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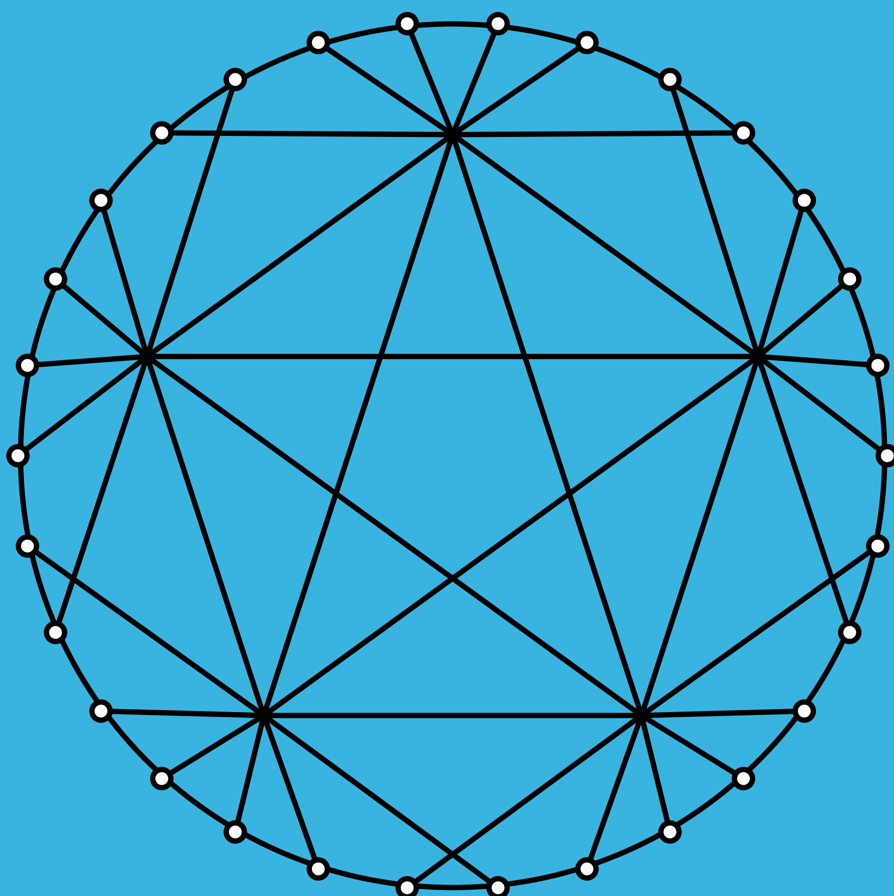

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Greedy triple systems

Alexander Rosa

McMaster University, Ontario, Canada
rosa@mcmaster.ca

Abstract: We determine the number of triples in the maximal partial triple system on n elements formed by a greedy algorithm.

1 Introduction

A *Steiner triple system* of order n (STS(n) is a pair $(V, \mathbf{B})$ where V is an n-set and $\mathbf{B}$ is a collection of 3-subsets of V called *triples* such that every 2-subset of V is contained in exactly one triple. If the word "exactly" is replaced with "at most" then $(V, \mathbf{B})$ is a *partial triple system* of order n (PTS(n)). The *leave* of a partial triple system is the graph $(V, \mathbf{E})$ with vertex-set V whose edges are those pairs of elements of V which are *not* contained in any triple of $\mathbf{B}$.

A PTS is *maximal* (MPT) if its leave is triangle-free. A PTS(n) $(V, \mathbf{B})$ is *maximum* (MPTS) if there exists no PTS(n) having more triples than $\mathbf{B}$.

It is well known [1] that an STS(n) exists if and only if $n \equiv 1$ or 3 (mod 6); clearly, the leave of an STS contains no edges. When $n \equiv 2$ or 4 (mod 6) the leave of a maximum PTS(n) is a 1-factor on V, when $n \equiv 5$ (mod 6) the leave of a maximum PTS(n) consists of a quadrangle (plus isolated vertices), and when $n \equiv 4$ (mod 6), the leave of any maximum PTS(n) is a factor consisting of a claw $K_{1,3}$ and isolated edges (cf., e.g., [1]).

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At the other extreme, the smallest (minimum) maximal PTS(n) has the least possible number of triples of a maximal PTS(n). This number, while more difficult to determine, was nevertheless determined exactly by Novak [5] (cf. [1]). When $n \equiv 2$ or 6 (mod 12), this number is exactly one-half of the number of triples in an MPT(n), in the remaining cases it is slightly more; for the exact number, see Theorem 9.8 in [1].

We note that, in fact, the complete spectrum for maximal partial triple systems is also known. It was determined almost completely by Severn [6] who left only a few cases unsettled; these were later settled in [2] completing the determination of all possible sizes of maximal partial triple systems (cf. Theorem 9.10 in [1]).

2 Greedy triple systems

Let $N = \{1, 2, \ldots, n\}$, let $\binom{N}{3}$ be the collection of all 3-subsets of N, and let the elements of N and those of $\binom{N}{3}$ be ordered lexicographically.

For every natural n, form now a PTS(n) $(N, \mathbf{B})$ greedily as follows: include into $\mathbf{B}$ repeatedly the lexicographically smallest element b of $\binom{N}{3}$ subject to the condition that no 2-subset of b is already contained in an element of $\mathbf{B}$; stop when this is no longer possible.

What results is the (uniquely determined) *greedy triple system* GTS(n) of order n, $(N, \mathbf{B})$. By definition, GTS(n) is a maximal PTS(n).

When is GTS(n) a *maximum* PTS? It was observed by many people (cf. [3]) that GTS(n) is an STS(n) if and only if $n = 2^m - 1$ for some m, and that in this case the STS(n) obtained is actually PG($m-1, 2$). It is also easy to see that when $n = 2^m - 2$ for some m, then GTS(n) is a maximum PTS(n), and the latter (with $n \equiv 0$ or 2 (mod 6)) is the "punctured" PG($m-1, 2$). It can be easily deduced from what follows that when $n \geq 6$, in no other case our GTS(n) is a maximum PTS.

The following question does not seem to have been treated in the literature: what is the size and the structure of the GTS(n) when n is "between" 2^m and 2^{m+1}?

First of all, it is easily observed that if $(N, \mathbf{B})$ is GTS(n) and $(N', \mathbf{B}')$ is GTS($n+1$) then $\mathbf{B} \subseteq \mathbf{B}'$. It follows that in any infinite sequence

$(\mathrm{GTS}(n))_i^\infty$, a GTS($k$) contains all GTS($j$) with $i \le j \le k$.

In particular, GTS(2^n) has the same triples as GTS($2^n - 1$) but contains one extra element that occurs in no triple.

So let $n = 2^m + t,\ \ t \in \{0, 1, \ldots, 2^m - 1\}$, thus $2^m \le n < 2^{m+1}$.

We have the following result concerning the number of triples in a GTS(n).

Theorem. Let $(N, \mathbf{B})$ be the GTS(n) with $n = 2^m + t,\ 0 \le t \le 2^m - 1$. Then

$$|\mathbf{B}| = (2^m - 1)(2^m - 2)/6 + \binom{t+1}{2}.$$

Proof. First we show that when $t < 2^m - 1$, the number of independent edges in the leave of GTS(n) is $t + 1$.

The leave of GTS(2^m) is the star $K_{1,2^m-1}$, and the leave of GTS($2^m + 1$) is the complete bipartite graph $K_{2,2^m-1}$; the number of independent edges in the leave is clearly 1, and 2, respectively. Now we proceed by induction. When $n < 2^{m+1} - 2$, maximum matching in the leave of GTS(n) is not a perfect matching, and so the leave contains a vertex x not in the maximum matching, thus the edge $\{x, n+1\}$ enlarges the maximum matching in GTS($n+1$) by one.

The proof of the theorem now follows by easy induction. □

One obvious corollary is that the GTS(n) can be embedded into PG($m - 1, 2$), where m is the smallest integer such that $n \le 2^m - 1$, and into no smaller STS.

Where does GTS(n) fit in the spectrum of maximal partial triple systems of order n? For example, when $n = 42$, any maximum PTS contains 280 triples, any smallest maximal PTS contains 140 triples, and the GTS(42) contains 210 triples, the exact average of the two.

On the other hand, for each of the intervals $[8, 15], [15, 31]$, and $[31, 63]$ it happens that for exactly four orders n in this interval is the number of triples in GTS(n) less than this average; in all other cases it is more. Is there a pattern here?

3 More on greedy triple systems

The automorphism group of GTS(n) acts 2-transitively on elements when $n = 2^m - 1$ and acts transitively when $n = 2^m - 2$; this is obvious since GTS in these cases is PG($m - 1, 2$)), and the punctured projective space, respectively. When $n \geq 4$, in no other case is the automorphism group of GTS(n) transitive.

What can one say about colouring GTSs? Both the chromatic number χ and the chromatic index χ' (for definitions, see [1]) of GTS(n) increase monotonically with n. Since the sequence of GTS(n)s includes the projective spaces PG($d, 2$), this alone indicates the difficulty of determining these parameters for individual GTSs.

The chromatic number χ of the projective spaces PG($d, 2$) remains undetermined. Since $\chi(PG(3,2))$, $\chi(PG(4,2))$, $\chi(PG(5,2))$ is known to equal $3, 4$ and 5, respectively, it has been conjectured that the chromatic number of PG($d, 2$) increases by 1 when dimension increases by 1. However, this was brilliantly disproved by A. Blokhuis (unpublished) who showed that the growth in chromatic number is slower.

Concerning the chromatic index χ', here the situation is somewhat better, because $\chi'(PG(d,2))$ has been recently completely determined: for d odd, it is well known that PG($d, 2$) is resolvable, and so its chromatic index χ' equals $2^d - 1$. On the other hand, for d even $\chi'(PG(d,2))$ was determined by Meszka [4]: when $d > 2$, $\chi'(PG(d,2)) = 2^d + 2$. So, for example, for each GTS(n) with $n \in \{7, 8, 9, 10, 11, 12, 13, 14, 15\}$ we have $\chi'(GTS(\mathrm{n}) = 7$, because $\chi'(PG(2,2))$ and $\chi'(PG(3,2))$ both equal 7. For an increase of 16 between orders 15 and 31, we have an increase of 11 (from 7 to 18) in chromatic index. Similarly, for an increase of 32 between orders 31 and 63, there is an increase of 13 (from 18 to 31) in chromatic index. But to determine exactly the chromatic index χ' of GTS(n) for individual orders n remains a challenge.

References

[1] C.J. Colbourn and A. Rosa, *Triple Systems*, Oxford Univ. Press, 1999.

[2] C.J. Colbourn, A. Rosa, and Š. Znám, The spectrum of maximal partial Steiner triple systems, *Des. Codes Cryptogr.*, **3** (1993), 209–219.

[3] A. Hartman and Z. Yehudai, Greedesigns, *Ars Combin.*, **29C** (1990), 69–76.

[4] M. Meszka, The chromatic index of projective triple systems, *J. Combin. Designs*, **21** (2013), 531–540.

[5] J. Novák, Edge bases of complete uniform hypergraphs, *Mat. Čas.*, **24** (1974) 43–57.

[6] E.A. Severn, *Maximal Partial Steiner Triple Systems*, Ph.D. Dissertation, Univ. of Toronto, 1984.