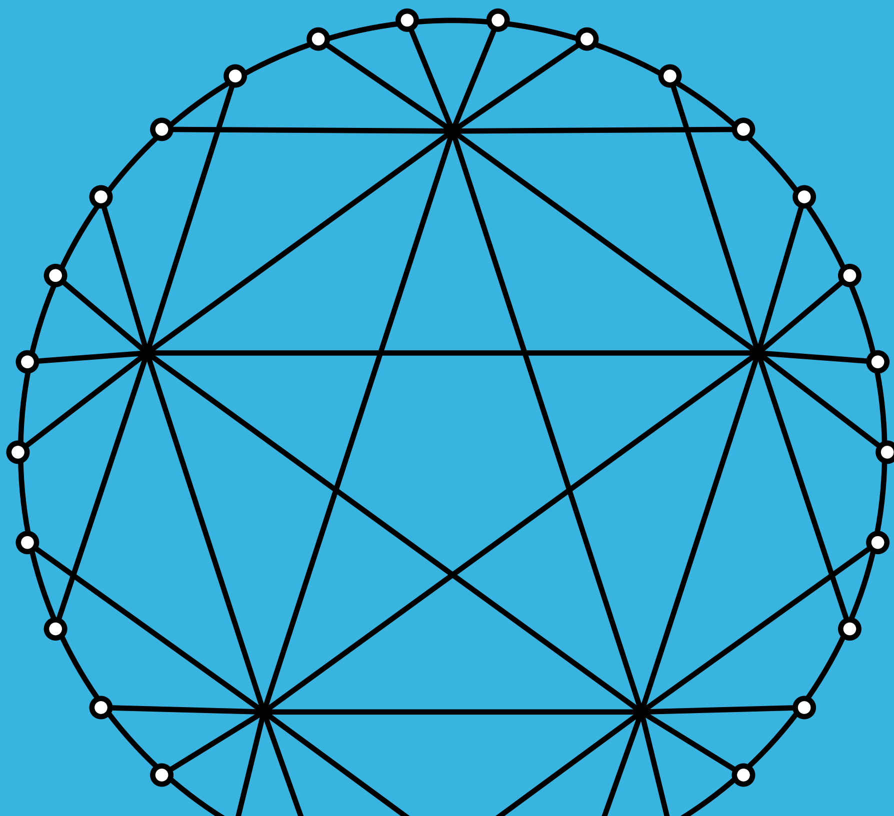
Volume 86
June 2019

BULLETIN of the INSTITUTE of COMBINATORICS and its APPLICATIONS



Boca Raton, FL, U.S.A.

ISSN 1182 - 1278

BULLETIN OF THE ICA
Volume **86** (2019), Pages 85–100

Palindromic graphs

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Abstract: A *palindromic labeling* on a graph G is a bijection $f : V(G) \to \{1, ..., |V(G)|\}$ such that if $uv \in E(G)$, then there exists $xy \in E(G)$ such that $f(x) = |V(G)| + 1 - f(u)$ and $f(y) = |V(G)| + 1 - f(v)$. A graph that admits a palindromic labeling is a *palindromic graph*. In this paper, we determine simple necessary and sufficient conditions for several well-known families of graphs to be palindromic. In addition, we give methods of constructing palindromic graphs. In our main result, we show that all palindromic trees are generated by these constructions. We also provide results on when the join and Cartesian product of graphs can be palindromic.

1 Introduction and preliminary results

In a recent paper [2], Buratti calls for papers on palindromes. Palindromes are words (sequences, numbers, sentences, etc.) that are the same regardless of whether they are read forwards or backwards. In other words, suppose that $a_1...a_n$ is a word in the alphabet $\mathcal{A}$. This word is a *palindrome* if for all $i = 1, ..., n$, $a_i = a_{n-i+1}$. Examples of English palindromes can be found at `http://www.palindromelist.net/`. In addition, Martin Gardner discusses several interesting properties of mathematical palindromes in [4].

We are inspired by the above discussion to consider palindromes on graphs. Let $G = (V, E)$ be a graph. A *palindromic labeling* is a bijection $f : V(G) \to \{1, ..., |V(G)|\}$ such that if $uv \in E(G)$, then there exists $xy \in E(G)$ such that $f(x) = |V(G)| + 1 - f(u)$ and $f(y) = |V(G)| + 1 - f(v)$. An example

Key words and phrases: graph labeling, palindromes, graph automorphisms
AMS (MOS) Subject Classifications: 05C78 (05C25, 05C75)

Received: 21 March 2018
Accepted: 7 March 2019

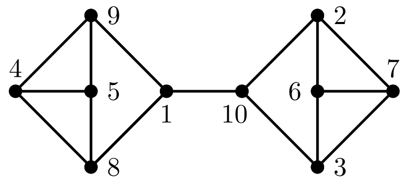

Figure 1: A palindromic labeling on a graph

of a palindromic labeling on a graph is given in Figure 1. A graph that admits a palindromic labeling is a *palindromic graph*. If f is a palindromic labeling of G and $v \in V(G)$, then $v' \in V(G)$ *corresponds* to v under f if $f(v') = |V(G)| + 1 - f(v)$. Likewise, if $uv \in E(G)$, then $u'v' \in E(G)$ *corresponds* to uv if $f(u') = |V(G)|+1-f(u)$ and $f(v') = |V(G)|+1-f(v)$. Note that a vertex, or an edge, can correspond to itself. All other graph theoretical notation and terminology will be consistent with West [9]. For an extensive survey of other types of graph labelings, the interested reader is referred to [3].

As we will see in our first theorem, there is a connection between a palindromic labeling on a graph and the automorphisms on its vertices. An *automorphism* on a graph G is a bijection $\phi : V(G) \to V(G)$ such that if $uv \in E(G)$, then $\phi(u)\phi(v) \in E(G)$. The set of all automorphisms on a graph G is the *automorphism group* of G. This is denoted $Aut(G)$. Note that graph automorphisms induce an equivalence relation on the vertices of a graph G. Namely, $u \in V(G)$ relates to $v \in V(G)$ if there an automorphism $\phi \in Aut(G)$ such that $\phi(u) = v$. Like all equivalence relations, this partitions the vertices into classes, called *automorphism classes*. For more information on the automorphism group of a graph, see [5, 8].

Theorem 1.1. *A graph G is palindromic if and only if there exists an automorphism ϕ on the vertices of G such that ϕ is an involution having at most one fixed point.*

Proof. Suppose that G is a palindromic graph on n vertices. Let f be a palindromic labeling of G. Define $\phi : V(G) \to V(G)$ as $\phi(v) = v'$, where $f(v') = n + 1 - f(v)$. Since f is a bijection, ϕ is likewise a bijection. If $uv \in E(G)$, then $u'v'$ satisfies $f(u') = n+1-f(u)$ and $f(v') = n+1-f(v)$. Therefore, $\phi(u)\phi(v) \in E(G)$. Thus, ϕ is a graph automorphism. Since $f(v') = n + 1 - f(v)$ it follows that $\phi(v) = v'$ and $\phi(v') = v$. Thus, ϕ is an involution. Suppose that there are distinct vertices u and v such that

$\phi(u) = u$ and $\phi(v) = v$. Thus, $f(u) = n+1-f(u)$ and $f(v) = n+1-f(v)$. This implies that $f(u) = f(v) = \frac{n+1}{2}$, contrary to f being a bijection. Ergo, ϕ leaves at most one vertex fixed.

Conversely, suppose that ϕ is an automorphism on the vertices of G such that ϕ is an involution having at most one fixed point. Consider

$$\{u_1, u_1'\}, \{u_2, u_2'\}, \ldots, \{u_k, u_k'\},$$

where $\phi(u_i) = u_i'$ and $k = \lceil \frac{n}{2} \rceil$. By definition of ϕ, this is a partition of $V(G)$. Further, at most one of these parts, say $\{u_k\}$, consists of a single vertex. For $i = 1, ..., k$, define $f(u_i) = i$ and $f(u_i') = n + 1 - i$. Note that if $\{u_k, u_k'\}$ consists of a single vertex, then $f(u_k) = \frac{n+1}{2}$ by construction. Clearly, f is a bijection mapping $V(G)$ to $\{1, ..., n\}$. If $uv \in E(G)$, then $\phi(u)\phi(v) \in E(G)$ satisfies $f(\phi(u)) = n+1-f(u)$ and $f(\phi(v)) = n+1-f(v)$. Thus, f is a palindromic labeling on G. □

Using automorphism classes, we now give another necessary condition for a graph to be palindromic.

Theorem 1.2. *Let G be a palindromic graph on n vertices.* (i) *If n is even, then every automorphism class has an even number of vertices.* (ii) *If n is odd, then exactly one automorphism class has an odd number of vertices.*

Proof. Let G be a palindromic graph on n vertices. Suppose to the contrary that X and Y are distinct automorphism classes of G such that $|X|$ and $|Y|$ are both odd. Let f be any palindromic labeling of G. As in the proof of Theorem 1.1, if $x \in X$, then there exists $x' \in X$ such that $f(x') = n + 1 - f(x)$. Since f is a bijection, these labels must be distinct. Since $|X|$ is odd, then one of the vertices of X must be labeled $f(x) = \frac{n+1}{2}$. If n is even, then this is a contradiction. Thus, (i) holds. Using a similar argument, if n is odd it follows that one of the vertices in Y must also be labeled $f(y) = \frac{n+1}{2}$. However, this is contrary to f being a bijection. Thus, (ii) holds. □

The necessary condition given in Theorem 1.2 is not sufficient. To see this, consider the Petersen graph illustrated in Figure 2. The automorphism group is generated by

$$(v_3, v_7)(v_4, v_{10})(v_8, v_9) \quad \text{and} \quad (v_1, v_2, v_3, v_4, v_5)(v_6, v_7, v_8, v_9, v_{10}).$$

Thus, the graph has a single automorphism class containing all ten vertices. Using a computer algebra system, we find that none of the 120 elements of

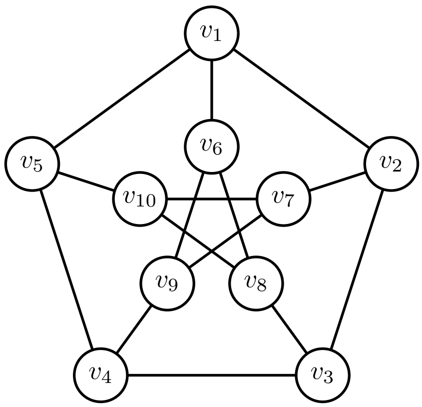

Figure 2: The Petersen Graph

the automorphism group satisfy Theorem 1.1. Hence, the Petersen graph is not palindromic.

A necessary condition for two vertices to be in the same automorphism class is that they have the same degree (see for example [5, 8]). Hence, the next corollary follows immediately. However, as shown above, the Petersen graph is an example where the condition given in Corollary 1.3 is not sufficient. We will later provide an infinite class of examples where this condition is not sufficient.

Corollary 1.3. *Let G be a palindromic graph on n vertices. Let d_k be the number of vertices in G of degree k. If n is even, then d_k must be even for all k. If n is odd, then exactly one d_k is odd.*

One consequence of a palindromic labeling is that it accentuates the inherit symmetry of the graph. This is in stark contrast to the distinguishing numbers introduced by Albertson and Collins in which the goal is to break the symmetries of the graph [1].

We note that a palindromic labeling is only unique when the graph is trivial. With this in mind, we give a lower bound on the number of palindromic labelings on palindromic graph in the next result.

Theorem 1.4. *If G is a palindromic graph on n vertices, then there are at least $\left\lfloor \frac{n}{2} \right\rfloor ! 2^{\lfloor n/2 \rfloor}$ palindromic labelings on G.*

Proof. Let G be a palindromic graph on n vertices. Let f be a palindromic labeling on G. As in the proof of Theorem 1.1, this induces a partition of the vertices of G into $\{u_1, u_1'\}$,...,$\{u_k, u_k'\}$, where $k = \lceil \frac{n}{2} \rceil$ and $f(u_i') = n + 1 - f(u)$. Without loss of generality, we will assume that if n is odd, then $u_k = u_k'$ and $f(u_k) = \frac{n+1}{2}$.

We claim that if the labels on two of the pairs $\{u_i, u_i'\}$ and $\{u_j, u_j'\}$ are swapped, then the resulting labeling is still palindromic. Define $g : V(G) \to \{1, ..., n\}$ as follows: $g(u_i) = f(u_j)$, $g(u_j) = f(u_j)$, $g(u_i') = f(u_j')$, $g(u_j') = f(u_j')$, and $g(v) = f(v)$ for all other $v \in V(G)$. Since f is a bijection, g is also a bijection. Suppose that $u_i w \in E(G)$. Under f, this corresponds to the edge $u_i' w'$. If suffices to show that correspondence holds under g. Note that $g(u_i') = f(u_j') = n+1-f(u_j) = n+1-g(u_i)$. If $w = u_i'$, then the same argument holds. If $w = u_j$, then $g(w') = g(u_j') = f(u_i') = n + 1 - f(u_i) = n + 1 - g(u_j)$. A similar argument holds if $w = u_j'$. If $w \notin \{u_i', u_j, u_j'\}$, then $g(w') = f(w') = n + 1 - f(w) = n + 1 - g(w')$. In any case, the new labeling remains palindromic. Analogously, if we swap the labels on u_i and u_i', then the resulting labeling is palindromic.

We will use the above partition to obtain our bound. This can be done as follows: If n is odd, then u_k must be assigned $\frac{n+1}{2}$. For all other pairs $\{u_i, u_i\}$, assign distinct elements of $\{\{\ell, n + 1 - \ell\} : 1 \leq \ell \leq \lfloor \frac{n}{2} \rfloor\}$. There are $\lfloor \frac{n}{2} \rfloor!$ ways to do this. For each such assignment, choose whether u_i or u_i' is assigned ℓ. The other vertex will be assigned $n + 1 - \ell$. There are $2^{\lfloor n/2 \rfloor}$ ways to do this. Thus, there are at least $\left\lfloor \dfrac{n}{2} \right\rfloor ! 2^{\lfloor n/2 \rfloor}$ palindromic labelings on G. □

One of our main results in Section 3 classifies palindromic trees. To aid in this proof, we provide an even simpler necessary condition for trees.

Corollary 1.5. *If T is a palindromic tree with even diameter, then T has an odd number of vertices. If T is a palindromic tree with odd diameter, then T has an even number of vertices.*

Proof. Let T be a palindromic tree. If T has even diameter, then its center is a single vertex (see for example [9]). This vertex must be in an automorphism class by itself. By Theorem 1.2, each of the remaining automorphism classes must have an even number of vertices. Thus, $|V(T)|$ must be odd.

If T has odd diameter, then its center consists of two adjacent vertices, say x and y (see for example [9]). Suppose to the contrary that $|V(T)|$ is odd.

By Theorem 1.2, there is an automorphism class C with an odd number of vertices. Let $z \in C$. Without loss of generality, suppose that the distance from z to x is strictly less than the distance from z to y. As in the proof of Theorem 1.1, there exists a graph automorphism ϕ such that $\phi(x) = y$. However, this same automorphism would map z to a vertex z' which is closer to y than to x. Hence, $z \neq z'$. Since z was chosen arbitrarily from C, this induces a partition of C into $\{z_1, z_1'\}$,...,$\{z_k, z_k'\}$, where $\phi(z_i) = z_i'$ and $z_i \neq z_i'$. This contradicts C having an odd number of vertices. Thus, T must have an even number of vertices. □

Note that the automorphism groups of a graph G and its complement $\overline{G}$ are the same (see for example [5, 8]). Hence the next result follows immediately.

Proposition 1.6. *A graph G is palindromic if and only if its complement $\overline{G}$ is palindromic.*

2 Specific families of graphs

In this section, we give simple necessary and sufficient conditions for several families of graphs to be palindromic. For each of the palindromic graphs in this section, we provide an explicit palindromic labeling. Let P_n, C_n, and K_n denote the path, cycle, and complete graph on n vertices, respectively. In all cases, suppose that the vertices are labeled v_0, v_1,...,v_{n-1} in the obvious way.

Theorem 2.1. *The path P_n, cycle C_n, and complete graph K_n are palindromic for all n.*

Proof. We begin by giving the required palindromic labeling on the path. Label $f(v_i) = i + 1$ for $i = 0, 1, ..., n - 1$. In each case, v_i and v_{n-i-1} are in the same automorphism class and are labeled $f(v_i) = i + 1$ and $f(v_{n-i-1}) = n - i = n + 1 - f(v_i)$. Hence, this labeling is palindromic.

To obtain the cycle from the path, we add the edge v_0v_{n-1}. Since this edge corresponds to itself under f, the graph remains palindromic. The complete graph is trivially palindromic. □

The *complete bipartite graph* $K_{n,m}$ is the graph with vertex set $x_1, ..., x_n$, $y_1, ..., y_m$ such that $x_iy_j \in E(K_{n,m})$ for all $i = 1, ..., n$ and $j = 1, ..., m$. In particular, when $n = 1$, this graph is called the *star*.

Theorem 2.2. *The complete bipartite graph $K_{n,m}$ is palindromic if and only if $n = m$ or at least one of n or m is even.*

Proof. Suppose that $n \neq m$ and n and m are both odd. In this case, $K_{n,m}$ has n vertices of degree m and m vertices of degree n. Since n and m are both odd and $n \neq m$, $K_{n,m}$ cannot be palindromic by Corollary 1.3.

It suffices to give the required palindromic labeling in the remaining cases. Suppose that $n = m$. The required labeling is $f(x_i) = i$ and $f(y_i) = 2n+1-i$ for all i. For all i, x_i corresponds to y_i under this labeling. Hence f is palindromic.

Suppose that m is even, say $m = 2t$. Label $f(x_i) = t + i$ for $i = 1, ..., n$. For $j = 1, .., t$, label $f(y_j) = j$ and $f(y_{m-j+1}) = n + m + 1 - j$. Under this labeling, x_i corresponds to x_{n-i+1} and y_j corresponds to $y_{n+m+1-j}$. Thus, f is palindromic. □

Theorem 2.2 illustrates that the join of two palindromic graphs need not be palindromic. As an example, the empty graphs $\overline{K_{2t+1}}$ and $\overline{K_{2k+1}}$ are palindromic by Proposition 1.6 and Theorem 2.1. Their join is $K_{2t+1,2k+1}$. However, if $t \geq k+1$, then this is not palindromic by Theorem 2.2. A more comprehensive result about whether the join of two graphs is palindromic is given in Theorem 4.2.

The *double star* is the tree in which every vertex is adjacent to one of two adjacent central vertices. The two center vertices are denoted x and y. The non-center vertices adjacent to x will be denoted x_1,...,x_n. The non-center vertices adjacent to y will be denoted y_1,...,y_m. This graph is denoted $S_{n,m}$.

Theorem 2.3. *The double star $S_{n,m}$ is palindromic if and only if $n = m$.*

Proof. Suppose that $n \neq m$. The automorphism classes of $S_{n,m}$ are $\{x\}$, $\{y\}$, $\{x_1, ..., x_n\}$, and $\{y_1, ..., y_m\}$. Since $S_{n,m}$ has at least two automorphism classes with an odd number of vertices, it is not palindromic by Theorem 1.2.

It suffices to give the required labeling in the case where $n = m$. Label $f(x_i) = i$, $f(x) = n + 2$, $f(y) = n + 1$, and $f(y_i) = 2n + 3 - i$. Under this labeling, x corresponds to y and x_i corresponds to y_i for all i. Hence this labeling is palindromic. □

The complete graph is palindromic by Theorem 2.1. It stands to reason that if a graph has enough edges, then it will be palindromic. By Proposition 1.6, any graph that has too few edges will likewise be palindromic. We are motivated by these comments to consider the following problem: For any n, determine the minimum m such that if G is a non-palindromic graph on n vertices and k edges, then k must satisfy $m \leq k \leq \frac{n(n-1)}{2} - m$. We are particularly interested in the case where G and $\overline{G}$ are both connected. With this in mind, we define P'_n to be the tree obtained from P_n by appending an additional pendant, v'_0, to v_1.

Proposition 2.4. *The graph P'_n is not palindromic for $n \geq 3$. Further, for $n \geq 4$ both P'_n and its complement $\overline{P'_n}$ are connected.*

Proof. Note that P'_3 is isomorphic to $K_{1,3}$. Hence it is not palindromic by Theorem 2.2. For $n \geq 4$, the automorphism classes of P'_n are $\{v_0, v'_0\}$, $\{v_1\}$,...,$\{v_{n-1}\}$. Hence, the graph is not palindromic by Theorem 1.2.

Clearly, P'_n is connected for all n. We now show that $\overline{P'_n}$ is connected for $n \geq 4$. Note that in $\overline{P'_n}$, v_0 is adjacent to $v'_0, v_2, ..., v_{n-1}$. Further, v_1 and v_{n-1} are adjacent in $\overline{P'_n}$. Thus $\overline{P'_n}$ is also connected for $n \geq 4$. □

From Proposition 2.4, it follows that for $n \geq 4$, if G is a non-palindromic graph on n vertices and k edges such that both G and $\overline{G}$ are both connected, then $n - 1 \leq k \leq \frac{(n-1)(n-2)}{2}$. Further, these bounds are sharp. Similarly, suppose that we restrict our attention to graphs with no isolated vertices. If n is even, then consider the disjoint union of $K_{1,3}$ and $\frac{n-4}{2}$ copies of K_2. This graph is not palindromic by Corollary 1.3 and has $1 + n/2$ edges. Likewise, if n is odd, then consider the disjoint union $K_{1,3}$, P_3, and $\frac{n-7}{2}$ copies of K_2. This graph is not palindromic by Corollary 1.3 and has $2 + \frac{n-1}{2}$ edges. Finally, suppose that we allow graphs with isolated vertices. Consider the disjoint union of $K_{1,3}$ and $n - 4$ isolated vertices. This graph is not palindromic by Corollary 1.3. Hence, the bound widens to $3 \leq k \leq \frac{n(n-1)}{2} - 3$.

3 Constructing palindromic graphs

In this section, we introduce two methods of constructing palindromic graphs. Let G be any graph, let $S \subseteq V(G)$, and let p be a non-negative

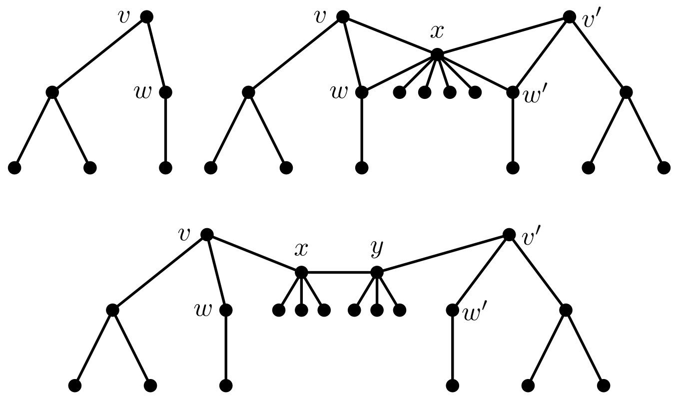

Figure 3: The graphs G, $Pal_1(G, \{v, w\}, 4)$, and $Pal_2(G, \{v\}, 3)$

integer. Define $Pal_1(G, S, p)$ to be the graph obtained by taking two isomorphic copies of G (labeled G_1 and G_2), a new vertex x, and p pendant vertices adjacent to x. For all $v \in S$, we add an edge from x to v and an edge from x to v', where $v' \in V(G_2)$ corresponds to $v \in V(G_1)$. Similarly, we define $Pal_2(G, S, p)$ to be the graph obtained by taking two isomorphic copies G (labeled G_1 and G_2), two adjacent vertices x and y, p pendants adjacent to x, and p pendants adjacent to y. For all $v \in S$, we add an edge from x to v, and an edge from y to the corresponding vertex v' in G_2. Examples of these graphs are given in Figure 3.

Theorem 3.1. *Let G be a graph, let $S \subseteq V(G)$, and let p be a non-negative integer. The graphs $Pal_1(G, S, 2p)$ and $Pal_2(G, S, p)$ are palindromic.*

Proof. Consider $Pal_1(G, S, 2p)$. In this graph, let x_1,...,x_{2p} be the pendants adjacent to x, let $V(G_1) = \{v_1, ..., v_n\}$, and $V(G_2) = \{v'_1, ..., v'_n\}$, where v'_i is the vertex in G_2 corresponding to v_i in G_1. Define g to be any bijection between $V(G)$ and the set $\{1, ..., n\}$.

It suffices to give the required palindromic labeling on $Pal_1(G, S, 2p)$. Define f on $V(Pal_1(G, S, 2p))$ as follows: $f(v_i) = g(v_i)$, $f(v'_i) = 2n + 2p + 2 - g(v_i)$, $f(x) = n+p+1$, $f(x_i) = n+p+1-i$ for $i = 1, ..., p$, and $f(x_i) = n+1+i$ for $i = p + 1, ..., 2p$. Clearly, f is a bijection. If $uv \in E(G_1)$, then $u'v' \in E(G_2)$ satisfies $f(u') = 2n + 2p + 2 - g(u) = |V(Pal_1(G, S, 2p))| + 1 - f(u)$ and $f(v') = 2n + 2p + 2 - g(v) = |V(Pal_1(G, S, 2p))| + 1 - f(v)$. A similar argument holds if $u'v' \in E(G_2)$. If $v \in S_1$, then xv' satisfies $f(x) = n + 1 + p = 2n + 2p + 2 - (n + 1 + p) = |V(Pal_1(G, S, 2p))| - f(x)$

and $f(v') = |V(Pal_1(G,S,2p))| + 1 - f(v)$. A similar argument holds if $v' \in V(G_2)$. For $i = 1, ..., p$, note that $xx_i \in E(Pal_1(G,S,2p))$. The edge xx_{p+i} satisfies $f(x) = |V(Pal_1(G,S,2p))| + 1 - f(x)$ and $f(x_{p+i}) = n+1+p+i = 2n+2p+2-(n+p+1-i) = |V(Pal_1(G,S,2p))|+1-f(x_i)$. A similar argument holds for the case where $i \in \{p+1, ..., 2p\}$.

The proof for $Pal_2(G,S,p)$ follows in a similar manner. Let $x_1, ..., x_p$ be the pendants adjacent to x, let y_1,...,y_p be the pendants adjacent to y, let $V(G_1) = \{v_1, ..., v_n\}$, and $V(G_2) = \{v'_1, ..., v'_n\}$. Define g to be any bijection between $V(G)$ and the set $\{1, ..., n\}$.

It suffices to give the required labeling. Define f on $V(Pal_2(G,S,p))$ as follows: $f(v_i) = g(v_i)$, $f(v'_i) = 2n+2p+3-g(v_i)$, $f(x) = n+p+1$, $f(y) = n+p+2$, $f(x_i) = n+p+2+i$ for $i = 1, ..., p$, and $f(y_i) = n+p+1-i$ for $i = 1, ..., p$. For $v \in S$, note that $vx \in E(Pal_2(G,S,p))$ and $v'y \in E(Pal_2(G,S,p))$. Further, $f(y) = n+p+2 = 2n+2p+3-(n+p+1) = |V(Pal_2(G,S,p))|+1-f(x)$ and $f(v') = 2n+2p+3-g(v) = |V(Pal_2(G,S,p))|+1-f(v)$. Likewise, for $xx_i \in E(Pal_2(G,S,p))$, yy_i satisfies $f(y) = |V(Pal_2(G,S,p))|+1-f(x)$ and $f(y_i) = n+p+1-i = 2n+2p+3-(n+p+2+i) = |V(Pal_2(G,S,p))|+1-f(x_i)$. The rest of the proof is analogous to above. □

If G is non-trivial, then two of the automorphism classes of $Pal_1(G,S,2p+1)$ are $\{x\}$ and $\{x_1, ..., x_{2p+1}\}$. Hence, it cannot be palindromic by Theorem 1.2. However, $Pal_1(G,S,0)$ remains palindromic when x is deleted. To see this, simply subtract one from each of the labels on the vertices of G_2. Similarly, $Pal_2(G,S,p)$ is palindromic if the edge xy is deleted. Finally, note that in either graph we may add any number of edges between corresponding vertices in G_1 and G_2.

Combining Corollary 1.3 and Theorem 3.1 allows us to characterize additional palindromic graphs. A *caterpillar* is obtained from the path on n vertices by appending pendant vertices to the existing vertices of the path. The vertices of the original path are labeled v_0, v_1,...,v_{n-1} in the obvious way. We append a_i pendant vertices to v_{i-1}. These pendants are denoted $v_{i-1,1}$,...,v_{i-1,a_i}. Without loss of generality, we may assume that $a_1 \geq 1$ and $a_n \geq 1$. A caterpillar is illustrated in Figure 4.

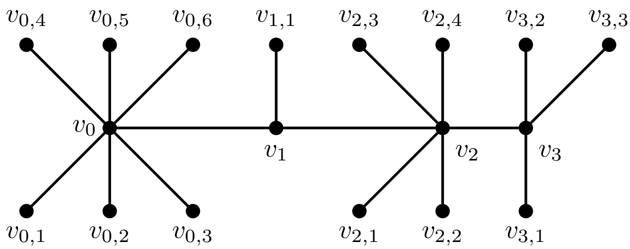

Figure 4: The caterpillar $P_4(6, 1, 4, 3)$

Theorem 3.2. *The graph $P_{2t}(a_1, ..., a_{2t})$ is palindromic if and only if $a_i = a_{2t-i+1}$ for all $i = 1, ..., t$. The graph $P_{2t+1}(a_1, ..., a_{2t+1})$ is palindromic if and only if $a_i = a_{2t-i+2}$ and a_{t+1} is even.*

Proof. Suppose that $a_i \neq a_{n-i+1}$ for some i. Two automorphism classes of the graph are $\{v_{i-1}\}$ and $\{v_{n-i}\}$. Thus, $P_n(a_1, ..., a_n)$ is not palindromic by Theorem 1.2. Likewise, suppose that $n = 2t + 1$ and a_{t+1} is odd. Two automorphism classes of the graph are $\{v_t\}$ and $\{v_{t,1}, ..., v_{t,a_{t+1}}\}$. Since both of these classes have an odd number of vertices, the graph is not palindromic by Theorem 1.2.

Suppose $n = 2t$ and $a_i = a_{2t-i+1}$ for $i = 1, ..., t$. Note that $P_{2t}(a_1, ..., a_{2t})$ is isomorphic to $Pal_2(P_{t-1}(a_1, ..., a_{t-1}), \{v_{t-2}\}, a_t)$. Thus, the graph is palindromic by Theorem 3.1. Likewise, suppose that $n = 2t + 1$, $a_i = a_{2t-i+2}$, and $a_{t+1} = 2p$ for some p. Note that $P_{2t+1}(a_1, ..., a_{2t+1})$ is isomorphic to $Pal_1(P_t(a_1, ..., a_t), \{v_{t-1}\}, 2p)$. Hence, the graph is palindromic by Theorem 3.1. □

One of the interesting implications of Theorem 3.2 is that a caterpillar is palindromic if and only if its parameters form a palindrome. The caterpillar also provides an infinite family of graphs in which the necessary condition given in Corollary 1.3 is not sufficient. To see this, consider the caterpillar $P_n(a_1, ..., a_n)$, where $n \geq 6$, $a_2 = a_{n-2} = 2$, $a_3 = a_{n-1} = 3$, and $a_i = 1$ for all other i. Since $a_2 \neq a_{n-1}$, this graph is not palindromic by Theorem 3.2. The technique employed in the proof of Theorem 3.2 can be used for other trees as well. With this in mind, we now prove our main result that all palindromic trees can be obtained via the construction given in Theorem 3.1.

Theorem 3.3. *If T is a palindromic tree, then there exists a forest F and a set of vertices $S \subseteq V(F)$ such that $T = Pa1_1(F, S, 0)$ or $T = Pal_2(F, S, 0)$.*

Proof. Suppose that T is a palindromic tree with n vertices and even diameter. Let ϕ be the associated automorphism. Since T has even diameter, the center consists of a single vertex x (see for example, [9]). This vertex is in an automorphism class by itself. This being the case, $\phi(x) = x$. Note that deleting the vertex x separates T into a set of connected components.

Suppose that C is a connected component of $T - x$ such that $\phi[C] = C$. Let $v \in C$ such that $vx \in E(T)$. Since trees are acyclic, this vertex is unique. Graph automorphisms are distance preserving (see for example [8]). Thus $\phi(v) = v$. However, $\phi(x) = x$ as well. This contradicts ϕ having at most one fixed point. Thus, there exists a connected component C' such that $\phi[C] = C'$ and $C \neq C'$.

Partition the connected components of $T - x$ as $\{C_1, C_1'\}$,...,$\{C_k, C_k'\}$, where $\phi[C_i] = C_i'$ and $C_i \neq C_i'$. Let F be the disjoint union of the C_i. Since T is a tree, it follows that F is a forest. Let v_i be the unique vertex in C_i adjacent to x in T. Let $S = \{v_1, ..., v_k\}$. Now, $T = Pal_1(F, S, 0)$.

The case when the tree is of odd diameter follows in a similar manner. Let T be a palindromic tree with n vertices and odd diameter. Let f be a palindromic labeling of T. Since T has odd diameter, the center consists of a pair of adjacent vertices x and y (see for example [9]). Note that $T - \{x, y\}$ is a forest. Suppose that C_1,...,C_k are the connected components of $T - \{x, y\}$ such that there is $x_i \in V(C_i)$ that is adjacent to x in T. Since T is palindromic, each component C_i corresponds to a component C_i' such that there is $x_i' \in V(C_i')$ that is adjacent to y in T. Let F be the disjoint union of C_1,...,C_k and let $S = \{x_1, ..., x_k\}$. Now, $T = Pal_2(F, S, 0)$. □

Whether an analog of Theorem 3.3 holds for all *graphs* is unknown at this time.

4 The join and cartesian product

In the previous section, we gave a method for constructing palindromic graphs. Another method of constructing such graphs would be to use graph operations, such as the join and the Cartesian product.

The *join* of graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. This graph is denoted $G \vee H$. As our results will involve the automorphism group of the join, we first establish some elementary properties of $Aut(G \vee H)$. Clearly, $Aut(G) \times Aut(H)$ is isomorphic to a subgroup of $Aut(G \vee H)$. Our next lemma establishes when an automorphism of $Aut(G \vee H)$ maps a vertex of $V(G)$ to a vertex of $V(H)$.

Lemma 4.1. *Let $\phi \in Aut(G \vee H)$ such that ϕ^2 is the identity isomorphism. Let G' be the subgraph induced by the set $\{g \in V(G) : \phi(g) \in V(H)\}$. It follows that:*

(i) *The image of G' under ϕ is a subgraph of H.*

(ii) *The subgraph induced by this image, H', is isomorphic to G'.*

(iii) *For all $x \in V(G')$ and for all $y \in V(G - G')$, $xy \in E(G)$.*

(iv) *For all $w \in V(H')$ and for all $z \in V(H - H')$, $wz \in E(H)$.*

Proof. Let ϕ and G' be defined as above. Clearly, the image of G' under ϕ is a subgraph of H. As above, this subgraph is denoted H'. Suppose that $g_1, g_2 \in V(G')$ such that $g_1g_2 \in E(G')$. Since ϕ is an automorphism on $G \vee H$, it follows that $\phi(g_1)\phi(g_2) \in E(G \vee H)$. By definition, $\phi(g_1), \phi(g_2) \in V(H')$. Thus, the edge $\phi(g_1)\phi(g_2) \in E(H')$. Ergo, H' is isomorphic to G' and (ii) follows.

Let $x \in V(G')$ and $y \in V(G - G')$. By definition, $\phi(x) \in V(H')$. As for $\phi(y)$, it may either be in $V(G - G')$, $V(G')$, or $V(H)$. If $\phi(y) \in V(H)$, then $y \in V(G')$, a contradiction. If $\phi(y) \in V(G')$, then $\phi^2(y) \in V(G - G')$, contrary to the definition of G'. Hence, $\phi(y) \in V(G - G')$. It follows from the definition of $G \vee H$ that $\phi(x)$ is adjacent to $\phi(y)$ in $G \vee H$. Because ϕ is an automorphism such that ϕ^2 is the identity, $xy \in E(G)$. Thus (iii) holds. The proof of (iv) is analogous. □

With this in mind, we are prepared to provide necessary and sufficient conditions for the join to be palindromic.

Theorem 4.2. *Let G and H be graphs. The join $G \vee H$ is palindromic if and only if all of the following conditions hold:*

(i) *There exists a subgraph G' of G such that for all $x \in V(G')$ and for all $y \in V(G - G')$, $xy \in E(G)$.*

(ii) *There exists a subgraph H' of H such that for all $w \in V(H')$ and for all $z \in V(H - H')$, $wz \in E(H)$.*

(iii) *G' and H' are isomorphic.*

(iv) *$G - G'$ and $H - H'$ are both palindromic graphs such that $G - G'$ or $H - H'$ have an even number of vertices.*

Proof. Suppose that (i)-(iv) hold. By (iv), $G - G'$ and $H - H'$ are palindromic with associated automorphisms ϕ and θ, respectively. By (iii), there is an isomorphism ψ mapping $V(G')$ to $V(H')$. We define $\xi : V(G \vee H) \to V(G \vee H)$ as follows: $\xi(v) = \phi(v)$ for $v \in V(G - G')$, $\xi(v) = \theta(v)$ for $v \in V(H - H')$, $\xi(v) = \psi(v)$ for $v \in V(G')$, and $\xi(v) = \psi^{-1}(v)$ for $v \in V(H')$. We claim that ξ is the required automorphism on $G \vee H$. If $xy \in E(G - G')$, then $\xi(x)\xi(y)$ is an edge in $E(G - G')$ because ϕ is an automorphism on $G - G'$. A similar argument holds if $xy \in E(H - H')$. If $xy \in E(G')$, then $\xi(x)\xi(y) \in E(H')$ because ψ is an isomorphism. A similar argument holds if $xy \in E(H')$. If $x \in V(G)$ and $y \in V(H)$, then $xy \in E(G \vee H)$ and their adjacency is preserved by the join. If $x \in V(G')$ and $y \in V(G - G')$, then $xy \in E(G \vee H)$ by (i). Further, $\xi(x) \in V(H')$ and $\xi(y) \in V(G - G')$ implies that $\xi(x)\xi(y) \in E(G \vee H)$. A similar argument holds for the case where $x \in V(H')$ and $y \in V(H - H')$. Note that by definition of ϕ, θ, and ψ, ξ^2 is the identity automorphism. Further, ξ leaves at most one vertex fixed by (iv). Thus, ξ is the required automorphism. Ergo, $G \vee H$ is palindromic by Theorem 1.1.

Conversely, suppose that $G \vee H$ is palindromic with associated automorphism ξ. Let G' be the subgraph induced by the set $\{g \in V(G) : \xi(g) \in V(H)\}$. By Lemma 4.1, the image of G' under ξ is H', a subgraph of H. Further, (i), (ii), and (iii) hold by Lemma 4.1. Using a similar argument as above, it follows that $\xi : V(G - G') \to V(G - G')$ and $\xi : V(H - H') \to V(H - H')$. Thus, $G - G'$ and $H - H'$ are palindromic. Since $G \vee H$ is also palindromic, at most one of $G - G'$ and $H - H'$ can have an odd number of vertices by Theorem 1.2. Hence (iv) follows. □

Note that the *wheel graph* is defined as the join $K_1 \vee C_n$. With this in mind, the next corollary follows from Theorem 2.1 and Theorem 4.2.

Corollary 4.3. *The wheel $K_1 \vee C_n$ is palindromic if and only if n is even.*

We now turn our attention to the Cartesian product. The Cartesian product of graphs G and H is the graph with vertex set $\{(g, h) : g \in V(G), h \in$

$V(H)\}$ and (g, h) is adjacent to (g', h') if and only if either $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. This graph is denoted $G \square H$. For more information on the Cartesian product as well as other graph products, refer to [6, 7].

Theorem 4.4. *If G and H are both palindromic graphs, then $G \square H$ is palindromic. If G is a palindromic graph with an even number of vertices, then $G \square H$ is palindromic.*

Proof. Suppose that G and H are both palindromic with associated automorphisms ϕ and θ, respectively. Consider the mapping $\xi : V(G \square H) \to V(G \square H)$ defined by $\xi((g, h)) = (\phi(g), \theta(h))$. Note that

$$\xi^2((g, h)) = (\phi^2(g), \theta^2(h)) = (g, h).$$

Further, the vertex (g, h) is a fixed point under ξ if and only if $\phi(g) = g$ and $\theta(h) = h$. Hence, $G \square H$ has at most one fixed point under ξ.

It suffices to check that ξ is a graph automorphism on $G \square H$. Suppose that (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$. If $h_1 = h_2$, then $\theta(h_1) = \theta(h_2)$. Further, g_1 and g_2 would be adjacent in G. It follows that $\phi(g_1)$ and $\phi(g_2)$ are adjacent in G. Since $\xi((g_1, h_1)) = (\phi(g_1), \theta(h_1))$ and $\xi((g_2, h_2)) = (\phi(g_2), \theta(h_2))$, it follows that $\xi((g_1, h_1))$ and $\xi((g_2, h_2))$ are adjacent in $G \square H$. A similar argument holds if $g_1 = g_2$ and h_1 is adjacent to h_2 in H. Thus, ξ is the required automorphism on $G \square H$.

Suppose that G is a palindromic graph with an even number of vertices. Again, let ϕ be the associated automorphism on G. Define $\xi : V(G \square H) \to V(G \square H)$ by $\xi((g, h)) = (\phi(g), h)$. Since G has an even number of vertices, $\phi(g) \neq g$ for all $g \in V(G)$. The remaining details of showing that ξ is the required automorphism follows in a similar manner to above. □

Theorem 4.4 allows us to show additional families of graphs are palindromic. Note that the n-dimensional hypercube Q_n is defined recursively by $Q_1 = P_2$ and $Q_n = Q_{n-1} \square P_2$ for $n \geq 2$. Hence the next corollary follows immediately from Theorem 2.1 and Theorem 4.4.

Corollary 4.5. *The n-dimensional hypercube Q_n is palindromic.*

Whether the conditions given in Theorem 4.4 are necessary for a Cartesian product to be palindromic is unknown at this time.

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