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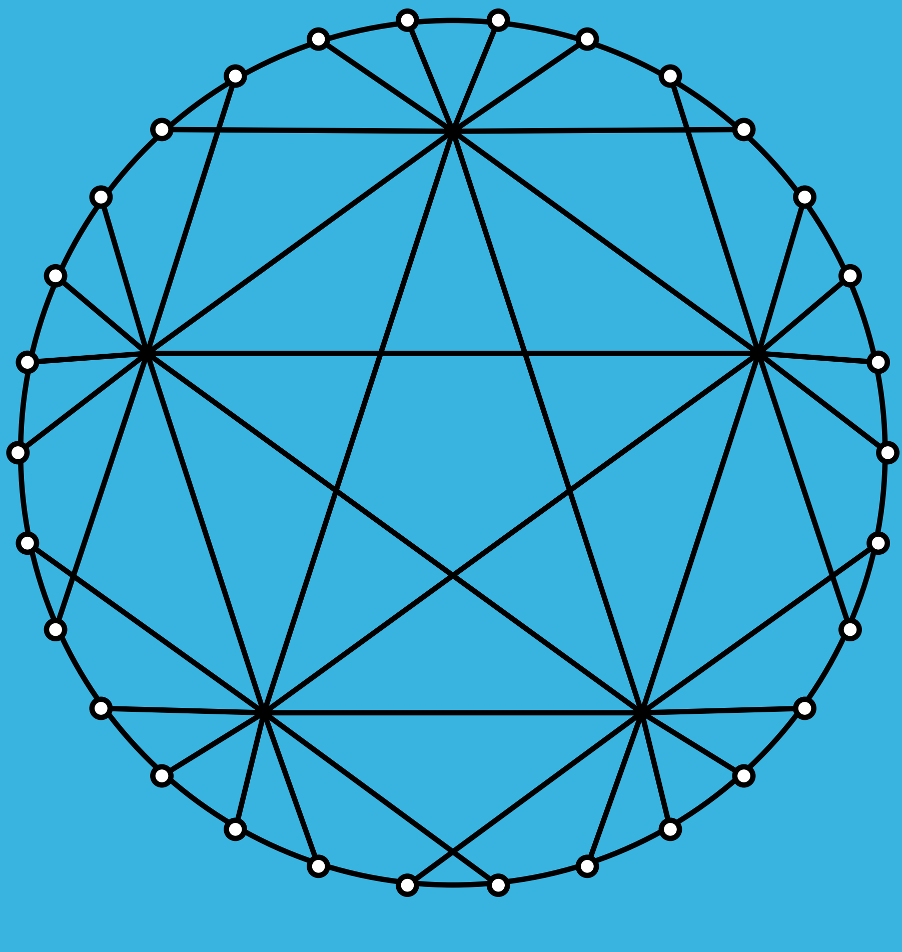

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Edge Grundy numbers of the regular Turán graphs

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Abstract: Suppose $r \geq 3$ and $k \geq 2$ are integers. If $k(r-1)$ is even then the edge Grundy number of the Turán graph $T_{kr,r}$ is $2k(r-1)-1$. Otherwise, the edge Grundy number of $T_{kr,r}$ is $2k(r-1)-m$ for some $m \in \{1,\ldots,\min[g(k),2r-1]\}$, where $g(k)=2$ if $k=3$ and $g(k)=k-2$ otherwise.

1 Introduction

A *Grundy coloring* of a finite simple graph G is a proper coloring of the vertices of G with positive integers such that if $v \in V(G)$ is colored with $c > 1$ then all the colors $1,\ldots,c-1$ appear on vertices adjacent to v. (As usual, *proper* means that adjacent vertices bear different colors.) The

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Grundy number of G, denoted $\Gamma(G)$, is the greatest number of colors in a Grundy coloring of G.

A *greedy coloring* of a G is a coloring of the vertices of G, with positive integers, obtained as follows: order $V(G)$ somehow, say $v_1, \ldots, v_n$, and color v_1 with 1; having colored $v_1, \ldots, v_{j-1}$, $1 < j \leq n$, color v_j with the smallest positive integer not appearing on neighbors of v_j among $v_1, \ldots, v_{j-1}$.

The following are basic results about Grundy colorings and Grundy numbers; see [2] and [3]. As usual, $\chi(G)$ denotes the chromatic number of G and $\Delta(G)$ the maximum degree in G.

1.1 Every greedy coloring of G is Grundy, and every Grundy coloring of G is greedy, meaning that for each Grundy coloring of G there is an ordering of $V(G)$ the greedy coloring with respect to which is the given Grundy coloring.

    A corollary: any *partial Grundy coloring* of G, an assignment of positive integers to some of the vertices of G so that whenever a vertex v bears a color $c > 1$ then the colors $1, \ldots, c-1$ appear on neighbors of v, and no two adjacent vertices bear the same color, can be extended to a Grundy coloring of G: order the vertices of G so that the colored vertices precede all the uncolored vertices, and so that the sequence of colors on the colored vertices is non-decreasing. Then the greedy coloring of $V(G)$ with respect to this ordering will reproduce the given partial coloring.

1.2 $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$.

1.3 If k is an integer satisfying

$$\chi(G) \leq k \leq \Gamma(G)$$

    then there is a Grundy coloring of G with k being the largest integer appearing.

1.4 If H is an induced subgraph of G then every Grundy coloring of H can be extended to a Grundy coloring of G (see 1.1) and so $\Gamma(H) \leq \Gamma(G)$.

1.5 For any Grundy coloring of G, the set of vertices colored 1 is an independent dominating (maximal independent) subset of $V(G)$.

The line graph of a graph G is the graph $L(G)$ such that $V(L(G)) = E(G)$ and distinct edges e, $f \in E(G)$ are adjacent in $L(G)$ if and only if they are adjacent as edges in G; i.e., they are both incident to some vertex. An edge Grundy coloring of G is, or corresponds to, a Grundy coloring of $L(G)$. As with any edge coloring of G, such a coloring can be thought of as a coloring of the edges of G or as a coloring of the vertices of $L(G)$; the former is usually preferred.

The edge Grundy number of G, provided $E(G) \neq \emptyset$, is $\Gamma'(G) = \Gamma(L(G))$. Let $\chi'(G) = \chi(L(G))$, the *edge-chromatic number* or *chromatic index* of G. Let d_G denote vertex degree in G, and let $\Delta'(G) = \Delta(L(G)) = \max_{uv \in E(G)}[d_G(u) + d_G(v) - 2]$. From the elementary results 1.1 - 1.5 about Grundy colorings and Γ we have the following for any finite simple graph G such that $E(G) \neq \emptyset$.

1.1′ Each edge Grundy coloring of G is obtainable by greedily coloring the edges with respect to an ordering of $E(G)$. Any partial edge Grundy coloring of G can be extended (greedily) to an edge Grundy coloring of G.

1.2′ $\chi'(G) \leq \Gamma'(G) \leq \Delta'(G) + 1 = \max_{uv \in E(G)}[d_G(u) + d_G(v) - 1]$

1.3′ For any integer $k \in [\chi'(G), \Gamma'(G)]$ there is an edge Grundy coloring of G with k being the largest color appearing.

1.4′ If H is a subgraph of G then every edge Grundy coloring of H can be extended to an edge Grundy coloring of G. Consequently, $\Gamma'(H) \leq \Gamma'(G)$.

1.5′ For any edge Grundy coloring of G, the set of edges colored 1 is an edge-dominating (maximal) matching in G.

It has proved to be not too easy to evaluate χ', Γ, and now Γ', even at common garden-variety graphs. For instance, the values of $\chi'(G)$ as G ranges over the complete multipartite graphs was not known until 1992, when the problem was solved in [5]. In [4] the values $\Gamma(Q_n)$, $n = 1, 2, 3, \ldots$, where Q_n is the n-cube, are determined. In [1] the values $\Gamma'(G)$ are determined (with a few small exceptions) when G is a grid, cylindrical grid, toroidal grid, cube, complete bipartite graph, or complete graph. Of these, the complete bipartite and the complete graphs were the most challenging. Because we will be using the results in these cases here, we state the results for those cases. In the following, m and n are integers.

Theorem 1.1 (Anderson et.al. [1]). *If $1 \leq m < n$ then $\Gamma'(K_{m,n}) = \Delta'(K_{m,n}) + 1 = m + n - 1$.*

If $n > 1$, $\Gamma'(K_{n,n}) = 2n - 2$.

Theorem 1.2 (Anderson et.al. [1]). *If* $n \geq 3$ *is odd, then* $\Gamma'(K_n) = \Delta'(K_n) + 1 = 2n - 3$.

If $n \geq 6$ *is even, then* $\Gamma'(K_n) = 2n - 4$.

Finally, $\Gamma'(K_1) = 0$, $\Gamma'(K_2) = 1$, *and* $\Gamma'(K_4) = 3$.

We would like very much to extend these results by determining $\Gamma'(G)$ for all complete multipartite graphs G, but this goal seems very far off, at present. In the next section we will state and prove what we know about the regular complete multipartite graphs. These have, for some positive integers r and k, r parts with k vertices per part. They are also known as the regular Turán graphs; such a graph is denoted $T_{kr,r}$. In view of Theorems 1.1 and 1.2, we may as well confine our attention to pairs (k, r) satisfying $r \geq 3$ and $k \geq 2$.

Note for future reference: $|V(T_{kr,r})| = kr$, $|E(T_{kr,r})| = \frac{r(r-1)}{2}k^2$, $\Delta(T_{kr,r}) = k(r-1)$, and $\Delta'(T_{kr,r}) + 1 = 2k(r-1) - 1$.

2 $\Gamma'(T_{kr,r})$

Throughout, $k \geq 2$ and $r \geq 3$ will be integers, and $A_1, \ldots, A_r$ will be the parts of $T_{kr,r}$; so $|A_i| = k$, $i = 1, \ldots, r$.

Theorem 2.1. *If* $k(r-1)$ *is even, then* $\Gamma'(T_{kr,r}) = \Delta'(T_{kr,r}) + 1 = 2k(r-1) - 1$.

Proof. Since $\Gamma'(T_{kr,r}) \leq \Delta'(T_{kr,r}) + 1$, it suffices to prove the existence of a Grundy edge coloring of $G = T_{kr,r}$ with $2k(r-1) - 1$ appearing as a color.

Let H be the subgraph of G induced by $A_1 \cup \cdots \cup A_{r-1}$. Because $k(r-1) = |V(H)|$ is even, and $H \simeq T_{k(r-1),r-1}$ is a complete multipartite graph,

$$\chi'(H) = \Delta(H) = k(r-2)$$

by the main result of [5]. Let the edges of H be properly colored with $1, \ldots, k(r-2)$. Note that for each $v \in V(H)$, all of these colors appear "at" v—meaning, on the $k(r-2)$ edges of H incident to v.

The edges of G that remain to be colored are the edges of $X \simeq K_{k,k(r-1)}$, the complete bipartite graph with bipartition A_r, $A_1 \cup \cdots \cup A_{r-1}$. Because $r-1 \geq 2$, by Theorem 1.1 there is an edge Grundy coloring of X with the color $k + k(r-1) - 1 = kr - 1$ appearing. Add $k(r-2)$ to the color on each edge of X. Putting this new edge coloring of X together with the edge coloring of H results in an edge Grundy coloring of G with the color $k(r-2) + kr - 1 = 2k(r-1) - 1$ appearing on some edge. □

Theorem 2.2. *If $k(r-1)$ is odd then $\Gamma'(T_{kr,r}) \geq 2k(r-1) - g(k)$, where $g(3) = 2$ and $g(k) = k-2$, $k \geq 5$.*

Proof. Clearly it suffices to show that there is a partial edge Grundy coloring of $G = T_{kr,r}$ in which $2k(r-1) - g(k)$ appears.

As in the proof of Theorem 2.1, let the parts of G be $A_1, \ldots, A_r$, and let $H = G - A_r \simeq T_{k(r-1),r-1}$. Since $\Delta(H) = k(r-2)$, by Vizing's theorem ($\chi'(H) \leq \Delta(H) + 1$) and a main result of [6], there is a proper *equalized* edge coloring of H with $k(r-2)+1$ colors. "Equalized" means that the most frequently occurring color appears on at most one more edge than does the least frequently occurring color. Consider H to be so colored.

Since the average number of appearances of a color in this coloring is

$$\frac{|E(H)|}{k(r-2)+1} = \frac{\frac{k^2(r-1)(r-2)}{2}}{k(r-2)+1} = \frac{k(r-1)-1}{2} - \frac{k-1}{2[k(r-2)+1]},$$

it follows that the frequencies of the colors on the edges of H are $\frac{k(r-1)-1}{2}$ and $\frac{k(r-1)-1}{2} - 1$. Let a be the number of colors that each appear on exactly $\frac{k(r-1)-1}{2} - 1$ of the edges of H, and let b be the number of colors that each appear on $\frac{k(r-1)-1}{2}$ edges of H. Let A be the first set of colors, and B the latter set of colors, so $|A| = a$ and $|B| = b$.

Solving the two equations

$$a + b = k(r-2) + 1$$

and

$$a\left[\frac{k(r-1)-1}{2} - 1\right] + b\frac{k(r-1)-1}{2} = \left|E(H)\right| = \frac{k^2(r-1)(r-2)}{2}$$

for a and b, we obtain

$$a = \frac{k-1}{2} \text{ and } b = k(r-1) - \frac{3}{2}(k-1).$$

Let the colors be named so that

$$\begin{aligned} B = & \quad \{1, \ldots, b\} = \{1, \ldots, k(r-1) - \tfrac{3}{2}(k-1)\} \text{ and} \\ A = & \quad \{b+1, \ldots, b+a\} = \{k(r-1) - \tfrac{3}{2}(k-1) + 1, \ldots, k(r-2)+1\}. \end{aligned}$$

Each color class in the edge coloring of H is a matching. Therefore, the number of vertices at which each color appears is twice the number of edges on which the color appears. Therefore each $c \in B$ appears at $2 \cdot \frac{k(r-1)-1}{2} = k(r-1) - 1$ vertices of H; that is, c appears at all but one vertex of H.

Similarly, each $c \in A$ appears at $k(r-1) - 3$ vertices of H, missing exactly 3 vertices.

Meanwhile, since $k(r-2)+1$ colors appear, and each vertex of H has degree $k(r-2)$ in H, it follows that at each vertex of H, exactly one color is missing. Therefore, for each color in B, at the single vertex of H where that color is missing, all the other colors in $A \cup B$ appear, and for each color in A, at each of the 3 vertices of H where that color is missing, all the other colors are present.

Let $S_B \subseteq V(H)$ be the set of b vertices of H corresponding to the b colors of B; at each $v \in S_B$ some $c \in B$ is missing from the edges incident to v. Let $S_A = V(H) \backslash S_B$. Then S_A can be partitioned into a 3-sets (triples) corresponding to the a colors of A.

Suppose $k = 3$. Let $A_r = \{u, v, w\}$. Since $a = \frac{k-1}{2} = 1$, S_A is a single triple, $\{x, y, z\}$, at each vertex of which the single color in A, $3(r-2)+1 = 3(r-1)-2$, is missing in the edge coloring of H. Color the edges ux, vy, wz with that color.

For each vertex $q \in S_B$, color wq with the (single) color missing at q in the edge coloring of H. At this point all colors $1, \ldots, 3(r-2)+1 = 3(r-1)-2$ appear at each vertex of H, and only those colors.

The $3(r-1)-1$ edges of G incident to u that have not yet been colored will be colored with integers from the set $\{3(r-1)-1, \cdots, 6(r-1)-2\} \backslash \{6(r-1)-3\}$, and the $3(r-1)-1$ edges of G incident to v that have not yet been colored will be colored with integers from the set $\{3(r-1)-1, \ldots, 6(r-1)-3\}$. The assignment of colors to the edges must satisfy two requirements:

1. For some vertex $q \in V(H)\backslash\{x, y\}$, uq is colored $6(r-1)-2$, and vq is colored $6(r-1)-3$.

2. For no $q \in V(H)$ will the color on uq be the same as the color on vq.

Clearly there will be no great difficulty in satisfying these requirements. The result will be a partial edge Grundy coloring of $G = T_{3r,r}$ in which $6(r-1)-2 = 2 \cdot 3(r-1) - g(3)$ appears. This proves the claim of the theorem in the case $k = 3$.

Now suppose that $k \geq 5$, and let u, v, w, x, and y be distinct vertices in A_r. For a color $c \in A$ let $\{h(c), q(c), z(c)\}$ be the triple of vertices in S_A at which c is missing in the edge coloring of H. Color $wh(c)$, $xq(c)$ and $yz(c)$ with c and let this be done for every $c \in A$. Also, for each vertex $p \in S_B$, color wb with the color missing at p in the edge coloring of H. At this point, all colors $1, \ldots, k(r-2)+1 = k(r-1)-k+1$ appear at each vertex of H, and only those colors appear anywhere in G.

Now we will color the edges incident to u with

$$\{k(r-1)-k+2, \ldots, 2k(r-1)-k+2\}\backslash\{2k(r-1)-k+1\}$$

and the edges incident to v with $\{k(r-1)-k+2, \ldots, 2k(r-1)-k, 2k(r-1)-k+1\}$, constrained by:

1. For some vertex $q \in V(H)$, uq is colored $2k(r-1)-k+2$ and vq is colored $2k(r-1)-k+1$.

2. For no $q \in V(H)$ will the color on uq be the same as the color on vq.

The result is a partial edge Grundy coloring of G with the color $2k(r-1)-k+2 = 2k(r-1)-g(k)$ appearing. □

Corollary 2.3. *If $k(r-1)$ is odd and g is as in Theorem 2.2, then*

$$\Gamma'(T_{kr,r}) \geq \max[2k(r-1)-g(k), 2k(r-1)-(2r-1)].$$

Proof. By Theorem 2.1, since $k-1$ is even, $\Gamma'(T_{(k-1)r,r}) = 2(k-1)(r-1)-1 = 2k(r-1)-(2r-1)$. Since $T_{(k-1)r,r}$ is a subgraph of $T_{kr,r}$, the result follows. □

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