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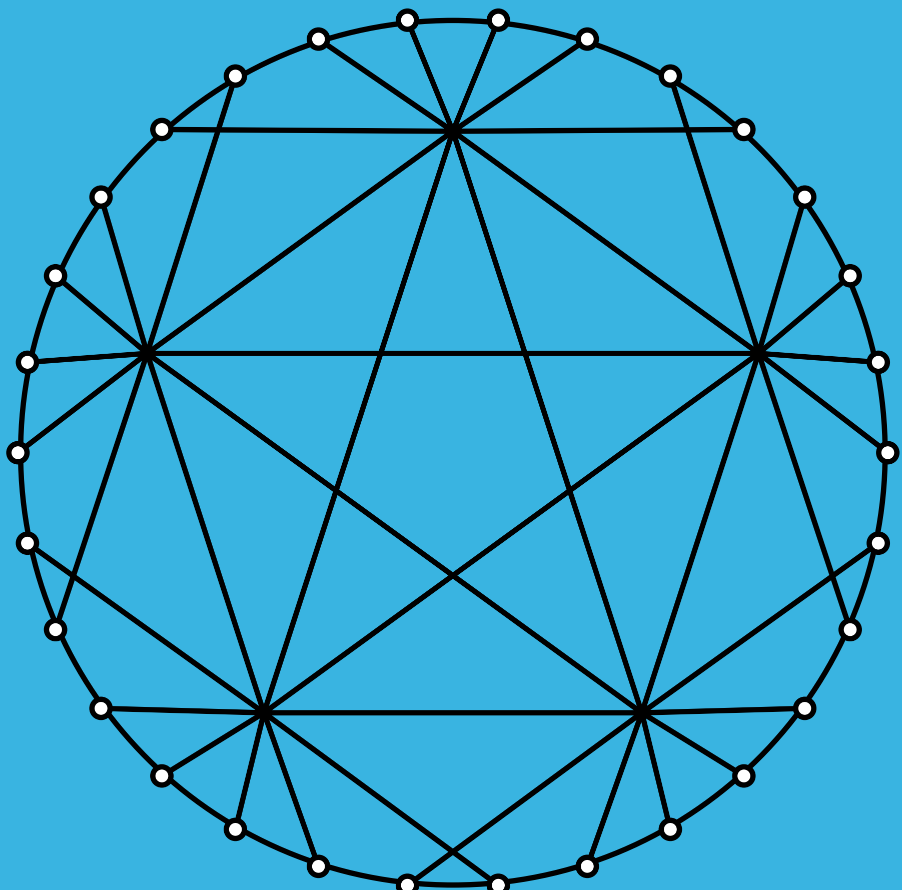

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On infection in hypergraphs

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Abstract: Eroh et al. proved that the zero-forcing number of a graph G, denoted by $Z(G)$, satisfies $Z(G) \leqslant 2Z(L(G))$ (where $L(G)$ denotes the line graph of G); and they conjectured that $Z(G) \leqslant Z(L(G))$. This conjecture was proven recently.

Hypergraph infection was introduced by Bergen et al. as a natural generalization of zero forcing in graphs. In [5], they determine the so-called *infection number* of a hypergraph H, denoted by $I(H)$, for several families of hypergraphs, and give bounds for $I(H)$ of many other families of hypergraphs. In particular, they show that $I(H) \leqslant kZ(L(H))$ for any reduced k-uniform hypergraph H with no isolated vertices (where $L(H)$ denotes the line graph of H).

In this note we improve this bound significantly by giving an algorithm to prove that $I(H) \leqslant Z(L(H))$ for any reduced hypergraph H with no isolated vertices. We note that this result not only completely eliminates the factor k, but it also applies to hypergraphs that are not uniform. We also show that the bound is tight.

1 Introduction

To describe the notion of zero forcing, we colour the vertices of a graph G with the colours black and white. Then a black vertex can *force* a

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white vertex to change its colour to black according to the following colour-changing rule: if v is black and w is a white neighbour of v, then v can force w to become black only if w is the only white vertex that is adjacent to v. Then a set of initially black vertices in G is a *zero-forcing set* for G if by applying the colour-changing rule repeatedly, all white vertices of G can eventually be forced to black. The *zero-forcing number* of G, denoted by $Z(G)$, is the size of a smallest zero-forcing set for G.

The notion of zero forcing was first introduced in [1] with the motivation to bound the minimum rank and therefore the maximum nullity of a graph. Along with the exact value of the zero-forcing number for a variety of graph classes it was indeed shown that the zero-forcing number of a graph is bounded below by the maximum nullity of the graph (see [1]).

This useful property generated much interest in the study of zero forcing since then. For example, it is now known that computing the zero-forcing number is an $\mathcal{NP}$-hard problem (see Theorem 3.1 in [10] and Theorems 6.3, 6.5, Corollary 6.6 in [16]). For additional background and results on zero forcing, also see [2], [3], [6], [7], [11], [12], [13], [14] and [15].

The *line graph of* G, denoted by $L(G)$, is the graph that has a vertex for each edge of G, with the property that two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. The zero-forcing number is known to be not minor-monotone [4]. That is, the operations of edge-deletion and edge-contraction may decrease, increase or not affect the zero-forcing number of a graph. This suggests that "small" graphs can potentially have a greater zero-forcing number than "large" graphs, which gives us an initial motivation to compare the zero-forcing number of a graph to the zero-forcing number of its line graph.

In [8] it is proven that

(I) $Z(G) \leqslant 2Z(L(G))$ for any non-trivial graph G.

They also prove that $Z(G) \leqslant Z(L(G))$ when G is a tree, or when G contains a Hamiltonian path and has a certain number of edges. Moreover they conjecture that $Z(G) \leqslant Z(L(G))$ for any non-trivial graph G. This conjecture was recently proven in [9]. In the current paper we extend this result to hypergraphs.

A *hypergraph* H is a pair $H = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a set of elements, called *vertices*, and $\mathcal{E}$ is a set of non-empty subsets of $\mathcal{V}$ called *edges* (or *hyperedges* whenever the context is not clear). Drawing terminology from the setting of graphs, we say that a vertex v is *adjacent* to a vertex w if v and w are contained together in some edge. Similarly two edges are said to be *adjacent* if their intersection is non-empty. An edge E is said to be *incident* with a vertex v if $v \in E$. A hypergraph where all edges are incident with exactly k vertices is called *k-uniform*. Consequently, a graph can be thought of as a 2-uniform hypergraph.

Infection in a hypergraph H is defined as follows: Initially each vertex of H is either *infected* or *uninfected*. Suppose A is a non-empty set consisting of infected vertices only. Then A can infect all vertices in an edge E (we also say that "A infects E") if

(i) $A \subsetneq E$, and

(ii) there exists no uninfected vertex $v \notin E$ with the property that $A \cup \{v\} \subseteq E'$ for some edge E'.

A set of initially infected vertices in a hypergraph H is an *infection set* if repeated application of the infection rule can eventually infect all initially uninfected vertices in the hypergraph. The *infection number* of H is then the size of a smallest infection set for H, and is denoted by $I(H)$.

Hypergraph infection was first defined in [5] in order to generalize the notion of zero forcing in graphs to the setting of hypergraphs. In [5] Bergen et al. determine the infection number for several families of hypergraphs and also give bounds on the infection number for many other families of hypergraphs. Many of their results justify that infection in hypergraphs as is defined is the natural extension of zero forcing to hypergraphs.

A hypergraph H is said to be *reduced* if no edge of H is a subset of another edge of H. It follows that in a reduced hypergraph, for each pair of distinct edges E_i and E_j, there exists a vertex in $E_i \setminus E_j$. In this note we prefer working with reduced hypergraphs, as they form a better basis for naturally extending results on zero forcing in graphs to the setting of infection in hypergraphs, as is also suggested in Section 2 of [5].

The line graph $L(H)$ of a hypergraph H is defined similarly to the line graph of a graph: Each edge of H corresponds to a vertex in $L(H)$, where two vertices are joined with an edge if the corresponding edges in H have a non-empty intersection. A hypergraph is said to be *connected* if it does not have any isolated vertices and its line graph is connected. A *connected* *component* of a hypergraph H is either an isolated vertex in H, or a hypergraph whose edge set is a subset of the edge set of H and which corresponds to a connected component of $L(H)$. In [5] it is shown that a bound similar to (I) can be achieved for reduced k-uniform hypergraphs for any k. More precisely, they show that

(II) $I(H) \leqslant kZ(L(H))$ for any reduced k-uniform hypergraph H with no isolated vertices.

Thinking of graphs as 2-uniform reduced hypergraphs, one sees that (II) is indeed a natural extension of (I) in the hypergraph setting. In this note, however, we improve this bound significantly by showing that $I(H) \leqslant Z(L(H))$ for any reduced hypergraph H with no isolated vertices, thereby affirming the prediction in [5] that a bound better than (II) exists. We note that the main result of this paper not only eliminates the factor k, but it also applies to hypergraphs that are not uniform. Moreover this result is tight (see Section 3).

2 Preliminaries

In this section we make some observations and set some notation that will be used in the proof of Theorem 3.2. We do so by adopting the terminology in [9] whenever it is useful.

First we observe that at each step in a zero-forcing process exactly one black vertex v forces exactly one white vertex w to change its colour to black, and there is no choice for which vertex w can be forced by v, as w is the only white neighbour of v at this step. Let Z be a zero-forcing set. Then the aforementioned observation suggests that we can order the vertices of a graph G in terms of $|Z|$ oriented paths $\mathcal{P}_i$, called *zero-forcing chains*, such that for each $1 \leqslant i \leqslant |Z|$, $w_{i,1}$ is in $\mathcal{P}_i$. Such a vertex $w_{i,1}$ is necessarily coloured black initially; moreover any vertex that is initially coloured black is of the form $w_{i,1}$ for some $1 \leqslant i \leqslant |Z|$. If a vertex is never forced by another vertex and itself does not force any other vertex, then it corresponds to a zero-forcing path having just one vertex. A vertex v forcing a vertex w is denoted by $v \to w$. A generic list of zero-forcing chains looks as follows:

$$\begin{aligned}
\mathcal{P}_1 =& w_{1,1} \to w_{1,2} \to \cdots \to w_{1,f(1)},\\
\mathcal{P}_2 =& w_{2,1} \to w_{2,2} \to \cdots \to w_{2,f(2)},\\
&\vdots\\
\mathcal{P}_{|Z|} =& w_{|Z|,1} \to w_{|Z|,2} \to \cdots \to w_{|Z|,f(|Z|)},
\end{aligned} \tag{1}$$

where f is a function from $\{1, \dots, |Z|\}$ to $\mathbb{Z}^+$.

We say that $w_{i,j}$ is the *active* vertex at some step of the zero-forcing process if at that step $w_{i,j}$ forces $w_{i,j+1}$. A vertex can force at most one other vertex, and after it does so we call it *used*. Similarly when the head $w_{i,1}$ of a zero-forcing chain $\mathcal{P}_i$ is used, we may mark $\mathcal{P}_i$ as *used* as well. A crucial observation is that if $w_{i,j}$ is the active vertex at some step of the zero-forcing process, then $w_{i,j+1}$ is the only white neighbour of $w_{i,j}$ at that step. This means that if at this step $w_{i,j}$ is adjacent to a vertex w distinct than $w_{i,j+1}$, then w must be black.

For infection in hypergraphs we note that the situation is somewhat different. At each step of the infection process, the infection is performed by a subset of the vertices of some edge E, and this results in all vertices in E being infected (we say "E is infected" or "E is entirely infected"). Nevertheless, we can draw some parallelism to zero forcing in graphs by observing that at the step when a set $A \subsetneq E$ infects the edge E, E is the only edge with the property that it has some uninfected vertices and also contains A as a subset. That is, at this step all sets other than E which contain A as a subset have only infected vertices.

3 Main result

In this section we prove our main result (Theorem 3.2), that $I(H) \leqslant Z(L(H))$ for any reduced hypergraph H with no isolated vertices.

The main strategy in proving Theorem 3.2 is to show that to each zero-forcing set Z of the line graph of a hypergraph H, there is an infection set of H of size at most $|Z|$. In order to obtain this one-to-one correspondence, an algorithm is given that defines certain infection steps in H corresponding to each zero-forcing step in $L(H)$.

Then we show that the bound given in Theorem 3.2 is tight, and we finish the section by illustrating the algorithm that is given in the proof of Theorem 3.2 with a concrete example.

Lemma 3.1. *For any hypergraph H consisting of $k \geqslant 1$ components $H_1, H_2, \ldots, H_k$,* $\quad I(H) = I(H_1) + I(H_2) + \ldots + I(H_k)$.

Proof: This is immediate by the definition of the infection number (also see, for example, Proposition 1.3 in [5]).

We introduce a vertex adding/switching procedure that will be useful in finding an infection set S of small size for a given hypergraph H.

Vertex adding/switching: Let H be a hypergraph and Z be a zero-forcing set for $L(H)$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_{|Z|}$ be the zero-forcing chains for a zero-forcing process starting with Z in $L(H)$ where the zero-forcing chains are ordered so that chains with just one vertex are all at the end of this collection. Label the vertices in $L(H)$ as in (1) of Section 2, and denote by $E_{i,j}$ the edge in H corresponding to the vertex $w_{i,j}$ in $L(H)$. For edges $E_{m,n}$ and $E_{k,l}$ in H, let ${}_{m,n}X_{k,l} = E_{m,n} \cap E_{k,l}$, and denote by ${}_{m,n}X$ the multiset of intersections $\{{}_{m,n}X_{k,l} \mid E_{k,l} \neq E_{m,n}$ is adjacent to $E_{m,n}$, and $E_{k,l}$ is currently not entirely infected$\}$. Note that each edge $E_{k,l}$ in H adjacent to $E_{m,n}$ corresponds to a vertex $w_{k,l}$ in $L(H)$ that is adjacent to $w_{m,n}$.

Suppose that $E_{1,1}$ is infected. Consider the set of uninfected edges adjacent to $E_{1,1}$. These are the edges that yield the elements of ${}_{1,1}X$. Rename these edges as $E^*_{1,1}, \ldots, E^*_{1,f(1)}, E^*_{2,1}, \ldots, E^*_{2,f(2)}, \ldots, E^*_{n,1}, \ldots, E^*_{n,f(n)}$ where $f : \{1, \ldots, n\} \to \mathbb{Z}^+$ is a function that takes on positive integer values, $E^*_{p,q}$ and $E^*_{r,s}$ have the same intersection with $E_{1,1}$ if and only if $p = r$, and if $(E^*_{i,1} \cap E_{1,1}) \supseteq (E^*_{j,1} \cap E_{1,1})$ then $i \leqslant j$.

Consider $E^*_{1,1}, \ldots, E^*_{1,f(1)}$. Add to a set S a vertex $v^*_{(1,1)\setminus(1,i)}$ from $E^*_{1,1} \setminus E^*_{1,i}$ for each $i \in \{2, \ldots, f(1)\}$ (note that $f(1) - 1$ vertices are being added). Do not delete potential multiple appearances of elements in S. At this point the choice of these vertices guarantees that $(E^*_{1,1} \cap E_{1,1}) \cup (S \cap E^*_{1,1})$ infects $E^*_{1,1}$. For all $E^*_{1,x}$ $(x \neq 1)$ such that $(E^*_{1,x} \cap E^*_{1,1}) \not\subseteq (E^*_{1,v} \cap E^*_{1,1})$ for any $v \notin \{1, x\}$, $E^*_{1,x} \cap E^*_{1,1}$ infects $E^*_{1,x}$. Now consider an uninfected edge $E^*_{1,w}$ $(w \geqslant 2)$ with the property that $E^*_{1,w} \cap E^*_{1,1}$ is not a proper subset of $E^*_{1,y} \cap E^*_{1,1}$ for any uninfected edge $E^*_{1,y}$. If no subset of the currently infected vertices in H can infect $E^*_{1,w}$, this must be because there is an uninfected edge $E^*_{1,t}$

such that $E^*_{1,w} \cap E^*_{1,1} = E^*_{1,t} \cap E^*_{1,1}$. For each such t exclude $v^*_{(1,1)\setminus(1,t)}$ from S, and instead include in S a vertex $v^*_{(1,w)\setminus(1,t)}$ from $E^*_{1,w} \setminus E^*_{1,t}$. Now $\bigcup_t \{v^*_{(1,w)\setminus(1,t)}\} \cup (E^*_{1,w} \cap E_{1,1}) \cup (E^*_{1,w} \cap E^*_{1,1})$ infects $E^*_{1,w}$ where t is such that $E^*_{1,w} \cap E^*_{1,1} = E^*_{1,t} \cap E^*_{1,1}$. It is crucial to note that this vertex-switching argument does not change the number of vertices included in S, and that the exclusion of $v^*_{(1,1)\setminus(1,t)}$ from S does not harm the infection step of $E^*_{1,1}$ – this is because $v^*_{(1,1)\setminus(1,w)} \in S$ plays the same role as $v^*_{(1,1)\setminus(1,t)}$ towards infecting $E^*_{1,1}$ as a consequence of the observation that $E^*_{1,1} \cap E^*_{1,w} = E^*_{1,1} \cap E^*_{1,t}$ in this scenario. Possibly some more uninfected edges from the list $E^*_{1,1}, \ldots, E^*_{1,f(1)}$ can be infected now by their respective intersections with $E^*_{1,1}$. If there are still some uninfected edges left in the list $E^*_{1,1}, \ldots, E^*_{1,f(1)}$, we apply the aforementioned vertex-switching argument to an appropriately chosen uninfected edge (like $E^*_{1,w}$ previously) and proceed similarly until there are no more uninfected edges in the list $E^*_{1,1}, \ldots, E^*_{1,f(1)}$.

Next consider $E^*_{2,1}, \ldots, E^*_{2,f(2)}$. First add to S a vertex $v^*_{(2,1)\setminus(2,i)}$ from $E^*_{2,1} \setminus E^*_{2,i}$ for each $i \in \{2, \ldots, f(2)\}$ (note that $f(2) - 1$ vertices are being added). Do not delete possible multiple appearances of elements in S. Using a similar argument whenever necessary, we can show that all edges in the list $E^*_{2,1}, \ldots, E^*_{2,f(2)}$ can be infected while having added $f(2) - 1$ vertices to S. In this respect, it is critical that the ordering $E^*_{1,1}, \ldots, E^*_{1,f(1)}, E^*_{2,1}, \ldots, E^*_{2,f(2)}$, $\ldots, E^*_{n,1}, \ldots, E^*_{n,f(n)}$ was imposed to have the property that if $(E^*_{i,1} \cap E_{1,1}) \supseteq (E^*_{j,1} \cap E_{1,1})$ then $i \leqslant j$. A repeated application of this procedure guarantees that the addition of at most $\sum_{i=1}^{n}(f(i) - 1)$ vertices to S is enough to infect all edges adjacent to $E_{1,1}$.

Initial Procedure given below shows the steps towards deciding which vertices to include in an initially empty set S (that will eventually become an infection set for H) to guarantee that $E_{1,1}$ in H is infected. In the proof of Theorem 3.2, this procedure will be called upon when necessary to describe the strategy for adding vertices to the set S at different steps in the zero-forcing process on $L(H)$.

Initial Procedure: Use the notation as in the vertex adding/switching procedure, and suppose that all elements in ${}_{1,1}X$ are mutually disjoint. For each element ${}_{1,1}X_{k,l}$ contained in the set ${}_{1,1}X$ take exactly one vertex $v_{(1,1)\setminus(k,l)} \in E_{1,1} \setminus E_{k,l}$ (observe that such a vertex exists for each edge $E_{k,l}$ since H is a reduced hypergraph). Let ${}_{1,1}W$ be the set consisting of these vertices. (If ${}_{1,1}W$ is initially a multiset, then reduce it to a set by deleting multiple copies of vertices in ${}_{1,1}W$.) Include in S the vertices of ${}_{1,1}W$. Then the choice of the elements in ${}_{1,1}W$ guarantees that ${}_{1,1}W \subsetneq E_{1,1}$ infects $E_{1,1}$ (or ${}_{1,1}W = E_{1,1}$). For each ${}_{1,1}X_{k,l}$ ($l \in \{1,2\}$) corresponding to a vertex $v_{(1,1)\setminus(k,l)} \in E_{1,1} \setminus E_{k,l}$ in ${}_{1,1}W$ mark the chain $\mathcal{P}_k$ as used. Remark that all elements in ${}_{1,1}X$ are of the form ${}_{1,1}X_{k,1}$ for some distinct $k \neq 1$, except for ${}_{1,1}X_{1,2}$. At this point, the assumption that all elements in ${}_{1,1}X$ are mutually disjoint guarantees that ${}_{1,1}X_{k,l} = E_{1,1} \cap E_{k,l}$ infects $E_{k,l}$ for each $E_{k,l}$ where $E_{1,1} \cap E_{k,l} = {}_{1,1}X_{k,l} \in {}_{1,1}X$ (i.e. for each $E_{k,l}$ that is adjacent to $E_{1,1}$ in H).

Now suppose that not all elements in ${}_{1,1}X$ are mutually disjoint. Exclude from the multiset ${}_{1,1}X$ any element ${}_{1,1}X_{s,t}$ with the property that ${}_{1,1}X_{s,t} \subsetneq {}_{1,1}X_{s',t'}$ for some ${}_{1,1}X_{s',t'}$ in ${}_{1,1}X$. Also exclude from ${}_{1,1}X$ all but one of the copies of each element so that the resulting collection is a set. Let this resulting set of intersections be ${}_{1,1}X'$. For each element ${}_{1,1}X_{k,l}$ contained in the set ${}_{1,1}X'$ take exactly one vertex $v_{(1,1)\setminus(k,l)} \in E_{1,1} \setminus E_{k,l}$ (observe that such a vertex exists for each such edge $E_{k,l}$ since H is a reduced hypergraph). Include these vertices in S (include each such vertex only if it is not already in S, i.e., at each step keep S as a set). Then the choice of the elements in S guarantees that $S \subsetneq E_{1,1}$ infects $E_{1,1}$ (or possibly $S = E_{1,1}$). Remark that all elements in ${}_{1,1}X'$ are of the form ${}_{1,1}X_{r,1}$ for some distinct $r \neq 1$, except possibly for ${}_{1,1}X_{1,2}$ which may or may not be in ${}_{1,1}X'$.

Theorem 3.2. *If H is a reduced hypergraph with no isolated vertices, then $I(H) \leqslant Z(L(H))$.*

Proof: First note that an isolated vertex in a hypergraph increases the infection number of the hypergraph by one, however it has no effect on the zero-forcing number of the line graph. By Lemma 3.1, we may assume that H is connected.

Let Z be a zero-forcing set for $L(H)$. Let $\mathcal{P}_1, \ldots, \mathcal{P}_{|Z|}$ be the zero-forcing chains for a zero-forcing process starting with Z in $L(H)$. Label vertices in $L(H)$ according to the notation of (1) in Section 2. We order the collection of zero-forcing chains so that no chain with a single vertex precedes a chain with multiple vertices.

We will present a strategy for choosing an infection set in H of size at most $|Z|$. At each step in the zero-forcing process on $L(H)$ we describe which vertices are added to a set S that will eventually become an infection set for H. For each of the $|Z|$ zero-forcing chains, at most one vertex in H will be added to S.

Denote by $E_{i,j}$ the edge in H that corresponds to the vertex $w_{i,j}$ in $L(H)$.

For edges $E_{m,n}$ and $E_{k,l}$ in H, let ${}_{m,n}X_{k,l} = E_{m,n} \cap E_{k,l}$, and ${}_{m,n}X$ be the multiset of intersections $\{{}_{m,n}X_{k,l} \mid E_{k,l} \neq E_{m,n}$ is adjacent to $E_{m,n}$, and $E_{k,l}$ is currently not entirely infected$\}$. An easy observation is that each edge $E_{k,l}$ in H adjacent to $E_{m,n}$ corresponds to a vertex $w_{k,l}$ in $L(H)$ adjacent to $w_{m,n}$.

In the initial step of the zero-forcing process on $L(H)$ $w_{1,1}$ is the active vertex; so $w_{1,1}$ forces $w_{1,2}$. At this step $w_{1,1}$ in $L(H)$ is black, as are all neighbours of $w_{1,1}$, except for $w_{1,2}$. In H consider the edges $E_{1,1}$ and $E_{1,2}$ corresponding to the vertices $w_{1,1}$ and $w_{1,2}$ in $L(H)$, respectively. Note that $E_{1,1} \cap E_{1,2} \neq \emptyset$.

Apply Initial Procedure to ${}_{1,1}X$, and for each ${}_{1,1}X_{k,l}$ ($l \in \{1, 2\}$) in ${}_{1,1}X$, mark the chain $\mathcal{P}_k$ as used. An easy counting argument shows that the number of vertices included in S is less than or equal to the number of chains that are being marked as used.

Now we apply the vertex adding/switching procedure to see that $w_{1,1}$ forcing $w_{1,2}$ in $L(H)$ corresponds to some infection steps in H in which no uninfected vertices are left incident with $E_{1,1}$ or with the edges adjacent to it. Towards showing that $I(H) \leqslant Z(L(H))$, it is crucial that in this process the number of vertices included in S is less than or equal to the number of zero-forcing chains that are marked as used, which is equal to the number of initially black vertices in $L(H)$ in the closed neighbourhood of the vertex $w_{1,1}$ (where in a graph the *closed neighbourhood* of a vertex v is defined as the set consisting of v together with all vertices that are adjacent to v).

We move on to a generic step of the zero-forcing process. Suppose that at some point in the zero-forcing process on $L(H)$ $w_{i,j}$ is the active vertex; so $w_{i,j}$ forces $w_{i,j+1}$. At this step $w_{i,j}$ in $L(H)$ is black, as are all neighbours of $w_{i,j}$, except for $w_{i,j+1}$ (some of these neighbours might be initially black vertices of $L(H)$, while others may have been forced at some earlier step). In H consider the edge $E_{i,j}$ and all the edges adjacent to it.

We consider two cases: (i) $w_{i,j}$ is not initially black, (ii) $w_{i,j}$ is an initially black vertex.

In case (i) $w_{i,j}$ must have been forced at some earlier step. Therefore, in H the edge $E_{i,j}$ must have been completely infected at the corresponding step.

In both cases (i) and (ii), the neighbours of $w_{i,j}$ in $L(H)$ which are not initially black but have been forced at an earlier step correspond to edges adjacent with $E_{i,j}$ in H that are entirely infected. The initially black neighbours of $w_{i,j}$ fall into two categories: (1) those that have already been considered at some earlier step in the zero-forcing process of $L(H)$ – these vertices are heads of zero-forcing chains in $L(H)$ that have been marked as used, and (2) those that have not been considered in the zero-forcing process of $L(H)$ yet – these vertices are heads of zero-forcing chains in $L(H)$ that have not been marked as used. The vertices of type (1) correspond to edges in H that are entirely infected at some earlier step. It follows that the only edges in H adjacent to $E_{i,j}$ that may have not been entirely infected are $E_{i,j+1}$ and those edges that correspond to initially black neighbours of $w_{i,j}$ of type (2). Apply Initial Procedure if necessary (that is, if $E_{i,j}$ is not yet infected) and the vertex adding/switching procedure with the roles of $w_{1,1}$ and $w_{1,2}$ being replaced by $w_{i,j}$ and $w_{i,j+1}$ to choose a set of at most x vertices that we include in S to guarantee that $E_{i,j}$ and all edges adjacent with $E_{i,j}$ are entirely infected, where x is the number of initially black vertices in the closed neighbourhood of $w_{i,j}$ in $L(H)$ that are heads of zero-forcing chains in $L(H)$ that have not been marked as used. If $w_{i,j} = w_{i,1}$ is an initially black vertex, for each ${}_{i,j}X_{k,l}$ $(l \in \{1, 2\})$ in ${}_{i,j}X$, mark the chain $\mathcal{P}_k$ as used. If, on the other hand, $w_{i,j}$ is not an initially black vertex, then for each ${}_{i,j}X_{k,1}$ in ${}_{i,j}X$, mark the chain $\mathcal{P}_k$ as used.

We showed that $w_{i,j}$ forcing $w_{i,j+1}$ in $L(H)$ corresponds to some infection steps in H in which $E_{i,j}$ and all edges adjacent with it are eventually infected. A key observation is that in both cases (i) and (ii) the steps that collectively lead to the infection of $E_{i,j}$ and all edges adjacent with it involve the inclusion of a number of vertices in S that is less than or equal to the number of initially black vertices in $L(H)$ in the closed neighbourhood of the vertex $w_{i,j}$ that are heads of zero-forcing chains that had not been marked as used.

We repeat this generic step for all zero-forcing process steps on $L(H)$. If there are any unused chains left, then each of these chains must consist of a single vertex, which is initially black. If $w_{t,1}$ is such a vertex of $L(H)$ $(1 \leqslant t \leqslant |Z|)$, then at this step in $L(H)$ all neighbours of $w_{t,1}$ are black; and for any edge E in H that is adjacent to $E_{t,1}$ and contains some uninfected vertices, the vertex in $L(H)$ corresponding to E is from an unused path of length 1. Suppose that $E_{t,1}$ contains some uninfected vertices. Apply Initial Procedure and the vertex adding/switching procedure as necessary – this time by granting the role of $w_{1,1}$ in Initial Procedure to $w_{t,1}$ and discarding all arguments pertaining to $w_{1,2}$ (since $w_{t,1}$ is not forcing any vertex, unlike $w_{1,1}$ forcing $w_{1,2}$ in Initial Procedure). Doing so results in the inclusion of a set of at most y vertices in S which guarantee that $E_{t,1}$ and all edges in H adjacent with $E_{t,1}$ are entirely infected, where y is the number of initially black vertices in the closed neighbourhood of $w_{t,1}$ in $L(H)$ that are heads of zero-forcing chains in $L(H)$ that have not been marked as used. For each ${}_{t,1}X_{u,1}$ in ${}_{t,1}X$ mark the chain $\mathcal{P}_u$ as used. Also mark the chain $\mathcal{P}_t$ as used.

Continue until all single-vertex zero-forcing chains (and therefore all vertices of $L(H)$) have been considered. At this point each edge in H is entirely infected. Delete multiples copies of vertices (if there are any) from S. Thus this procedure produces an infection set S for H of order at most $|Z|$. $\square$

Now we show that Theorem 3.2 yields a tight bound. Lemma 3.5 of [5] indicates that for a reduced hypergraph H that is a flower with p petals, $I(H) = p - 1$. It is easy to see that the line graph of H is the complete graph K_p, and therefore $Z(K_p) = p - 1$.

We conclude with an example that illustrates some of the ideas of the proof of Theorem 3.2.

Example 3.3. A hypergraph H on the vertex set $\{1, 2, \ldots, 25\}$ is given with the following edges:

$$\begin{aligned}
E_{1,1} &= \{1,2,3\}, & E_{1,2} &= \{1,2,4,8,9\},\\
E_{2,1} &= \{1,2,7,13\}, & E_{2,2} &= \{6,7,14\},\\
E_{2,3} &= \{6,9,10\}, & E_{2,4} &= \{9,10,11\},\\
E_{2,5} &= \{8,11\}, & E_{3,1} &= \{1,2,5,12\},\\
E_{4,1} &= \{2,3,4,8,9,15,24\}, & E_{5,1} &= \{7,12,25\},\\
E_{6,1} &= \{6,12,14\}, & E_{7,1} &= \{15,16,17,20,23\},\\
E_{8,1} &= \{15,16,17,18,21,22\}, & E_{9,1} &= \{15,16,17,19,21\}.
\end{aligned}$$

A zero-forcing set for $L(H)$ can be given as $Z = \{v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}, v_{5,1}, v_{6,1}, v_{7,1}, v_{8,1}, v_{9,1}\}$ with the following zero-forcing chains:

$$\begin{aligned}
\mathcal{P}_1 &= w_{1,1} \to w_{1,2},\\
\mathcal{P}_2 &= w_{2,1} \to w_{2,2} \to w_{2,3} \to w_{2,4} \to w_{2,5},\\
\mathcal{P}_3 &= w_{3,1},\\
\mathcal{P}_4 &= w_{4,1},\\
\mathcal{P}_5 &= w_{5,1},\\
\mathcal{P}_6 &= w_{6,1},\\
\mathcal{P}_7 &= w_{7,1},\\
\mathcal{P}_8 &= w_{8,1},\\
\mathcal{P}_9 &= w_{9,1}.
\end{aligned} \tag{2}$$

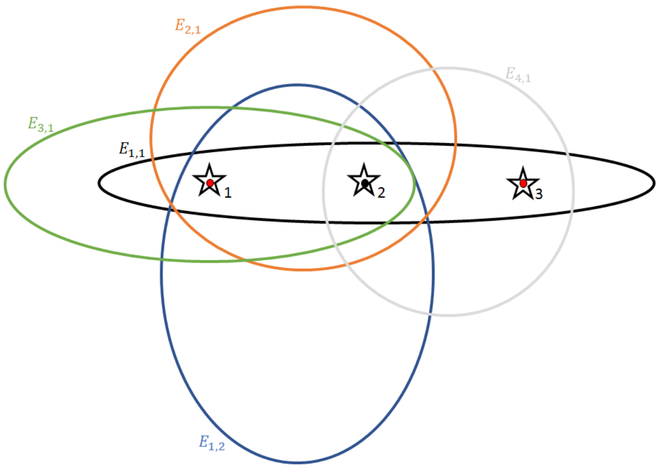

Figure 1: ${}_{1,1}X$ consists of the non-empty intersections of the edges with $E_{1,1}$: ${}_{1,1}X = \{\{1,2\},\{1,2\},\{1,2\},\{2,3\}\}$. Include 3 and 1 in S to infect $E_{1,1}$. Vertices in S coloured red; infected vertices indicated with a star.

Consider the first step of the zero-forcing procedure given in (2): $w_{1,1} \rightarrow w_{1,2}$. In H consider the edge $E_{1,1}$, and form ${}_{1,1}X = \{\{1,2\},\{1,2\},\{1,2\},\{2,3\}\}$. Elements of ${}_{1,1}X$ are not all mutually disjoint, so form ${}_{1,1}X' = \{\{1,2\},\{2,3\}\}$. Include in S the following vertices: $3 \in E_{1,1} \setminus E_{1,2}$ and $1 \in E_{1,1} \setminus E_{4,1}$. Then $\{1,3\}$ infects $E_{1,1}$. (See Figure 1.)

Now consider the set of edges $M = \{E_{1,2}, E_{2,1}, E_{3,1}\}$. All three of these edges have the same intersection with $E_{1,1}$, namely $\{1,2\}$. Therefore $\{1,2\}$ cannot infect $E_{1,2}$. Include in S the following vertices: $4 \in E_{1,2} \setminus E_{2,1}$ and $8 \in E_{1,2} \setminus E_{3,1}$. Then $\{1,2,4,8\}$ infects $E_{1,2}$. (See Figure 2.)

At this step $E_{2,1}$ cannot be infected (neither can $E_{3,1}$). Exclude $8 \in E_{1,2} \setminus E_{3,1}$ from S (doing so does not harm the previous infection steps), and include $7 \in E_{2,1} \setminus E_{3,1}$ in S. Then $\{1,2,7\}$ infects $E_{2,1}$. (See Figure 3.)

Now $\{1,2\}$ infects $E_{3,1}$. Moreover, $\{2,3\}$ infects $E_{4,1}$. (See Figure 4.)

Mark $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$ as used. Note that at this step all vertices in $E_{1,1}$ and all vertices in the edges adjacent with $E_{1,1}$ are infected; and observe that this is achieved by including only four vertices (namely, $1,3,4,7$) in S, where four is equal to the number of zero-forcing chains that are marked as used.

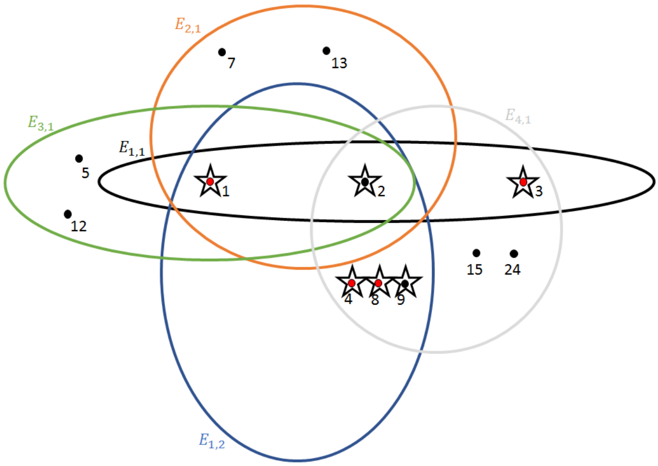

Figure 2: $E_{1,2} \cap E_{1,1} = E_{2,1} \cap E_{1,1} = E_{3,1} \cap E_{1,1} = \{1, 2\}$. Include 4 and 8 in S to infect $E_{1,2}$. Vertices in S coloured red; infected vertices indicated with a star.

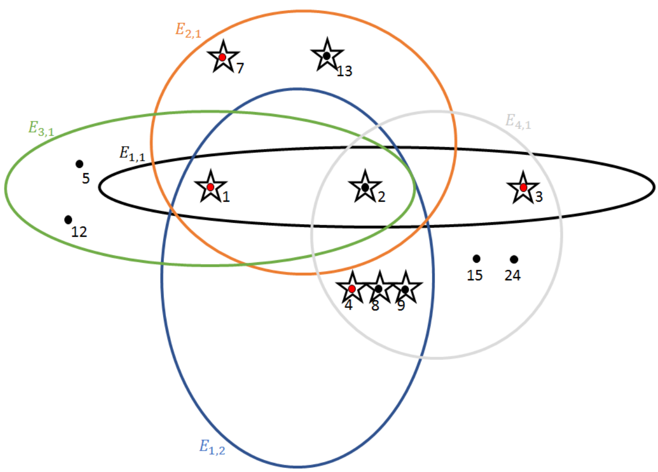

Figure 3: Exclude 8 from S; then include 7 in S to infect $E_{2,1}$. Vertices in S coloured red; infected vertices indicated with a star.

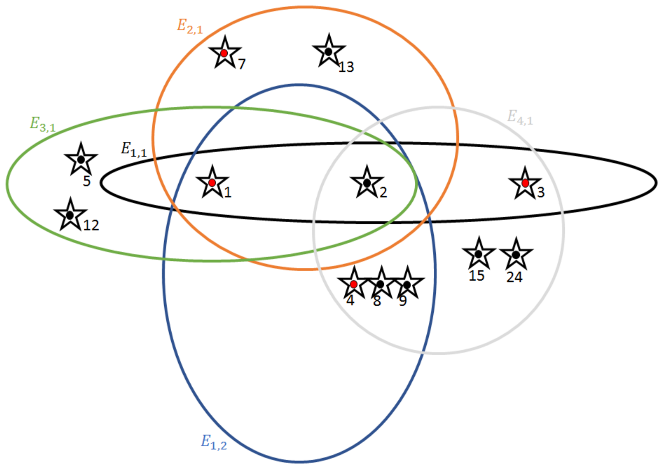

Figure 4: $\{1,2\}$ infects $E_{3,1}$; $\{2,3\}$ infects $E_{4,1}$. Vertices in S coloured red; infected vertices indicated with a star.

Consider the step in the zero-forcing procedure where $w_{2,1} \to w_{2,2}$. In H, $E_{2,1}$ is adjacent to $E_{1,1}$, $E_{1,2}$, $E_{2,2}$, $E_{3,1}$, $E_{4,1}$, $E_{5,1}$; and all of these edges except for $E_{2,2}$ and $E_{5,1}$ are entirely infected. $E_{2,2}$ and $E_{5,1}$ have the same intersection with $E_{2,1}$, namely $\{7\}$. Therefore $\{7\}$ cannot infect $E_{2,2}$. Include in S the vertex $6 \in E_{2,2} \setminus E_{5,1}$. Then $\{6,7\}$ infects $E_{2,2}$, and $\{7\}$ infects $E_{5,1}$. Mark $\mathcal{P}_5$ as used.

Next consider $w_{2,2} \to w_{2,3}$. In H, $E_{2,2}$ is adjacent to $E_{2,1}$, $E_{2,3}$, $E_{5,1}$, $E_{6,1}$; and all of these edges except for $E_{2,3}$ are entirely infected. $\{6\}$ infects $E_{2,3}$.

Consider $w_{2,3} \to w_{2,4}$. In H, $E_{2,3}$ is adjacent to $E_{1,2}$, $E_{2,2}$, $E_{2,4}$, $E_{4,1}$, $E_{6,1}$; and all of these edges except for $E_{2,4}$ are entirely infected. $\{9,10\}$ infects $E_{2,4}$.

The final zero forcing occurs at the step $w_{2,4} \to w_{2,5}$. In H, $E_{2,4}$ is adjacent to $E_{1,2}$, $E_{2,3}$, $E_{2,5}$, $E_{4,1}$; and all of these edges are entirely infected.

It only remains to consider the paths of length 1 in the zero-forcing chains. The first unused path of length 1 corresponding to an edge that contains some uninfected vertices is $\mathcal{P}_7$. At this step $16, 17, 20, 23 \in E_{7,1}$ are uninfected. $E_{8,1}$ and $E_{9,1}$ are the only edges adjacent to $E_{7,1}$ that contain some uninfected vertices. Then ${}_{7,1}X = \{\{15,16,17\},\{15,16,17\}\}$ and therefore ${}_{7,1}X' = \{\{15,16,17\}\}$. Include in S the vertex $20 \in E_{7,1} \setminus E_{8,1}$. $\{15,16,17,20\}$ infects $E_{7,1}$. Since $\{15,16,17\} = E_{7,1} \cap E_{8,1} = E_{7,1} \cap E_{9,1}$, neither $E_{8,1}$ nor $E_{9,1}$ can be infected at this point. We include in S the vertex $18 \in E_{8,1} \setminus E_{9,1}$. Then $\{15, 16, 17, 18\}$ infects $E_{8,1}$, and $\{15, 16, 17\}$ infects $E_{9,1}$. Mark $\mathcal{P}_8$ and $\mathcal{P}_9$ as used. $S = \{1, 3, 4, 6, 7, 18, 20\}$ is infection set for H of size 7. Observe that the given zero-forcing set for $L(H)$ was of size 9.

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