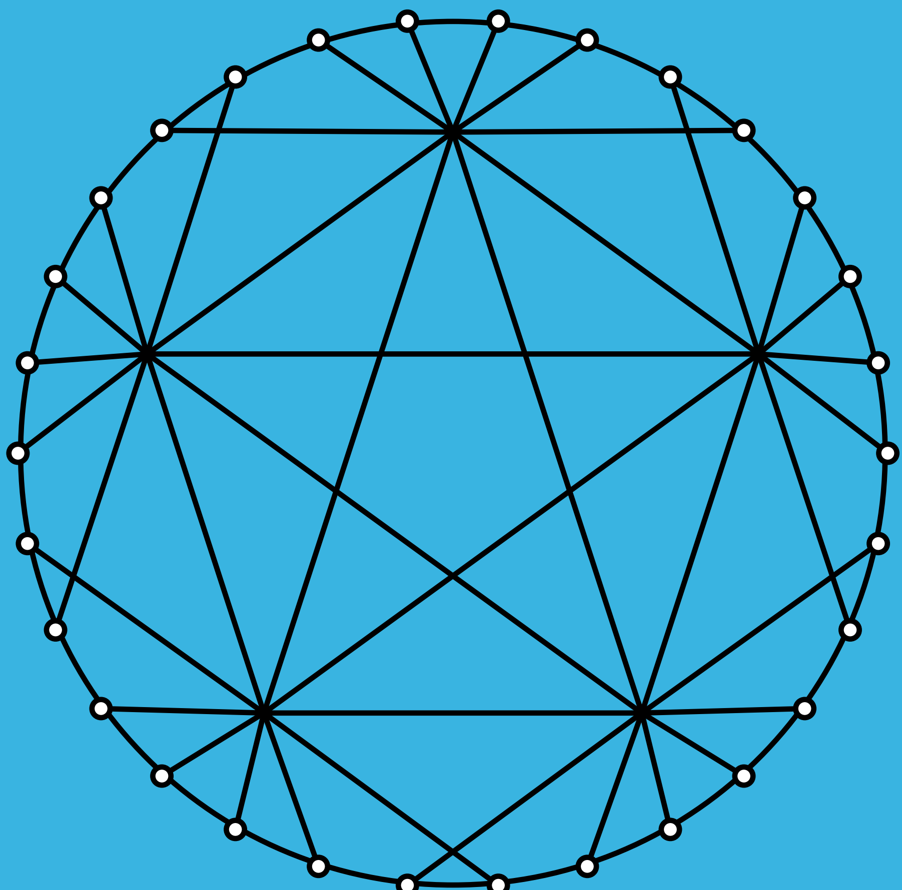
BULLETIN of The INSTITUTE of COMBINATORICS and its APPLICATIONS

Volume 87
October 2019



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **87** (2019), Pages 75–84

On loose 4-cycle decompositions of complete 3-uniform hypergraphs

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Abstract: The complete 3-uniform hypergraph of order v has a set V of size v as its vertex set and the set of all 3-element subsets of V as its edge set. A loose 4-cycle in such a hypergraph has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \subseteq V$ and edge set $\big\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_7, v_8, v_1\}\big\}$. We give necessary and sufficient conditions for the existence of a decomposition of the complete 3-uniform hypergraph of order v into isomorphic copies of a loose 4-cycle.

1 Introduction

A commonly studied problem in combinatorics concerns decompositions of graphs into edge-disjoint subgraphs. A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of pairwise edge-disjoint subgraphs of K such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_s) = E(K)$. If each element of Δ is isomorphic to a fixed graph G, then Δ is called a *G-decomposition* of K. A G-decomposition of K_v is also known as a *G-design of order v*. A K_k-design of order v is an *$S(2, k, v)$-design* or a *Steiner system*. An $S(2, k, v)$-design is also known as a *balanced incomplete block design of index 1* or a *$(v, k, 1)$-BIBD*. The problem of determining all v for which there exists a G-design of order v is of special interest (see [1] for a survey).

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AMS (MOS) Subject Classifications: 05C65, 05C51, 05C38
Key words and phrases: Hypergraph decomposition, loose cycle, spectrum problem

Received: 12 June 2019
Accepted: 28 August 2019

The notion of decompositions of graphs naturally extends to decompositions of uniform hypergraphs. A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of V called *hyperedges*. If for each $e \in E$ we have $|e| = t$, then H is said to be *t-uniform*. Thus graphs are 2-uniform hypergraphs. The complete t-uniform hypergraph on the vertex set V has the set of all t-element subsets of V as its edge set and is denoted by $K_V^{(t)}$. If $v = |V|$, then $K_v^{(t)}$ is called the *complete t-uniform hypergraph of order v* and is used to denote any hypergraph isomorphic to $K_V^{(t)}$. A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of pairwise edge-disjoint subgraphs of K such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$. If each element H_i of Δ is isomorphic to a fixed hypergraph H, then H_i is called an *H-block*, and Δ is called an *H-decomposition* of K. If there exists an H-decomposition of K, then we may simply state that H *decomposes* K. An H-decomposition of the complete t-uniform hypergraph of order v is called an *H-design of order v*. The problem of determining all v for which there exists an H-design of order v is called the *spectrum problem for H-designs*.

A $K_k^{(t)}$-design of order v is a generalization of Steiner systems and is equivalent to an $S(t, k, v)$-design. A summary of results on $S(t, k, v)$-designs appears in [7]. Keevash [13] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$-design are sufficient for sufficiently large values of v. Similar results were obtained by Glock, Kühn, Lo, and Osthus [8, 9] and extended to include the corresponding asymptotic results for H-designs of order v for all uniform hypergraphs H. These results for t-uniform hypergraphs mirror the celebrated results of Wilson [19] for graphs. Although these asymptotic results assure the existence of H-designs for sufficiently large values of v for any uniform hypergraph H, the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on G-decompositions of K_v where G is a graph with a relatively small number of edges (see [1] and [6] for known results). Some authors have investigated the corresponding problem for 3-uniform hypergraphs. For example, in [4], the spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. In [16], Mathon and Street give necessary conditions for the existence of decompositions of $K_v^{(3)}$ into copies of the projective plane $PG(2, 2)$ and into copies of the affine plane $AG(2, 3)$. They give sufficient conditions for several infinite classes in both cases. More recently, the spectrum problem was settled in [5] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [5], they also settle the spectrum

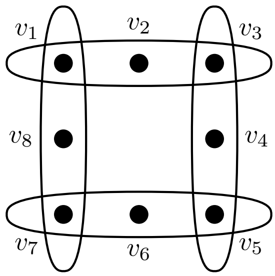

Figure 1: The loose 4-cycle LC_4 denoted $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$.

problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered H-designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let T, O, and I denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph T is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [10]. In another paper [11], Hanani settled the spectrum problem for O-designs and gave necessary conditions for the existence of I-designs. Perhaps the best known general result on decompositions of complete t-uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m. There are, however, several articles on decompositions of complete t-uniform hypergraphs (see [2] and [17]) and of t-uniform t-partite hypergraphs (see [14] and [18]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [12] and [15]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in the spectrum problem for H-designs where H is the hypergraph known as a loose 4-cycle. A *loose m-cycle* in $K_n^{(3)}$, denoted LC_m, is a hypergraph with vertex set $\{v_1, v_2, \ldots, v_{2m}\}$ and edge set $\big\{\{v_{2i-1}, v_{2i}, v_{2i+1}\} : 1 \leq i \leq m-1\big\} \cup \{v_{2m-1}, v_{2m}, v_1\}$. The spectrum problem for a loose 3-cycle was settled by Bryant, Herke, Maenhaut, and Wannasit in [5]. Let $H[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ denote the loose 4-cycle with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ and edge set $\big\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_7, v_8, v_1\}\big\}$. This hypergraph is shown in Figure 1.

1.1 Additional notation and terminology

If a and b are integers, we define $[a,b]$ to be $\{r \in \mathbb{Z} : a \le r \le b\}$. Let $\mathbb{Z}_n$ denote the group of integers modulo n. We next define some notation for certain types of 3-uniform hypergraphs.

Let U_1, U_2, U_3 be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of U_1, U_2, U_3 is denoted by $K^{(3)}_{U_1,U_2,U_3}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of U_1, U_2 is denoted by $L^{(3)}_{U_1,U_2}$. If $|U_i| = u_i$ for $i \in \{1,2,3\}$, we may use $K^{(3)}_{u_1,u_2,u_3}$ to denote any hypergraph that is isomorphic to $K^{(3)}_{U_1,U_2,U_3}$ and $L^{(3)}_{u_1,u_2}$ to denote any hypergraph that is isomorphic to $L^{(3)}_{U_1,U_2}$.

If H' is a subhypergraph of H, then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H'. We may refer to $H \setminus H'$ as the hypergraph H with a *hole* H'. The vertices in H' are called the vertices in the hole.

2 Some small examples

We give several examples of LC_4-decompositions that are used in proving our main result.

Example 2.1. Let $V\left(K^{(3)}_8\right) = \mathbb{Z}_8$ and let

$$\begin{aligned} B_1 &= \big\{H[0,5,1,7,2,3,6,4]\big\}, \\ B_2 &= \big\{H[0,1,2,3,4,5,6,7],\ H[0,7,2,1,4,3,6,5],\ H[0,5,2,7,4,1,6,3], \\ &\qquad H[1,2,3,4,5,6,7,0],\ H[1,0,3,2,5,4,7,6],\ H[1,6,3,0,5,2,7,4]\big\}. \end{aligned}$$

Then an LC_4-decomposition of $K^{(3)}_8$ consists of the orbit of the H-block in B_1 under the action of the map $j \mapsto j+1 \pmod 8$ along with the H-blocks in B_2.

Example 2.2. Let $V\left(K^{(3)}_9\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$ and let

$$\begin{aligned} B = \big\{&H[\infty_1,\infty_2,0,5,2,4,3,6],\ H[2,\infty_1,0,3,\infty_2,6,1,4], \\ &H[4,3,0,\infty_2,1,\infty_1,2,5]\big\}. \end{aligned}$$

Then an LC_4-decomposition of $K_9^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod 7$.

Example 2.3. Let $V\left(K_{10}^{(3)}\right) = \mathbb{Z}_{10}$ and let

$$B = \big\{H[0,8,1,3,2,9,5,7],\ H[4,0,2,8,6,3,7,9],\ H[0,1,3,4,8,2,9,5]\big\}.$$

Then an LC_4-decomposition of $K_{10}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $j \mapsto j+1 \pmod{10}$.

Example 2.4. Let $V\left(K_{12}^{(3)}\right) = \mathbb{Z}_{11} \cup \{\infty\}$ and let

$$B = \big\{H[9,1,0,\infty,5,4,3,6],\ H[2,5,0,\infty,3,7,1,4],\ H[0,\infty,1,5,2,4,9,6],\ H[6,0,1,\infty,8,4,9,2],\ H[8,1,0,\infty,2,6,4,3]\big\}.$$

Then an LC_4-decomposition of $K_{12}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{11}$.

Example 2.5. Let $V\left(K_{14}^{(3)}\right) = \mathbb{Z}_{13} \cup \{\infty\}$ and let

$$B = \big\{H[4,12,8,11,1,0,\infty,2],\ H[0,9,11,6,4,8,\infty,3],\ H[1,4,10,5,0,7,\infty,6],\ H[0,1,2,3,8,4,9,10],\ H[8,11,6,4,3,0,7,1],\ H[2,0,8,12,7,9,10,3],\ H[1,5,4,11,2,0,10,8]\big\}.$$

Then an LC_4-decomposition of $K_{14}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j+1 \pmod{13}$.

Example 2.6. Let $V\left(L_{8,8}^{(3)}\right) = \mathbb{Z}_{16}$ with vertex partition $\big\{\{0,2,4,6,8,10,12,14\},\ \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$B = \big\{H[0,5,1,2,8,14,15,11],\ H[0,2,5,9,6,8,13,7],\ H[1,6,0,4,3,10,15,12],\ H[0,8,1,2,10,4,7,5],\ H[0,14,1,10,3,11,6,5],\ H[7,0,3,8,13,4,1,14],\ H[1,0,2,5,3,7,12,4]\big\}.$$

Then an LC_4-decomposition of $L_{8,8}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $j \mapsto j+1 \pmod{16}$.

Example 2.7. Let $V\left(L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty\}$ with vertex partition $\big\{\{\infty\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$B = \big\{H[7,0,1,11,2,3,\infty,10],\ H[5,0,1,13,2,9,\infty,10],\\ H[0,2,5,9,6,8,13,7],\ H[1,6,0,4,3,10,15,12],\\ H[0,8,1,2,10,4,7,5],\ H[0,14,1,10,3,11,6,5],\\ H[7,0,3,8,13,4,1,14],\ H[1,0,2,5,3,7,12,4]\big\}.$$

Then an LC_4-decomposition of $L_{8,8}^{(3)} \cup K_{1,8,8}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j{+}1 \pmod{16}$.

Example 2.8. Let $V\left(K_{2,8,8}^{(3)}\right) = \mathbb{Z}_{16} \cup \{\infty_1, \infty_2\}$ with vertex partition $\big\{\{\infty_1,\infty_2\}, \{0,2,4,6,8,10,12,14\}, \{1,3,5,7,9,11,13,15\}\big\}$ and let

$$B = \big\{H[\infty_1,1,0,3,\infty_2,9,8,11],\ H[\infty_1,5,0,7,\infty_2,13,8,15]\big\}.$$

Then an LC_4-decomposition of $K_{2,8,8}^{(3)}$ consists of the orbits of the H-blocks in B under the action of the map $\infty_i \mapsto \infty_i$, for $i \in \{1,2\}$, and $j \mapsto j+1 \pmod{16}$.

Example 2.9. Let $V\left(K_{12}^{(3)} \backslash K_4^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1,\infty_2,\infty_3,\infty_4\}$ with $\infty_1, \ldots, \infty_4$ being the vertices in the hole and let

$$B_1 = \big\{H[2,0,4,\infty_3,5,6,\infty_1,\infty_2],\ H[0,2,5,\infty_2,4,\infty_4,\infty_1,\infty_3],\\ H[0,2,\infty_3,3,\infty_2,4,\infty_4,1],\ H[2,0,\infty_1,4,7,\infty_4,\infty_3,5],\\ H[3,0,\infty_2,5,7,4,\infty_4,1]\big\},$$

$$B_2 = \big\{H[2,1,0,\infty_1,4,5,6,\infty_2],\ H[3,2,1,\infty_1,5,6,7,\infty_2],\\ H[4,3,2,\infty_1,6,7,0,\infty_2],\ H[5,4,3,\infty_1,7,0,1,\infty_2],\\ H[3,1,0,\infty_3,4,5,7,\infty_4],\ H[4,2,1,\infty_3,5,6,0,\infty_4],\\ H[5,3,2,\infty_3,6,7,1,\infty_4],\ H[6,4,3,\infty_3,7,0,2,\infty_4],\\ H[0,5,1,6,2,7,3,4],\ H[4,0,1,5,2,6,3,7],\ H[6,1,0,3,2,5,4,7],\\ H[4,1,5,2,6,3,7,0],\ H[0,4,5,1,6,2,7,3],\ H[7,2,1,4,3,6,5,0]\big\}.$$

Then an LC_4-decomposition of $K_{12}^{(3)} \backslash K_4^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in [1,4]$, and $j \mapsto j+1 \pmod 8$ along with the H-blocks in B_2.

Example 2.10. Let $V\left(K_{14}^{(3)} \backslash K_6^{(3)}\right) = \mathbb{Z}_8 \cup \{\infty_1,\infty_2,\infty_3,\infty_4,\infty_5,\infty_6\}$ with $\infty_1, \ldots, \infty_6$ being the vertices in the hole and let

$$B_1 = \big\{H[\infty_1,0,\infty_2,2,\infty_3,4,\infty_5,6],\ H[\infty_6,7,\infty_5,5,\infty_4,3,\infty_2,1],\\ H[\infty_1,0,\infty_3,7,\infty_6,6,\infty_4,1],\ H[\infty_6,\infty_1,0,2,\infty_3,\infty_4,5,7],$$

$H[0, \infty_1, 1, \infty_2, 4, \infty_4, 5, \infty_5]$, $H[0, \infty_2, 1, \infty_3, 4, \infty_5, 5, \infty_6]$,
$H[0, \infty_3, 1, \infty_1, 4, \infty_6, 5, \infty_4]$, $H[0, \infty_1, 2, \infty_2, 4, \infty_4, 6, 3]$,
$H[\infty_5, \infty_2, 0, 4, 1, 3, 5, 7]\}$,

$B_2 = \{H[2, 1, 0, \infty_1, 4, 5, 6, \infty_2]$, $H[3, 2, 1, \infty_1, 5, 6, 7, \infty_2]$,
$H[4, 3, 2, \infty_1, 6, 7, 0, \infty_2]$, $H[5, 4, 3, \infty_1, 7, 0, 1, \infty_2]$,
$H[3, 1, 0, \infty_3, 4, 5, 7, \infty_4]$, $H[4, 2, 1, \infty_3, 5, 6, 0, \infty_4]$,
$H[5, 3, 2, \infty_3, 6, 7, 1, \infty_4]$, $H[6, 4, 3, \infty_3, 7, 0, 2, \infty_4]$,
$H[3, 2, 0, \infty_5, 4, 6, 7, \infty_6]$, $H[4, 3, 1, \infty_5, 5, 7, 0, \infty_6]$,
$H[5, 4, 2, \infty_5, 6, 0, 1, \infty_6]$, $H[6, 5, 3, \infty_5, 7, 1, 2, \infty_6]$,
$H[0, 5, 1, 6, 2, 7, 3, 4]$, $H[4, 1, 5, 2, 6, 3, 7, 0]\}$.

Then an LC_4-decomposition of $K_{14}^{(3)} \backslash K_6^{(3)}$ consists of the orbits of the H-blocks in B_1 under the action of the map $\infty_i \mapsto \infty_i$, for $i \in [1, 6]$, and $j \mapsto j + 1 \pmod 8$ along with the H-blocks in B_2.

3 Main results

We begin by giving necessary conditions for the existence of an LC_4-decomposition of $K_v^{(3)}$. An obvious necessary condition is that 4 must divide the number of edges in $K_v^{(3)}$, and thus we must have $v \equiv 0$, 1, 2, 4, or 6 (mod 8). Since $K_1^{(3)}$ and $K_2^{(3)}$ contain no edges, it is vacuously true that LC_4 decomposes $K_1^{(3)}$ and $K_2^{(3)}$. Also since LC_4 has order 8, there is no LC_4-decomposition of $K_4^{(3)}$ or $K_6^{(3)}$. Thus we have the following.

Lemma 1. *There exists an LC_4-decomposition of $K_v^{(3)}$ only if $v \equiv 0$, 1, 2, 4, or 6 (mod 8) and $v \notin \{4, 6\}$.*

We will show that the above conditions are sufficient by showing how to construct LC_4-decompositions of $K_v^{(3)}$ for all $v \equiv 0$, 1, 2, 4, or 6 (mod 8) with $v \geq 8$. Our constructions are dependent on the many small examples given in Section 2.

We begin by proving a lemma that is fundamental to our constructions.

Lemma 2. *Let n, x, and r be nonnegative integers such that $nx + r \geq 3$. There exists a decomposition of $K_{nx+r}^{(3)}$ that is comprised of isomorphic copies of each of the following under the given conditions:*

- $K_r^{(3)}$ *if* $x = 0$,
- $K_{n+r}^{(3)}$ *if* $x \geq 1$,
- $K_{n+r}^{(3)} \setminus K_r^{(3)}$ *if* $x \geq 2$,
- $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$ *if* $x \geq 2$,
- $K_{n,n,n}^{(3)}$ *if* $x \geq 3$.

Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Similarly, if $n = 0$, the result is trivial because $K_r^{(3)} = K_{n+r}^{(3)} = K_{nx+r}^{(3)}$ while $K_{n+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,n,n}^{(3)} \cup L_{n,n}^{(3)}$, and $K_{n,n,n}^{(3)}$ are all empty (i.e., contain no edges). For the remainder of the proof, we assume that $x \geq 2$ and $n \geq 1$.

Let $V_0, V_1, \ldots, V_x$ be pairwise disjoint sets of vertices with $|V_0| = r$ and $|V_1| = |V_2| = \cdots = |V_x| = n$. Then, the result follows from the fact that the complete 3-uniform hypergraph on the vertex set $V_0 \cup V_1 \cup \cdots \cup V_x$, which is $nx + r$ vertices, can be viewed as the (edge-disjoint) union

$$K_{V_1 \cup V_0}^{(3)} \cup \bigcup_{2 \leq i \leq x} \left(K_{V_i \cup V_0}^{(3)} \setminus K_{V_0}^{(3)} \right) \cup \bigcup_{1 \leq i < j \leq x} \left(K_{V_0, V_i, V_j}^{(3)} \cup L_{V_i, V_j}^{(3)} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left(K_{V_i, V_j, V_k}^{(3)} \right). \qquad \square$$

We now give our main result.

Theorem 3. *There exists an LC_4-decomposition of $K_v^{(3)}$ if and only if $v \equiv 0$, 1, 2, 4, or 6 $\pmod 8$ and $v \notin \{4, 6\}$.*

Proof. The necessary conditions for the existence of an LC_4-decomposition of $K_v^{(3)}$ are established in Lemma 1. Thus we need only to establish their sufficiency. Let $v = 8x + r$ where $x \geq 1$ and $r \in \{0, 1, 2, 4, 6\}$. By Lemma 2 it suffices to find LC_4-decompositions of $K_{8+r}^{(3)}$, $K_{8+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,8,8}^{(3)} \cup L_{8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. We note that if $r \in \{0, 1, 2\}$ then $K_{8+r}^{(3)} \setminus K_r^{(3)}$ is isomorphic to $K_{8+r}^{(3)}$. Also, $K_{0,8,8}^{(3)}$ is empty, and $K_{2,8,8}^{(3)}$ decomposes $K_{4,8,8}^{(3)}$, $K_{6,8,8}^{(3)}$, and $K_{8,8,8}^{(3)}$. Thus, it suffices to find LC_4-decompositions of $K_8^{(3)}$, $K_9^{(3)}$, $K_{10}^{(3)}$, $K_{12}^{(3)}$, $K_{14}^{(3)}$, $K_{12}^{(3)} \setminus K_4^{(3)}$, $K_{14}^{(3)} \setminus K_6^{(3)}$, $K_{1,8,8}^{(3)} \cup L_{8,8}^{(3)}$, $K_{2,8,8}^{(3)}$, and $L_{8,8}^{(3)}$, which are each shown to exist in Examples 2.1–2.10. $\square$

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