \deg(v) = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}(2^{n-k} - 1) = \frac{1}{2}(3^n - 2^{n+1} + 1)$. Next note that for each $1 \leq k \leq n-1$ there are $\binom{n}{k}$ vertices having k elements and these constitute all of the vertices of G_n. So precisely $\binom{n}{k}$ vertices have degree $2^{n-k} - 1$. The result follows.

To prove that G_n is connected, note that if u, v are two nonadjacent vertices of G_n and $u \subset v$ then $(u, X - v, v)$ is a path. If neither is contained in the other then $(u, v - u, u - v, v)$ is a path. For $n \geq 3$, this also shows that $\operatorname{diam}(G_n) = 3$. $\square$

Lemma 1.2. The graph G_n has chromatic number $\chi(G_n) = n$ and clique number $\omega(G_n) = n$.

Proof. Note that since K_n is a subgraph of G_n, so $n = \chi(K_n) \leq \chi(G_n)$. For the reverse inequality, first let $X = \{x_1, \ldots, x_n\}$. Now, color all vertices containing x_1 by color 1 and then for each $2 \leq i \leq n$ color all vertices containing x_i but not containing $x_1, \ldots, x_{i-1}$, by color i. This proves $\chi(G_n) \leq n$.

Now note that K_n is the subgraph induced by the vertices $\{x_1\}, \ldots, \{x_n\}$ in G_n. We claim that K_{n+1} is not a subgraph of G_n. Suppose otherwise that K_{n+1} was a subgraph induced by the vertices $v_1, \ldots, v_{n+1}$. Let $y_i \in v_i$. Now since v_i are mutually disjoint so y_i are $n+1$ distinct elements, which is absurd. $\square$

We now recall the definitions of independence number, vertex cover number, matching number and edge cover number of a graph. Let G be a graph. The independence number of G, denoted by $\alpha(G)$, is the size of the largest independent set of G, where by an independent set we mean a set of vertices which are mutually not adjacent to each other. The vertex cover number of G, denoted by $\tau(G)$, is the size of the smallest set of vertices of G such that every edge is incident on at least one vertex in that set. The matching number of G, denoted by $\nu(G)$, is the size of the largest matching of G, where by a matching we mean a set of edges which mutually have no common vertices. Finally, the edge cover number of G, denoted by $\varrho(G)$, is the size of the smallest set of edges of G such that every vertex belongs to at least one edge in that set.

Lemma 1.3. $\alpha(G_n) = \tau(G_n) = \nu(G_n) = \varrho(G_n) = 2^{n-1} - 1.$

Proof. We first show $\alpha(G_n) = 2^{n-1} - 1$ which is actually a well known result in the theory of set intersecting families. Clearly any independent set $\mathcal{I}$ cannot exceed $2^{n-1} - 1$ in size since both a subset and its complement cannot simultaneously be in $\mathcal{I}$. On the other hand, if $X = \{x_1, \ldots, x_n\}$, the family of all those subsets of X which contain x_1, excluding X, is an independent set of $2^{n-1} - 1$ vertices in G_n. Now to show $\tau(G_n) = 2^{n-1} - 1$ note that for any graph G, we have $\alpha(G) + \tau(G) = |V(G)|$.

Next consider $\nu(G_n)$. The set of edges $\{xx^c : \emptyset \neq x \subsetneq X\}$ is a matching consisting of $2^{n-1} - 1$ edges in G_n. If there existed a matching with 2^{n-1} edges then this would have implied that G_n had 2^n vertices, an absurdity. So $\nu(G_n) = 2^{n-1} - 1$. Finally, note that for any graph G without isolated vertices, we have $\nu(G) + \varrho(G) = |V(G)|$ by Gallai's theorem [13]. $\square$

Now we define the notion of a reduced graph. For a vertex v of a graph G, the open neighborhood of v is the set $N(v) = \{w \in V(G) : vw \in E(G)\}$. A graph G is reduced if no two vertices of G have the same open neighborhoods. For example, the path graph $P_4 = (v_1, v_2, v_3, v_4)$ is a reduced graph whereas the path graph $P_3 = (v_1, v_2, v_3)$ is not.

Lemma 1.4. G_n is a reduced graph.

Proof. Let u, v be distinct vertices. If $u \cap v = \emptyset$ then they are in each other's open neighborhoods and the result is clear. If $u \cap v \neq \emptyset$ then, without loss of generality, there exists some $x \in u$ such that $x \notin v$. Then $\{x\} \in N(v)$ and $\{x\} \notin N(u)$. $\square$

Finally, we also have the following result.

Lemma 1.5. G_n is an induced subgraph of G_{n+1}.

Proof. Let $|X| = n + 1$. Then X yields G_{n+1}. Let $x \in X$. The graph induced by all proper subsets of $X - \{x\}$ is G_n. □

2 Clique partition number of G_n

The definition of clique partition number is given below.

Definition 2. Let G be a simple graph where $G \neq \overline{K_n}$. A clique partition of G is a partition $\mathcal{P} = \{S_1, \ldots, S_k\}$ of $E(G)$ such that each element S_i of $\mathcal{P}$ induces a clique in G. We let $cp(G)$ denote the minimum of $\{|\mathcal{P}| : \mathcal{P}$ is a clique partition of $G\}$, and refer to it as the clique partition number of G. A clique partition for which this minimum is achieved is called a minimum clique partition. We define $cp(\overline{K_n})$ as 0.

We require a result given by R. Rees in his doctoral dissertation to obtain the clique partition number of G_n [11] (see also Theorem 2.1.7 of [8]).

Proposition 2.1. Let G be any graph and let I be an independent set of G. Suppose H is the graph obtained by removing all the vertices in I from G and the edges incident to them. Further let $G \setminus H$ be the graph obtained by deleting the edges but not the vertices of H from G. Then for every integer $f > 0$,

$$cp(G) \geq \frac{2(f|E(G \setminus H)| - |E(H)|)}{f^2 + f}$$

with equality if and only if G admits a clique partition in which every clique has either $f + 1$ or $f + 2$ vertices and has a vertex in I.

Theorem 2.2. $cp(G_n) = \frac{1}{2}(3^{n-1} - 1)$.

Proof. Let $X = \{x_1, \ldots, x_n\}$ as usual and I be the independent set given by the set of all those subsets of X which contain x_n, excluding X. It is easy to see, as in Lemma 1.3, that this is an independent set containing $2^{n-1} - 1$ vertices. If H is the graph obtained by removing all the vertices in I from G and also the edges incident to them, then H is isomorphic to the graph G_{n-1} together with an isolated vertex $\{x_1, \ldots, x_{n-1}\}$. So,

$|E(H)| = \frac{1}{2}(3^{n-1} - 2^n + 1)$ and $|E(G \setminus H)| = 3^{n-1} - 2^{n-1}$. Hence using Rees' bound for $f = 1$ we get $cp(G_n) \geq \frac{1}{2}(3^{n-1} - 1)$.

We now show that there exists a clique partition of G_n in $\frac{1}{2}(3^{n-1} - 2^n + 1)$ triangles and $2^{n-1} - 1$ edges. This is easy to see if $n = 2$. For $n \geq 2$ for each edge uv of G_{n-1} (here $G_{n-1} = G(\{x_1, \ldots, x_{n-1}\})$ is considered as an induced subgraph of G_n) we consider the triangle $\langle u, v, X - (u \cup v)\rangle$ in G_n. This yields $\frac{1}{2}(3^{n-1} - 2^n + 1)$ triangles.

We claim that none of these triangles share an edge. Suppose otherwise and let two distinct triangles $\langle u, v, X - (u \cup v)\rangle$ and $\langle v, w, X - (v \cup w)\rangle$ share an edge, where u, v, w are vertices in G_{n-1}. Clearly, we must have u, v, w distinct. Now the edge uv does not coincide with vw since $u \neq w$ and with $\langle w, X - (v \cup w)\rangle$ as $u \neq v \neq w$. Also, uv doesn't coincide with $\langle v, X - (v \cup w)\rangle$ as otherwise $u = X - (v \cup w)$ which is not possible as $x_n \in X - (v \cup w)$ but $x_n \notin u$. Next note that $\langle v, X - (v \cup w)\rangle$ cannot coincide with $\langle v, X - (u \cup v)\rangle$ as $X - (u \cup v) \neq X - (v \cup w)$. All other cases are similar. There remain $2^{n-1} - 1$ edges required to complete the clique partition. Hence we have $cp(G_n) \leq \frac{1}{2}(3^{n-1} - 1)$ and the result follows.

Note that our clique partition also meets the requirements for equality to prevail in Rees' bound for $f = 1$. $\square$

It may be interesting to obtain $cp(\overline{G_n})$ as well. To understand the reason for this we first recall the following definitions.

Definition 3. Let $\mathcal{F}$ be a family of sets (allowing sets in $\mathcal{F}$ to be repeated). The intersection graph of $\mathcal{F}$ is the graph whose vertices are the sets in $\mathcal{F}$ and two vertices are adjacent if and only if the corresponding sets have a nonempty intersection. The intersection number of G is the smallest positive integer $\theta'(G)$ such that there exists a representation of this type for which the union of $\mathcal{F}$ has $\theta'(G)$ elements [4]. The intersection number coincides with the minimum number of complete subgraphs needed to cover $E(G)$ and so coincides with the clique cover number $cc(G)$ ([1],[13]).

Note that by definition, $\overline{G_n}$ is the intersection graph of the subsets of the n-set X (excluding ϕ, X) and this is optimal in the sense that they yield the intersection number. In other words, the clique cover number $cc(\overline{G_n}) = n$. To prove this let $\mathcal{C}_i$, for any $i, 1 \leq i \leq n$, denote the clique induced in $\overline{G_n}$ by all the vertices containing x_i. Then $\{\mathcal{C}_i : 1 \leq i \leq n\}$ is a clique cover of $\overline{G_n}$. Also any clique cover of $\overline{G_n}$ contains at least n cliques because the vertices $\{x_1\}, \ldots, \{x_n\}$ will belong to distinct cliques.

Now, a clique partition is clearly a special type of a clique cover and hence it will be of interest to determine $cp(\overline{G_n})$. Obviously as $cc(G) \leq cp(G)$ for any graph G, we have $cp(\overline{G_n}) \geq n$.

3 Dimension of G_n

We first recall the definition of tensor product of graphs.

Definition 4. Let G, H be two graphs. The tensor product $G \otimes H$ of G and H is the graph determined by the following two rules:

1. The vertices of $G \otimes H$ are the elements of $V(G) \times V(H)$.
2. The edges of $G \otimes H$ are characterized by the following rule: $(u_1, v_1)(u_2, v_2) \in E(G \otimes H)$ if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

It can be shown that the tensor product is associative. If $\{G_i : i \in I\}$ is a family of graphs, we denote their tensor product as $\bigotimes_{i \in I} G_i$.

Next recall the definition of dimension of a graph as given by Nešetřil and Rödl in 1977.

Definition 5. Let G be any graph and let I be the multiset of minimum cardinality such that G can be embedded in $\bigotimes_{i \in I} K_i$. Then the dimension of G (also called as the product dimension of G), denoted by $\operatorname{pdim} G$, is defined as $|I|$.

An equivalent definition of dimension may be given as follows: Let G be a graph. For any positive integer n, we define an encoding of G in $\mathbb{N}^n$ as an injection $l : V(G) \to \mathbb{N}^n$ where

1. $uv \in E(G)$ implies $l(u)$ and $l(v)$ differ in all coordinates;
2. $uv \notin E(G)$ implies $l(u)$ and $l(v)$ agree in at least one coordinate.

Here $\mathbb{N}^n$ denote the Cartesian product of the set of positive integers with itself n times. The dimension of G is the least positive integer $\operatorname{pdim} G$ such that an encoding of G in $\mathbb{N}^{\operatorname{pdim} G}$ exists.

It has been proved in [10] that the dimension of every graph is a well defined positive integer. We will next obtain $\operatorname{pdim} G_n$. Clearly $\operatorname{pdim} G_2 = \operatorname{pdim} K_2 = 1$.

We require the following result from [6].

Proposition 3.1. Let G be a graph and $u_1, \ldots, u_k \in V(G)$ be distinct vertices such that for some vertices $v_1, \ldots, v_k \in V(G)$, which are not necessarily distinct, $u_i v_i \in E(G)$ and $u_i v_j \notin E(G)$ for $i < j$. Then $\operatorname{pdim} G \geq \lceil \log_2 k \rceil$.

Our main theorem in this section is the following.

Theorem 3.2. $\operatorname{pdim} G_2 = 1$ and $\operatorname{pdim} G_n = n$ if $n \geq 3$.

Proof. The result is trivial for G_2 and hence let $n \geq 3$ in what follows. We first establish the lower bound.

Let $X = \{1, 2, \ldots, n\}$. Now define $n-1$ vectors M_j of length $\binom{n}{j}$ as follows. For any j, where $1 \leq j \leq n-1$, consider all j-subsets of X and consider each of them as a j-tuple by ordering its elements according to the usual integer ordering. The successive entries of M_j are now obtained by the lexicographic ordering of these j-tuples. For example, if $n = 5$, the vector M_2 is given by:

$$\big((1,2) \quad (1,3) \quad (1,4) \quad (1,5) \quad (2,3) \quad (2,4) \quad (2,5) \quad (3,4) \quad (3,5) \quad (4,5)\big)$$

Now we choose $2^n - 2$ distinct vertices u_i as follows. Consider M_{n-k-1} for each k, where $k = 0, 1, \ldots, n-2$. Starting with $k = 0$, for each such M_{n-k-1}, choose u_i's successively from its consecutive entries (under the natural correspondence of such an entry with a subset), and after this increase k by 1. Now let $v_i = u_i^c = X - u_i$. We shall show that u_i, v_j satisfy the conditions of Proposition 3.1. It is clear that each $u_i v_i$ is an edge and we need only establish that there is no edge between u_i and v_j where $i < j$.

Consider any two vertices u_i and v_j where $i < j$. If u_i is coming from some $M_{\alpha+\beta}$ and u_j from some M_α and, $u_i \cap v_j = \emptyset$ are disjoint, then their union has at least $n+1$ elements, which is impossible. So we will confine our attention to the case when u_i and u_j are coming from the same M_α. In this case since u_j will contain at least one element not in u_i, so v_j will contain at least one element in u_i, following which they are not adjacent. So the u_i, v_j satisfy the requisite conditions of the mentioned Proposition

3.1. We can therefore conclude that $\operatorname{pdim} G_n \geq \log_2(2^n - 2) > n - 1$ (as $n \geq 3$).

We now turn to the upper bound. Consider each vertex of X as a string of 0's and 1's where the ith coordinate is 1 if and only if x_i is in the corresponding subset. We consider this string as a number in the binary base and replace each 0 in the string by the decimal base equivalent of the number +1. This modified string is the encoding of the particular vertex. For example, if $n = 5$, the vertex $\{x_1, x_3\}$ corresponds to the string $(1, 0, 1, 0, 0)$ and is encoded as $(1, 21, 1, 21, 21)$. Note that since there is no string consisting entirely of 0's so each 0 is converted into a number which is at least 2.

Now if two subsets u, v were adjacent, then they will differ in each coordinate. The coordinates of u which were 1 would differ from the corresponding coordinates of v which would be all at least 2. The coordinates of u which were 0 for u but 1 for v, would all become at least 2 and hence would differ from the corresponding coordinates for v. The coordinates which are 0 for both u and v would differ since as $u \neq v$, their strings are different, following which the numbers expressed in the binary base coming from their strings are different, following which the corresponding decimal base equivalents of those numbers are different. Finally, if u, v are not adjacent then they will have at least one 1 in the same coordinate after encoding.

Thus we have a valid encoding in n coordinates, and so $\operatorname{pdim} G_n \leq n$. □

4 Representation number

For a finite graph G, with vertices $v_1, \ldots, v_n$, a representation of G modulo m is a set $\{a_1, \ldots, a_n\}$ of distinct, non-negative integers, $0 \leq a_i < m$ satisfying $\gcd(a_i - a_j, m) = 1$ if and only if v_i is adjacent to v_j. The representation number of G, denoted by $\operatorname{Rep}(G)$, is the smallest m such that G has a representation modulo m. It has been proved in [2] that $\operatorname{Rep}(G)$ is well defined.

We wish to establish an estimate for $\operatorname{Rep}(G_n)$. Clearly Rep $(G_2) = 2$ since we may take $a_1 = 0, a_2 = 1$.
The general lower bound follows directly from the following result from [3]. In this section, $p_1 = 2, p_2 = 3, \ldots$ denotes the sequence of primes in the natural order.

Proposition 4.1. If G is a reduced graph then $\operatorname{Rep}(G) \geq p_i p_{i+1} \cdots p_{i+d-1}$, where p_i is any prime satisfying $p_i \geq \chi(G)$ and $d = \operatorname{pdim} G$.

Corollary 4.2. Let $n \geq 3$ and p_i be the smallest prime $\geq n$. Then,

$$\operatorname{Rep}(G_n) \geq p_i p_{i+1} \cdots p_{i+n-1}.$$

Proof. Note that it has already been proved in Lemma 1.4 that G_n is a reduced graph and in Lemma 1.2, Theorem 3.2 that $\chi(G_n) = \operatorname{pdim}(G_n) = n$. Now apply Proposition 4.1. □

From a general upper bound for all graphs (Theorem 3, [9]) we get

$$\operatorname{Rep}(G_n) \leq p_i p_{i+1} \cdots p_{i+2^n-3}$$

where p_i is the smallest prime $\geq 2^n - 3$. We improve this result for G_n in the following theorem.

Theorem 4.3. Let $n \geq 3$ and p_k be the smallest prime $\geq 2^n - 1$. Then,

$$\operatorname{Rep}(G_n) \leq p_k p_{k+1} \cdots p_{k+n-1}.$$

Proof. Let $X = \{x_1, \ldots, x_n\}$. Consider each vertex u of X as a string of 0's and 1's where the j^{th} coordinate is 1 if and only if x_j is in the corresponding subset. In other words, identify u with the n-tuple having j^{th} coordinate $\chi_u(x_j)$, where χ_u is the characteristic function with respect to u. Also let $\hat{u}$ be the decimal value of the string identified with u considered as a number in the binary base. Now a representation of G_n can be obtained by encoding any vertex u as the unique solution modulo $p_k \cdots p_{k+n-1}$ (call it u^*) of the system $x \equiv \chi_u^*(x_j) \pmod{p_j}$, $k \leq j \leq k+n-1$, where $\chi_u^*(x_j) = 1$ if $\chi_u(x_j) = 1$ and $\chi_u^*(x_j) = \hat{u} + 1$ otherwise. Note that each 0 is converted into a number which is at least 2 and at most $2^n - 1$, since the largest $\hat{u}$ with u having a zero is $2^n - 2$. Since $\chi_u^*(x_j)$ takes values between 1 and $2^n - 1$, these are distinct modulo $(2^n - 1)$.

This encoding establishes the requisite upper bound. If u and v were adjacent then for each p_j, the difference $u^* - v^* \not\equiv 0 \pmod{p_j}$ by our encoding. This is clear if x_j is in one of the vertices and not in the other. In case, x_j is in neither vertex, then as by our choice of k, both $\hat{u}+1$, $\hat{v}+1$ are distinct and $\in \{2, \ldots, p_j\}$, so they are different in $\mathbb{Z}_{p_j}$ as well, following which $u^* \equiv \hat{u} + 1 \not\equiv \hat{v} + 1 \equiv v^* \pmod{p_j}$. So $\gcd(u^* - v^*, p_j) = 1$ and hence $\gcd(u^* - v^*, \prod p_j) = 1$. If, on the other hand, u and v were not adjacent then there exists some p_j such that $u^* \equiv v^* \equiv 1 \pmod{p_j}$. Hence, $\gcd(u^* - v^*, p_j) \neq 1$, and so, $\gcd(u^* - v^*, \prod p_j) \neq 1$. □

The gap between the upper bound and the lower bound is significantly large. We have verified by hand that for $n = 3$, $n = 4$ the actual value of $\mathrm{Rep}(G_n)$ coincides with the lower bound obtained by Evans et al (Proposition 4.1). In other words, $\mathrm{Rep}(G_3) = 3 \cdot 5 \cdot 7 = 105$ and $\mathrm{Rep}(G_4) = 5 \cdot 7 \cdot 11 \cdot 13 = 5005$. A representation of G_3 can be obtained by labelling each vertex as the product of the numbers in the set corresponding to that vertex, as shown in the graph of G_3 of Figure 1 (see Section 1). Representations of G_3 and G_4 are given in Table 1. It would be interesting to find $\mathrm{Rep}(G_n)$ for a general n.

Table 1: Representations of G_3 and G_4

Set	Labelling for G_3
$\{x_1\}$	3
$\{x_2\}$	5
$\{x_3\}$	7
$\{x_1, x_2\}$	$3 \cdot 5 = 15$
$\{x_1, x_3\}$	$3 \cdot 7 = 21$
$\{x_2, x_3\}$	$5 \cdot 7 = 35$

Set	Labelling for G_4
$\{x_1\}$	5
$\{x_2\}$	7
$\{x_3\}$	11
$\{x_4\}$	13
$\{x_1, x_2\}$	$2 \cdot 5 \cdot 7 = 70$
$\{x_1, x_3\}$	$2 \cdot 5 \cdot 11 = 110$
$\{x_1, x_4\}$	$5 \cdot 13 = 65$
$\{x_2, x_3\}$	$7 \cdot 11 = 77$
$\{x_2, x_4\}$	$2 \cdot 7 \cdot 13 = 182$
$\{x_3, x_4\}$	$11 \cdot 13 = 143$
$\{x_1, x_2, x_3\}$	$5 \cdot 7 \cdot 11 = 385$
$\{x_1, x_2, x_4\}$	$5 \cdot 7 \cdot 13 = 455$
$\{x_1, x_3, x_4\}$	$5 \cdot 11 \cdot 13 = 715$
$\{x_2, x_3, x_4\}$	$7 \cdot 11 \cdot 13 = 1001$

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