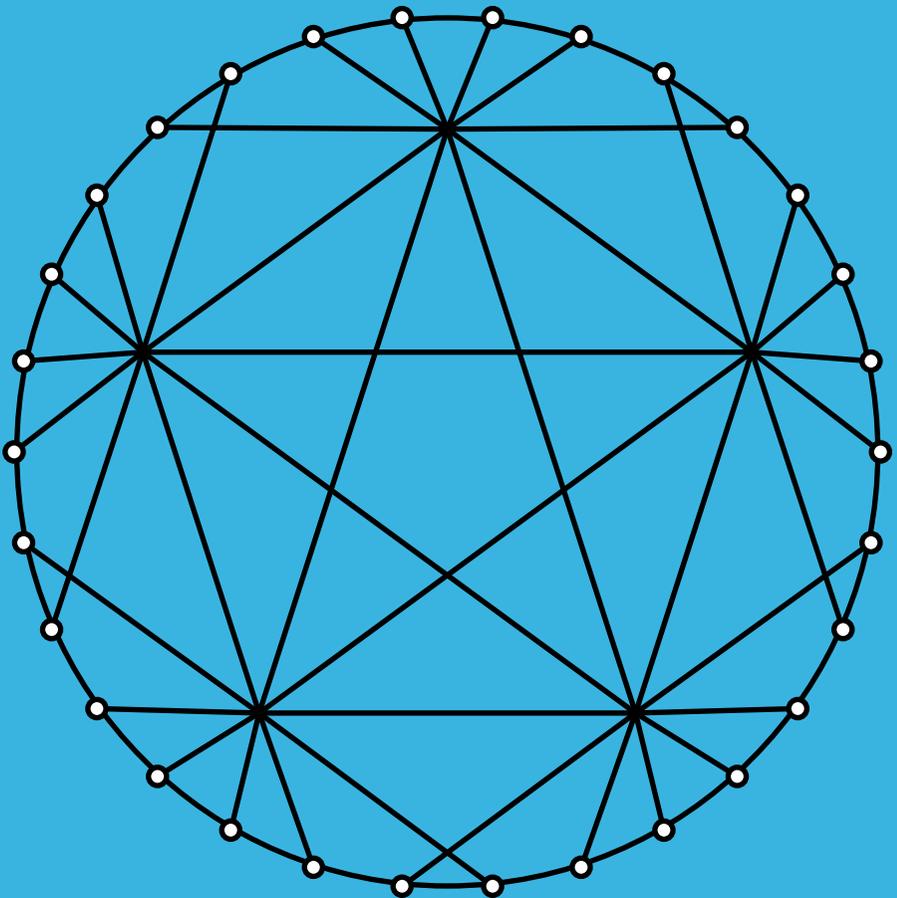


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Balanced 3-nearly Platonic graphs

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Abstract: A balanced 3-nearly Platonic graph of type (k, d) is a k -regular planar graph with three faces of the same degree m , and all $f - 3$ remaining faces of degree $d \neq m$. We present constructions of infinite families of balanced 3-nearly Platonic graph of type (k, d) for all admissible pairs (k, d) , whose existence was claimed by Deza, Dutour Sikirič, and Shtogrin in 2013.

1 Introduction

The five Platonic solids have been known since antiquity, and their discovery is often attributed to Pythagoras (cca 570–495 B.C.).

It is well-known that the graph representations of the five Platonic solids are the only finite regular planar graphs with all faces of the same size. These graphs are often called *Platonic graphs*. We say that a Platonic graph is of type (k, d) if it is k -vertex-regular and all its faces are of size d (sometimes also called degree). The five Platonic graphs—tetrahedron, cube, dodecahedron, octahedron, and icosahedron—are of types $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$, respectively.

A k -regular planar graph with f faces is a t -nearly Platonic graph of type (k, d) if $f - t$ of its faces are of size d and the remaining t faces are of sizes other than d . The faces of size d are often called *common faces*, and

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the remaining ones *exceptional* or *disparate* faces. When $t > 1$ and all disparate faces are of the same size, then the graph is called a *balanced t -nearly Platonic graph*.

Keith, Froncek, and Kreher [4, 5] and Froncek and Qiu [3] proved recently that there are no 1-nearly Platonic graphs. There are several well-known families of balanced 2-nearly Platonic graphs (see, e.g., [4]). In [4], the authors asked about existence of balanced 3-nearly Platonic graphs.

We constructed the graphs (except one, adapted from [2]) presented in this paper before we discovered that this question had been studied earlier by Deza and Dutour Sikirič [1] and Deza, Dutour Sikirič, and Shtogrin [2]. Because their proofs are not easily accessible and the publications are largely unknown to combinatorial community, we present detailed constructive proofs in this paper, obtaining infinite classes of balanced 3-nearly Platonic graphs.

We show that the only admissible types of 3-nearly Platonic graphs are $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$, and for each pair construct at least one infinite family of such graphs. All our constructions are based on surgeries of Platonic graphs.

2 Necessary conditions and small cases

Denote the order of the 3-nearly Platonic graph of type (k, d) by v , the number of edges by e , and number of faces by f . The degree (or size) of the exceptional faces will be denoted m . By vertex regularity the number of edges is $\frac{kv}{2}$. Since the graph is planar, we have $e \leq 3v - 6$ according to Euler's formula. Therefore, the inequality

$$\frac{kv}{2} \leq 3v - 6$$

holds. This gives

$$kv \leq 6v - 12,$$

which implies

$$12 \leq 6v - kv = (6 - k)v.$$

Since the right-hand side must be positive, we observe that $k = 3, 4$, or 5 .

Now we show the possible values of d for each k . We can compute the number of faces based on Euler's formula, that is, $v - e + f = 2$. Therefore,

$$\begin{aligned}
 f &= e - v + 2 \\
 &= \frac{kv}{2} - v + 2 \\
 &= \left(\frac{k}{2} - 1\right)v + 2.
 \end{aligned} \tag{1}$$

Since the sum of all face degrees equals twice the number of edges and we have $f - 3$ faces of degree d and three of degree m , we obtain

$$(f - 3)d + 3m = 2e \tag{2}$$

and by substituting (1)

$$\begin{aligned}
 \left(\left(\frac{k}{2} - 1\right)v + 2 - 3\right)d + 3m &= kv \\
 \left(\left(\frac{k}{2} - 1\right)v - 1\right)d + 3m &= kv.
 \end{aligned} \tag{3}$$

Solving for d , we get

$$\begin{aligned}
 d &= \frac{kv - 3m}{\left(\frac{k}{2} - 1\right)v - 1} \\
 &= \frac{2kv - 6m}{(k - 2)v - 2}.
 \end{aligned} \tag{4}$$

We plug in $k = 3, 4, 5$ separately into the equation (4) to determine the possible values of d . First, for $k = 3$, we obtain

$$\begin{aligned}
 d &= \frac{6v - 6m}{v - 2} \\
 &= \frac{6(v - 2) + 12 - 6m}{v - 2} \\
 &= 6 + \frac{12 - 6m}{v - 2}.
 \end{aligned}$$

Since $m \geq 3$, we have $\frac{12-6m}{v-2} < 0$, and then $d < 6$. Possible values of d for $k = 3$ are therefore 3, 4, or 5.

When $k = 4$, then

$$\begin{aligned} d &= \frac{8v - 6m}{2v - 2} \\ &= \frac{4v - 3m}{v - 1} \\ &= \frac{4(v - 1) + 4 - 3m}{v - 1} \\ &= 4 + \frac{4 - 3m}{v - 1}. \end{aligned}$$

Since the exceptional faces are of degree at least 3, we have $4 - 3m < 0$, which leads to $d < 4$. Hence, $d = 3$ is the only possibility for $k = 4$.

Finally, for $k = 5$,

$$\begin{aligned} d &= \frac{10v - 6m}{3v - 2} \\ &= \frac{\frac{10}{3}(3v - 2) + \frac{20}{3} - 6m}{3v - 2} \\ &= \frac{10}{3} + \frac{\frac{20}{3} - 6m}{3v - 2}. \end{aligned}$$

Again, we have $m \geq 3$. Then $\frac{20}{3} - 6m < 0$ and because $3v - 2$ is positive, it follows that $d < 4$ for $k = 5$.

Therefore, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, $(5, 3)$ are the only possible combinations for (k, d) . We formalize our findings as follows.

Proposition 2.1. *If there exists a balanced 3-nearly Platonic graph of type (k, d) , then $(k, d) \in \{(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)\}$.*

Now we show minimum balanced 3-nearly Platonic graphs for each possible (k, d) . We plug each (k, d) into (3) to obtain the lower bound for the order of the graph and present a smallest graph for each (k, d) .

For $(k, d) = (3, 3)$ we have

$$\begin{aligned} \left(\left(\frac{3}{2} - 1 \right) v - 1 \right) \cdot 3 + 3m &= 3v \\ \left(\frac{1}{2}v - 1 \right) \cdot 3 + 3m &= 3v \\ \frac{3}{2}v - 3 + 3m &= 3v \\ \frac{3}{2}v &= 3m - 3 \\ v &= 2m - 2. \end{aligned}$$

The length of three exceptional faces should differ from the common faces, therefore $m \geq 4$. Thus the minimum order is six, and such graph is shown in Figure 1. Here and in all figures in this section, the common faces are shaded and the three exceptional faces are left white. Notice that except the last case $(5, 3)$ in Figure 5, the outer face is always one of the exceptional faces.

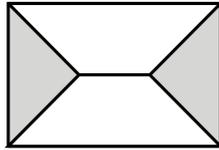


Figure 1: Minimum balanced 3-nearly Platonic graph of type $(3,3)$

For type $(3, 4)$,

$$\begin{aligned} \left(\left(\frac{3}{2} - 1 \right) v - 1 \right) \cdot 4 + 3m &= 3v \\ \left(\frac{1}{2}v - 1 \right) \cdot 4 + 3m &= 3v \\ 2v - 4 + 3m &= 3v \\ v &= 3m - 4. \end{aligned}$$

Since the graph is 3-regular, the number of vertices must be even, thus $m \geq 6$ and $v \geq 14$. Hence we obtain a smallest graph shown in Figure 2.

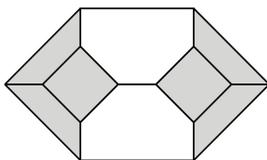


Figure 2: Minimum balanced 3-nearly Platonic graph of type (3, 4)

For type (3, 5),

$$\begin{aligned} \left(\left(\frac{3}{2} - 1 \right) v - 1 \right) \cdot 5 + 3m &= 3v \\ \frac{5}{2}v - 5 + 3m &= 3v \\ \frac{1}{2}v &= 3m - 5 \\ v &= 6m - 10 \end{aligned}$$

We claim that $m \neq 3$. Suppose $m = 3$, $v = 8$, and $e = \frac{kv}{2} = 12$, $f = e - v + 2 = 6$. Consider the dual graph, which is of order six. Then three of the vertices of the dual graph are of degree five, others are of degree three. Therefore each vertex of degree five must be adjacent to all the vertices of degree three, which forms a $K_{3,3}$. Since $K_{3,3}$ is not planar, the original graph cannot be planar. Thus $m \neq 3$. Then $m \geq 4$, and $v \geq 14$. A smallest graph is shown in Figure 3.

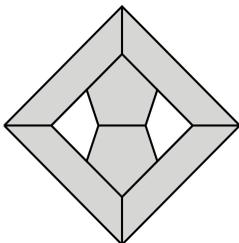


Figure 3: Minimum balanced 3-nearly Platonic graph of type (3, 5)

For type $(4, 3)$,

$$\begin{aligned} \left(\left(\frac{4}{2} - 1 \right) v - 1 \right) \cdot 3 + 3m &= 4v \\ (v - 1) \cdot 3 + 3m &= 4v \\ 3v - 3 + 3m &= 4v \\ v &= 3m - 3. \end{aligned}$$

Again, the minimum value for m will be 4 since the common faces are triangles, which yields $v \geq 9$. A smallest graph is shown in Figure 4.

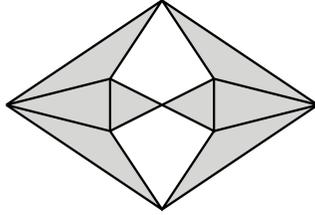


Figure 4: Minimum balanced 3-nearly Platonic graph of type $(4, 3)$

For type $(5, 3)$,

$$\begin{aligned} \left(\left(\frac{5}{2} - 1 \right) v - 1 \right) \cdot 3 + 3m &= 5v \\ \left(\frac{3}{2}v - 1 \right) \cdot 3 + 3m &= 5v \\ \frac{9}{2}v - 3 + 3m &= 5v \\ 9v - 6 + 6m &= 10v \\ v &= 6m - 6. \end{aligned}$$

Since $m \geq 4$, we have $v \geq 18$. A smallest graph is shown in Figure 5.

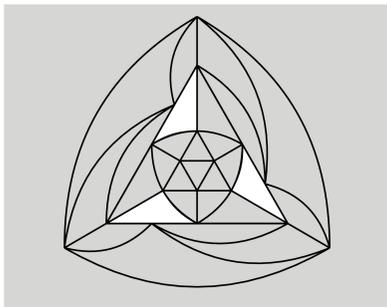


Figure 5: Minimum balanced 3-nearly Platonic graph of type $(5, 3)$

3 Pseudoblocks and connectors

As mentioned in Introduction, our constructions are based on surgeries performed on Platonic graphs. In this section, we build graphs with connectivity 2 using two types of structures, called *pseudoblocks* and *connectors*.

Definition 3.1. A 2-connected planar graph $B_{a,b}(k, d)$ with $a, b < k$ is called a *pseudoblock* if it has one vertex of degree a , one vertex of degree b , all remaining vertices of degree k , and all inner faces of degree d .

A 2-connected planar graph $D_{a,b,c}(k, d)$ with $a, b, c < k$ is called a *connector* if it has one vertex of degree a , one vertex of degree b , one vertex of degree c , all remaining vertices of degree k , and all inner faces of degree d .

The vertices with exceptional degrees a, b and possibly c all belong to the exceptional face. The graph K_1 will be denoted as $D_{0,0,0}$ and K_2 as $B_{1,1}$.

In Figure 6 we show a balanced 3-nearly Platonic graph of type $(3, 3)$ with six pseudoblock on 4 vertices $B_{2,2}(3, 3)$ isomorphic to $K_4 - e$, 9 pseudoblocks $B_{1,1}$ isomorphic to K_2 , one connector $D_{2,2,2}(3, 3)$ isomorphic to K_3 , and one connector $D_{0,0,0}$ isomorphic to K_1 .

All pseudoblocks and connectors used in this section arise from Platonic solids be either removing an edge or *splitting a vertex*.

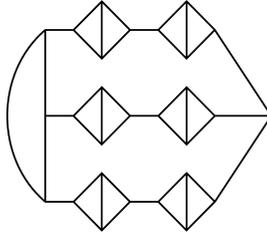


Figure 6: A balanced 3-nearly Platonic graph of type $(3, 3)$

Definition 3.2. Let G be a graph and x a vertex of G of degree at least 3. Let $N_G(x) = \{x_1, x_2, \dots, x_s\}$ be the neighborhood of x in G and sets N^1, N^2, \dots, N^p form a partition of $N_G(x)$ for some p , $2 \leq p \leq s$. We construct a graph G^x by taking G , removing vertex x and adding vertices x^1, x^2, \dots, x^p and edges from x^i to all vertices in N^i . We say that graph G^x arises from G by *splitting vertex x* .

Type $B_{a,b}(3, d)$

All pseudoblocks of type $B_{a,b}(3, d)$ arise from tetrahedron, cube, or dodecahedron. In this case, we will be only using edge removal. Splitting a vertex would result in the same pseudoblock with a pendant edge attached.

Pseudoblock $B_{2,2}(3, 3)$ arises from the tetrahedron and is shown in Figure 7.

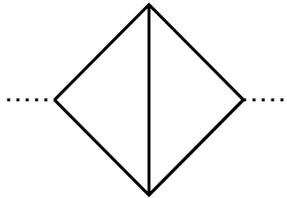


Figure 7: Pseudoblock $B_{2,2}(3, 3)$

We use two connectors, $D_{2,2,2}(3, 3)$ arising from the tetrahedron by removing a vertex (isomorphic to triangle K_3 and shown in Figure 8), and the trivial connector $D_{0,0,0}$, that is, a single vertex.

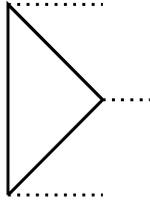


Figure 8: Connector $D_{2,2,2}(3,3)$

Using pseudoblocks $B_{2,2}(3,3)$ and $B_{1,1}$ and combining them with the two connectors, we obtain three infinite classes of balanced 3-nearly Platonic graphs of type $(3,3)$ shown in Figures 9, 10, 11. It is easy to verify that the sizes of the three exceptional faces are $2, 3, 4 \pmod{6}$, respectively.

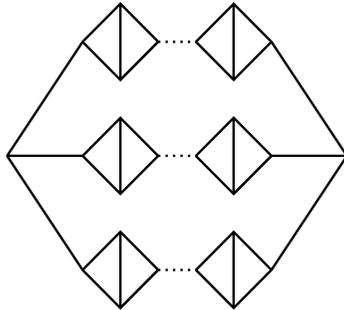


Figure 9: Type $(3,3)$ with exceptional faces of size $2 \pmod{6}$

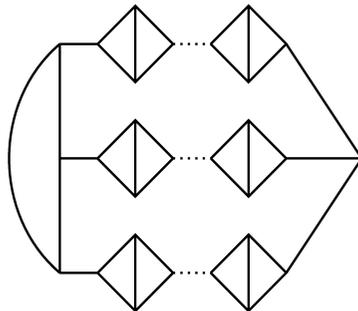


Figure 10: Type $(3,3)$ with exceptional faces of size $3 \pmod{6}$

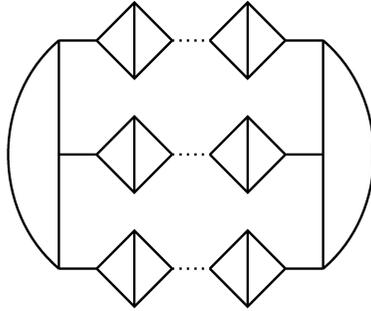


Figure 11: Type (3,3) with exceptional faces of size 4 (mod 6)

Proposition 3.3. *There exist balanced 3-nearly Platonic graphs of type (3,3) with connectivity 2 and exceptional faces of size m for every $m \equiv 2, 3, 4 \pmod{6}$, $m \geq 4$.*

Pseudoblock $B_{2,2}(3,4)$ arises from the cube by deleting an edge and is shown in Figure 12.

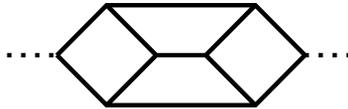


Figure 12: Pseudoblock $B_{2,2}(3,4)$

We again have two connectors, $D_{2,2,2}(3,4)$ arising from the cube by removing a vertex (shown in Figure 13), and the trivial connector $D_{0,0,0}$.

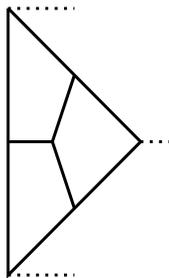


Figure 13: Connector $D_{2,2,2}(3,4)$

We use pseudoblocks $B_{2,2}(3,4)$ and $B_{1,1}$ and combine them with the two connectors in three different ways. Thus, we obtain three infinite classes of balanced 3-nearly Platonic graphs of type $(3,4)$ shown in Figures 14, 15, 16. One can check that the sizes of the exceptional faces are $2, 4, 6 \pmod{8}$, respectively.

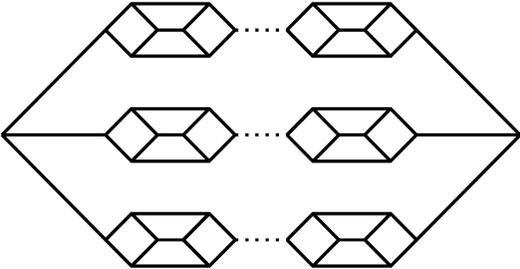


Figure 14: Type $(3,4)$ with exceptional faces of size $2 \pmod{8}$

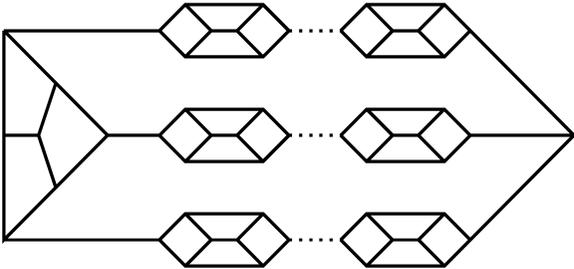


Figure 15: Type $(3,4)$ with exceptional faces of size $4 \pmod{8}$

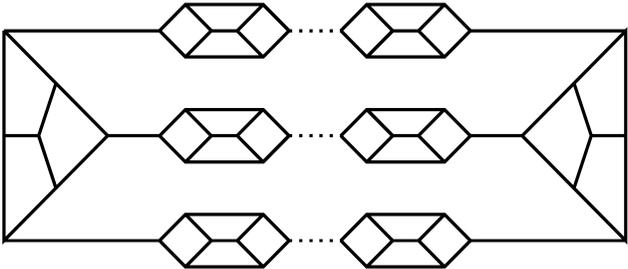


Figure 16: Type $(3,4)$ with exceptional faces of size $6 \pmod{8}$

Proposition 3.4. *There exist balanced 3-nearly Platonic graphs of type $(3, 4)$ with connectivity 2 and exceptional faces of size m for every $m \equiv 2, 4, 6 \pmod{8}, m \geq 6$.*

Pseudoblock $B_{2,2}(3, 5)$ shown in Figure 17 arises from the dodecahedron by deleting an edge.

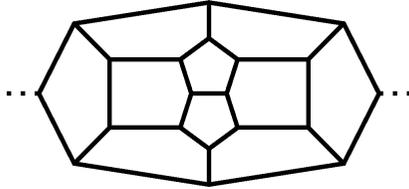


Figure 17: Pseudoblock $B_{2,2}(3, 5)$

We have once more two connectors, $D_{2,2,2}(3, 5)$ arising from the dodecahedron by removing a vertex (shown in Figure 18), and the trivial connector $D_{0,0,0}$.

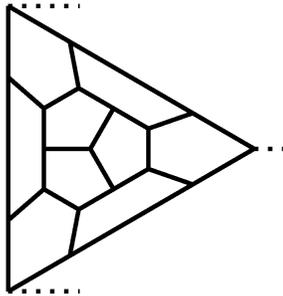


Figure 18: Connector $D_{2,2,2}(3, 5)$

Combining pseudoblocks $B_{2,2}(3, 5)$ and $B_{1,1}$ with the two connectors, we obtain three infinite classes of balanced 3-nearly Platonic graphs of type $(3, 5)$ shown in Figures 19, 20, 21. It is easy to show that the exceptional faces are of sizes $2, 5, 8 \pmod{10}$, respectively.

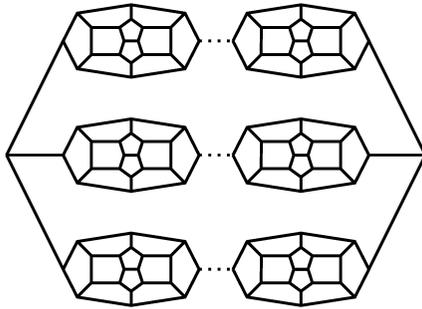


Figure 19: Type (3,5) with exceptional faces of size 2 (mod 10)

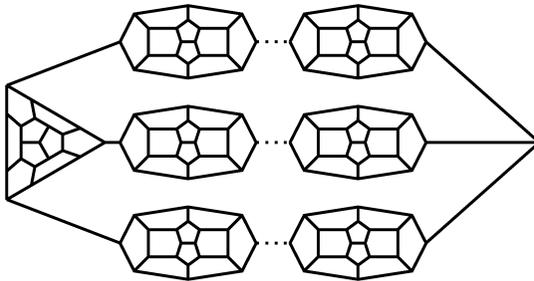


Figure 20: Type (3,5) with exceptional faces of size 5 (mod 10)

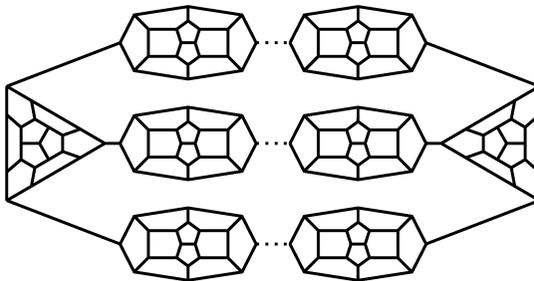


Figure 21: Type (3,5) with exceptional faces of size 8 (mod 10)

Proposition 3.5. *There exist balanced 3-nearly Platonic graphs of type (3, 5) with connectivity 2 and exceptional faces of size m for every $m \equiv 2, 5, 8 \pmod{10}, m \geq 8$.*

Type $B_{a,b}(4, 3)$

There are two pseudoblocks arising from the octahedron. Namely $B_{2,2}(4, 3)$ is obtained by splitting a vertex and $B_{3,3}(4, 3)$ is obtained by removing an edge. They are shown in Figures 22 and 23.

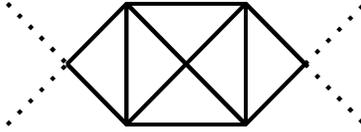


Figure 22: Pseudoblock $B_{2,2}(4, 3)$

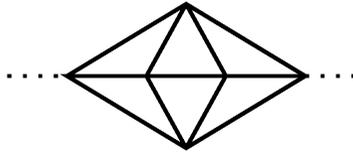


Figure 23: Pseudoblock $B_{3,3}(4, 3)$

We use three different connectors, $D_{2,2,2}(4, 3)$ arising from the octahedron by removing all edges of one triangle, $D'_{2,2,2}(4, 3)$ isomorphic to K_3 arising from the tetrahedron by removing a vertex, and $D_{3,3,2}(4, 3)$ arising from the tetrahedron by removing two adjacent edges. They are shown in Figures 24, 25, and 26.

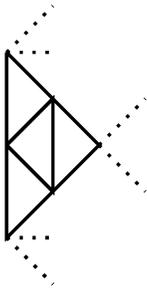


Figure 24: Connector $D_{2,2,2}(4, 3)$

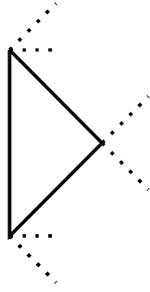


Figure 25: Connector $D'_{2,2,2}(4,3)$

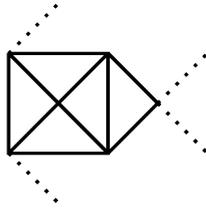


Figure 26: Connector $D_{3,3,2}(4,3)$

Combining pseudoblocks $B_{2,2}(4,3)$ with the connectors $D_{2,2,2}(4,3)$ and $D'_{2,2,2}(4,3)$ in three ways, we get three infinite classes of balanced 3-nearly Platonic graphs of type $(4,3)$ shown in Figures 27, 28, 29. The exceptional faces are of sizes $m \equiv 2, 3, 4 \pmod{6}$, respectively.

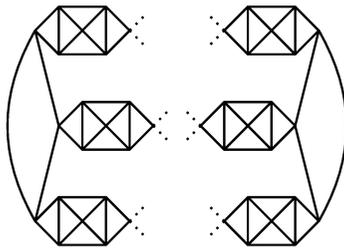


Figure 27: Type $B_{2,2}(4,3)$ with exceptional faces of size $2 \pmod{6}$

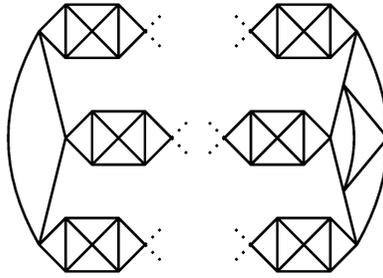


Figure 28: Type $B_{2,2}(4,3)$ with exceptional faces of size $3 \pmod{6}$

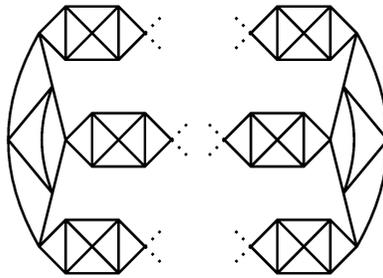


Figure 29: Type $B_{2,2}(4,3)$ with exceptional faces of size $4 \pmod{6}$

Also, we can combine two chains of pseudoblocks $B_{3,3}(4,3)$ and $B_{1,1}$ with a single chain using $B_{2,2}(4,3)$ with connector $D_{3,3,2}(4,3)$ and another one isomorphic to $B_{2,2}(4,3)$ to obtain another class with exceptional faces of sizes $m \equiv 3 \pmod{6}$, as shown in Figure 30.

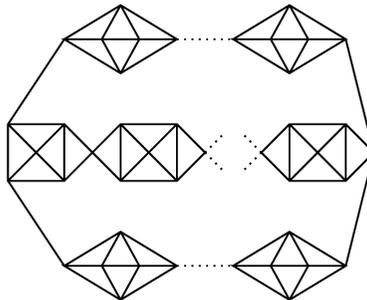


Figure 30: Combined chains $B_{a,b}(4,3)$

Proposition 3.6. *There exist balanced 3-nearly Platonic graphs of type $(4, 3)$ with connectivity 2 and exceptional faces of size m for every $m \equiv 2, 3, 4 \pmod{6}$, $m \geq 4$.*

Type $B_{a,b}(5, 3)$

We use two pseudoblocks arising from the icosahedron. By splitting a vertex, we obtain $B_{3,2}(5, 3)$, and by removing an edge we get $B_{4,4}(5, 3)$, both shown in Figures 31 and 32.

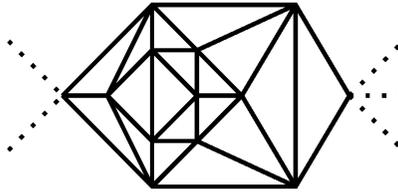


Figure 31: Pseudoblock $B_{3,2}(5, 3)$

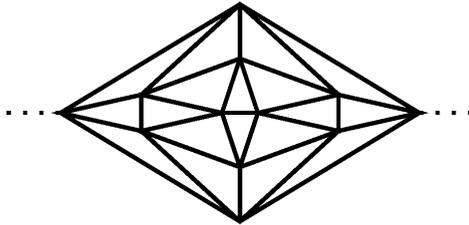


Figure 32: Pseudoblock $B_{4,4}(5, 3)$

We combine pseudoblocks $B_{3,2}(5, 3)$ and use two connectors, the triangle which we now for consistency call $D_{2,2,2}(5, 3)$, and $D_{3,3,3}(5, 3)$ obtained from the dodecahedron by removing all edges of one triangle (shown in Figure 33).

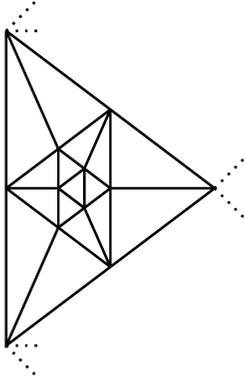


Figure 33: Connector $D_{3,3,3}(5,3)$

We obtain an infinite class of balanced 3-nearly Platonic graphs of type $(5,3)$ shown in Figure 34. The exceptional faces are of size $m \equiv 3 \pmod{6}$.

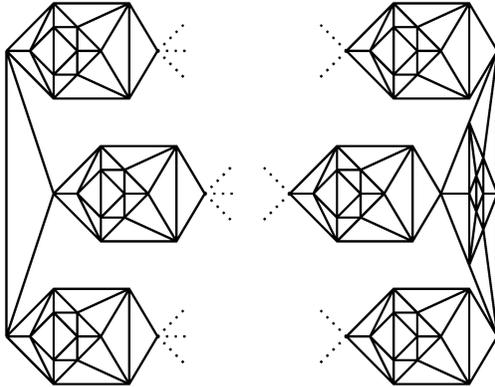


Figure 34: Type $(5,3)$ with exceptional faces of size $3 \pmod{6}$

Or we can build two chains consisting of pseudoblocks $B_{4,4}(5,3)$ and $B_{1,1}$ and one chain consisting only of copies of $B_{3,2}(5,3)$. One connector is $D_{3,1,1}(5,3)$ arising from the icosahedron by splitting a vertex. One can also instead use as a connector its subgraph, $D_{4,4,3}(5,3)$, arising from the icosahedron by removing two adjacent edges (shown in Figure 35). This would mean that the two chains made of $B_{4,4}(5,3)$ and $B_{1,1}$ would contain one extra copy of $B_{1,1}$ each.

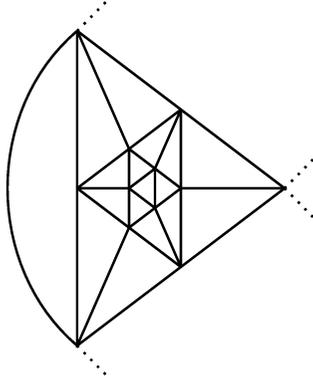


Figure 35: Connector $D_{4,4,3}(5, 3)$

The other connector $D_{2,1,1}(5, 3)$ obtained from the icosahedron by first splitting a vertex and adding two pendant edges at one of the two new vertices, namely the one of degree 3, is shown in Figure 36. It can be also viewed as the pseudoblock $B_{3,2}(5, 3)$ with two copies of $B_{1,1}$ attached to the single vertex of degree 3. The exceptional faces are also of size $m \equiv 3 \pmod{6}$.

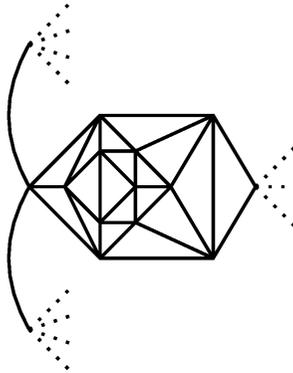


Figure 36: Connector $D_{2,1,1}(5, 3)$

We get another infinite class of balanced 3-nearly Platonic graphs of type $(5, 3)$ shown in Figure 37. The exceptional faces are again of size $m \equiv 3 \pmod{6}$.

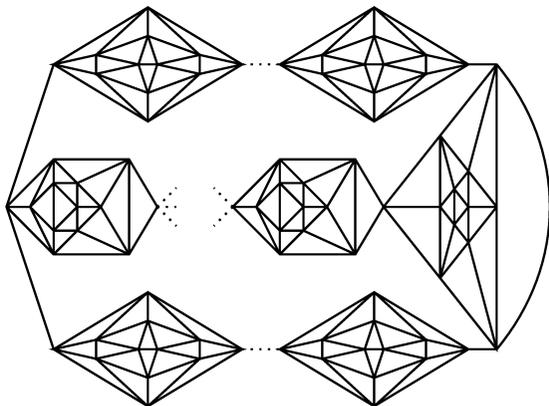


Figure 37: Type (5, 3) with exceptional faces of size 3 (mod 6)

Proposition 3.7. *There exist balanced 3-nearly Platonic graphs of type (5, 3) with connectivity 2 and exceptional faces of size m for every $m \equiv 3 \pmod{6}$, $m \geq 9$.*

4 Bars and joints

In the previous section, the graphs had connectivity 2. In this section, we construct 3-connected graphs. The building ingredients are again arising from the Platonic solids by either *splitting an edge* or by deleting vertices.

Definition 4.1. A 2-connected planar graph $R_{a,b}(k, d)$ is called a *bar* if the outer face has two edges e_1 and e_2 , each with endvertices of degree a and b , where $a, b < k$, all remaining vertices of degree k , and all inner faces of degree d .

All bars used in this section arise from Platonic solids by *splitting an edge*.

Definition 4.2. Let G be a graph and $e = xy$ an edge of G with both vertices x, y of degree at least 3. Let the *neighborhood* $N_G(e)$ of $e = xy$ in G be the set of all vertices other than x and y that are adjacent to at least one of x, y . Let N^1, N^2 form a partition of $N_G(e)$. We construct a graph G^e by taking G , removing edge e , adding edges $e^1 = x^1y^1, e^2 = x^2y^2$ and x^iz and y^iz to vertices in N^i whenever there was an edge xz or yz in G . We say that graph G^e arises from G by *splitting edge* $e = xy$.

By amalgamating bars at the split edges we obtain *strips*, and we will connect strips by subgraphs of Platonic graphs called *joints*. The joints are too heterogeneous to introduce a unified notation. However, the joints that we use again all arise from Platonic graphs by splitting or removing edges and/or vertices. We will use notation related only to the type of graph, that is, $J(k, d)$ will be a joint used for building 3-nearly Platonic graphs of type (k, d) .

We have three infinite classes of 3-nearly Platonic graphs of type $(3, 5)$ and three of type $(5, 3)$ constructed only from bars and joints.

Type $R_{2,2}(3, 5)$

The graph shown in Figure 38 obtained by splitting an edge of the dodecahedron will be denoted $R_{2,2}(3, 5)$.

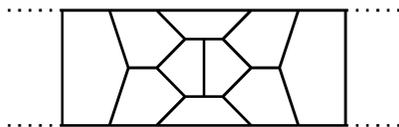


Figure 38: Bar $R_{2,2}(3, 5)$

We use two different joints, $J(3, 5)$ on 10 vertices and $J'(3, 5)$ on 16 vertices, shown in Figure 39. Notice that they arise from the dodecahedron by splitting three edges.

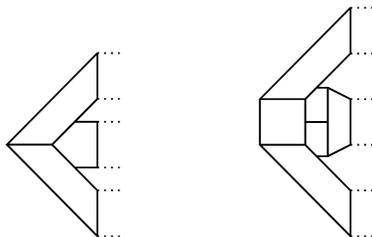


Figure 39: Joints $J(3, 5)$ and $J'(3, 5)$

Combining bars $R_{2,2}(3, 5)$ with joints $J_{2,2}(3, 5)$ and $J'_{2,2}(3, 5)$ in three different ways we obtain 3-nearly Platonic graphs of type $(5, 3)$ with the exceptional faces of sizes $4, 5, 6 \pmod{10}$, respectively, shown in Figures 40, 41, and 42.

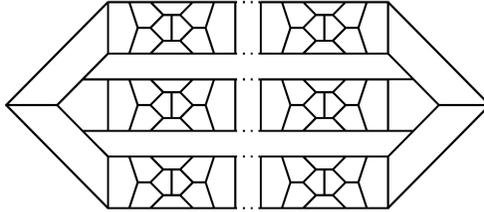


Figure 40: Bars and joints $(3, 5)$ with exceptional faces of size $4 \pmod{10}$

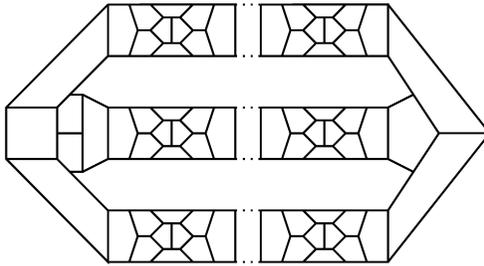


Figure 41: Bars and joints $(3, 5)$ with exceptional faces of size $5 \pmod{10}$

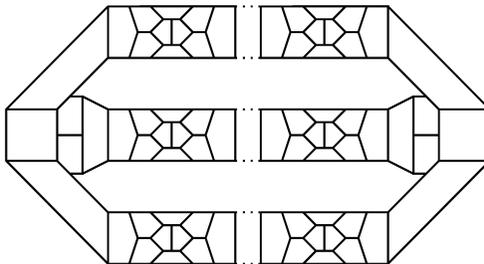


Figure 42: Bars and joints $(3, 5)$ with exceptional faces of size $6 \pmod{10}$

Proposition 4.3. *There exist balanced 3-nearly Platonic graphs of type $(3, 5)$ with connectivity 3 and exceptional faces of size m for every $m \equiv 4, 5, 6 \pmod{10}$, except when $m = 5$.*

Type $R_{4,2}(5, 3)$

The graph shown in Figure 43 obtained by splitting an edge of the dodecahedron will be denoted $R_{4,2}(5, 3)$. We also use joints $J(5, 3)$ on 12 vertices and $J'(5, 3)$ on 6 vertices arising from the dodecahedron, shown in Figures 44 and 45.

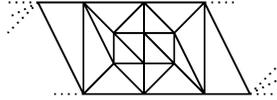


Figure 43: Bar $R_{4,2}(5, 3)$

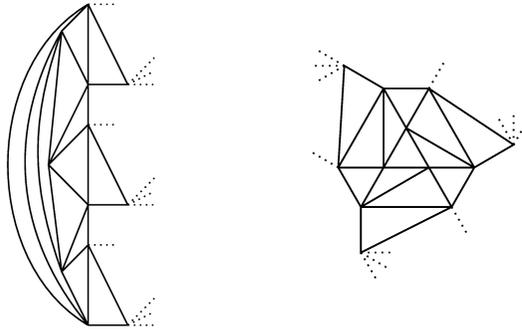


Figure 44: Two drawings of joint $J(5, 3)$

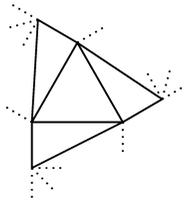


Figure 45: Joint $J'(5, 3)$

Amalgamating bars $R_{4,2}(5, 3)$ with joints $J(5, 3)$ and $J'(5, 3)$ we obtain 3-nearly Platonic graphs of type $(5, 3)$ with the exceptional faces of size 2, 3 and 4 (mod 6) shown in Figures 46, 47, and 48, respectively. The graph in Figure 5 has the exceptional face of size 4 and consists of two joints $J(5, 3)$.

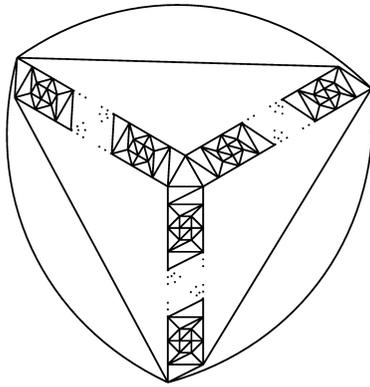


Figure 46: Bars and joints of type $(5, 3)$, exceptional faces of size $2 \pmod 6$

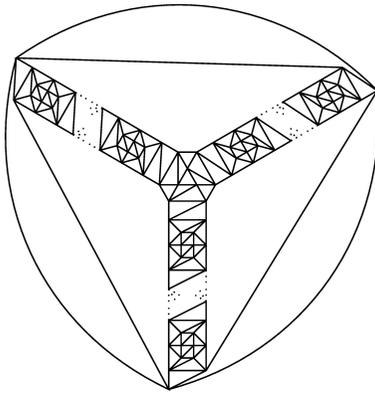


Figure 47: Bars and joints of type $(5, 3)$, exceptional faces of size $3 \pmod 6$

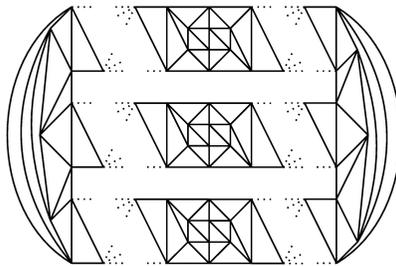


Figure 48: Bars and joints of type $(5, 3)$, exceptional faces of size $4 \pmod 6$

Proposition 4.4. *There exist balanced 3-nearly Platonic graphs of type $(5, 3)$ with connectivity 4 and exceptional faces of size m for every $m \equiv 2, 3, 4 \pmod{6}$, $m \geq 4$.*

5 Mixed constructions

In this section, we combine the techniques from the previous two sections to obtain more families of balanced 3-nearly Platonic graphs.

Mixed type $(3, 4)$

For graphs of type $(3, 4)$ shown in Figure 49, both the bars and joints are rectangles C_4 . The pseudoblocks are $B_{2,2}(3, 4)$ combined with $B_{1,1}$. The exceptional faces have size $m \equiv 4 \pmod{8}$.

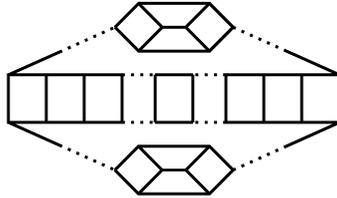


Figure 49: Mixed construction for type $(3, 4)$

Proposition 5.1. *There exist balanced 3-nearly Platonic graphs of type $(3, 4)$ with connectivity 2 and exceptional faces of size m for every $m \equiv 4 \pmod{8}$, $m \geq 12$.*

Mixed type $(3, 5)$

For graphs of type $(3, 5)$, we use bars $R_{2,2}(3, 5)$, pseudoblocks $B_{2,2}(3, 5)$ combined with $B_{1,1}$ and joints $J''(3, 5)$ and C_5 , shown in Figure 50. Notice that the joints arise from the dodecahedron by removing an edge and splitting two other edges. The exceptional faces have size $m \equiv 5 \pmod{10}$. The graph is shown in Figure 50.

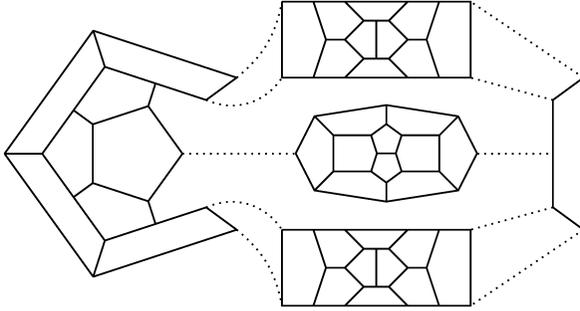


Figure 50: Mixed construction for type $(3, 5)$ with joints $J''(3, 5)$ and C_5

Proposition 5.2. *There exist balanced 3-nearly Platonic graphs of type $(3, 5)$ with connectivity 2 and exceptional faces of size m for every $m \equiv 5 \pmod{10}, m \geq 15$.*

Mixed type $(4, 3)$

For graphs of type $(4, 3)$, we use bars $R_{2,2}(4, 3)$ arising from the octahedron by splitting an edge, pseudoblocks $B_{2,2}(4, 3)$, $B_{3,3}(4, 3)$ combined with $B_{1,1}$ and joints $J(4, 3)$ and $J'(4, 3)$ (shown in Figure 51). The exceptional faces have size $m \equiv 3 \pmod{6}$. The graph is shown in Figure 52.

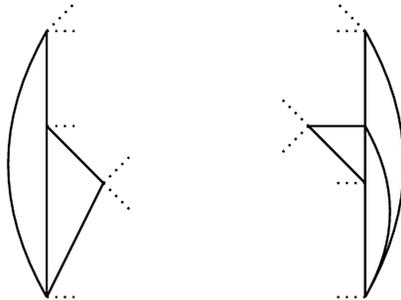


Figure 51: Joints $J(4, 3)$ and $J'(4, 3)$

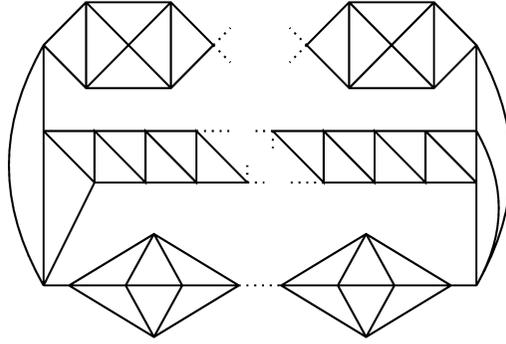


Figure 52: Mixed construction for type $(4, 3)$

Proposition 5.3. *There exist balanced 3-nearly Platonic graphs of type $(4, 3)$ with connectivity 2 and exceptional faces of size m for every $m \equiv 3 \pmod{6}$, $m \geq 9$.*

Mixed type $(5, 3)$

For graphs of type $(5, 3)$, we use bars $R_{3,3}(5, 3)$ arising from the icosahedron by splitting an edge, pseudoblocks $B_{3,2}(5, 3)$, and joints $J''(5, 3)$ and $J'''(5, 3)$ (shown in Figure 53). The exceptional faces have size $m \equiv 3 \pmod{6}$. The graph is shown in Figure 54.

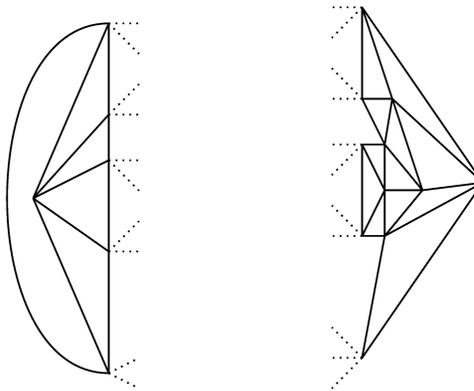


Figure 53: Joints $J''(5, 3)$ and $J'''(5, 3)$

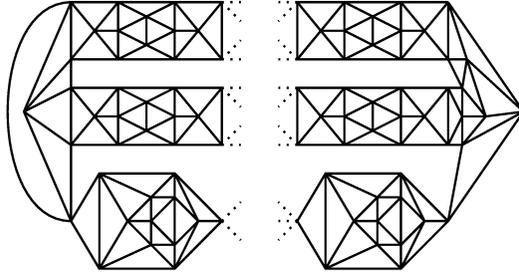


Figure 54: Mixed construction for type $(5, 3)$

Proposition 5.4. *There exist balanced 3-nearly Platonic graphs of type $(5, 3)$ with connectivity 2 and exceptional faces of size m for every $m \equiv 3 \pmod{6}$, $m \geq 9$.*

6 Main result

We summarize our constructions in our main result as follows.

Theorem 6.1. *There exist balanced 3-nearly Platonic graphs with exceptional faces of size m of type*

- $(3, 3)$ for every $m \equiv 2, 3, 4 \pmod{6}$, $m \geq 4$,
- $(3, 4)$ for every $m \equiv 2, 4, 6 \pmod{8}$, $m \geq 6$,
- $(3, 5)$ for every $m \equiv 2, 4, 5, 6, 8 \pmod{10}$, $m \neq 2, 5$,
- $(4, 3)$ for every $m \equiv 2, 3, 4 \pmod{6}$, $m \geq 4$,
- $(5, 3)$ for every $m \equiv 2, 3, 4 \pmod{6}$, $m \geq 4$.

Notice that Deza, Dutour Sikirič, and Shtogrin [2] published the above list as a necessary and sufficient condition. Because with our methods we are unable to easily verify the necessity, we present their result separately.

Theorem 6.2 ([2]). *The above conditions are both necessary and sufficient.*

References

- [1] M. Deza, M. Dutour Sikirić, *Geometry of Chemical Graphs: Polycycles and Two-Faced Maps*. Cambridge University Press, 2008.
- [2] M. Deza, M. Dutour Sikirić, and M. Shtogrin, *Fullerene-like spheres with faces of negative curvature*, Chapter 13 in “Diamond and Related Nanostructures” M.V. Diudea and C.L. Nagy ed., Springer, 2013.
- [3] D. Froncek, J. Qiu, A note on non-existence of nearly Platonic graphs with connectivity one, *unpublished manuscript*.
- [4] W. Keith, D. Froncek, and D. Kreher, A note on nearly Platonic graphs, *Australas. J. Combin.*, **70** (2018), 86–103.
- [5] W. Keith, D. Froncek, and D. Kreher, Corrigendum to: A note on nearly Platonic graphs, *Australas. J. Combin.*, **72** (2018), 163.