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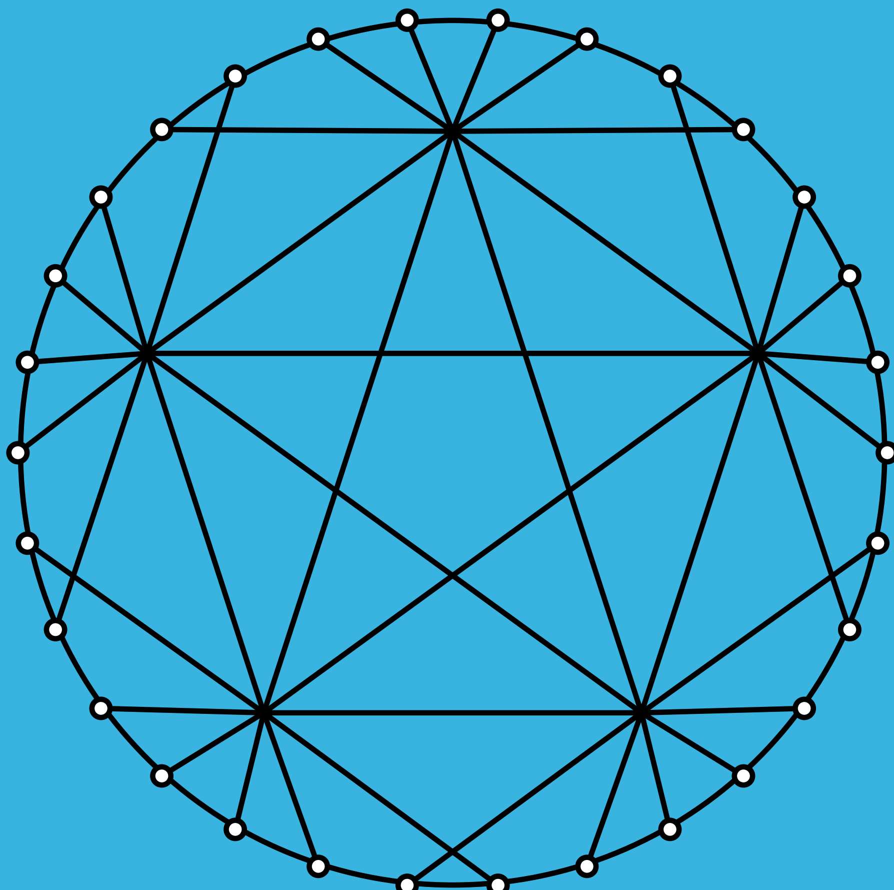

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Skolem circles

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Abstract: Skolem sequences and Skolem labelled graphs have been described and examined for several decades. This paper explores the weak Skolem labelling of cycle graphs, which we call Skolem circles. The relationship between Skolem sequences and Skolem cirlces is explored, and Skolem circles of small sizes are enumerated, with some loose general bounds established.

1 Introduction

A Skolem-type sequence of order m is a sequence $S = (s_1, s_2, ..., s_{2m})$ of positive integers taken from a set D with $|D| = m$ such that for each $s \in D$, there are exactly two subscripts $i, j \in [1, 2m]$ such that $s_i = s_j$ and $|i - j| = s$. If D is the set $\{1, 2, 3, \dots, m\}$ then S is a *Skolem sequence* or order m, if the set $D = \{d, d+1, \dots, d+m-1\}$ then S is a *Langford sequence* of order m and defect d [16]. A sequence $S' = (s_1, s_2, ..., s_{2m+1})$ is an *extended Skolem-type sequence* when it contains a null element often denoted as 0. A Skolem-type sequence is k *extended* if it contains a null symbol its k^{th} position. If the null element appears in the penultimate position of the sequence the null may be called a *hook*. We consider (0) to be the extended Skolem-type sequence of order 0.

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Skolem and Langford sequences were first introduced by Langford in [9] and Skolem [19] in the 1950s. Skolem and Langford sequences have been used to construct Steiner Triple Systems [3, 19], difference sets [2, 15] and have many applications in graph theory [5, 7, 13] and cryptography [8].

There are many Skolem like sequences, each interesting on its own or for particular applications [6, 16] and many interesting variations [17]. The variation of interest here is a Skolem labelling of a graph.

A *Skolem labelled graph* [12] is a graph G with $2m$ nodes, each node having a label from the set $\{1, 2, \ldots, m\}$ such that

1. each label appears exactly twice,
2. for any two nodes v_1, v_2, which have the same label, x, the length of the shortest path from v_1 to v_2 is x,
3. removing any edge from the graph violates criterion 2.

Criterion 2 is called the *Skolem property.* In this paper we are interested in Skolem labelling of cycle graphs. However we are also interested in the cases that violate criterion 3. A *weak Skolem labelled graph* is a labelling of a graph which meets criterion 1 and 2 of a Skolem labelled graph, but may not meet criterion 3.

A Skolem sequence of order m may be represented as a sequence of $2m$ symbols or as a set of m ordered pairs $\{(a_1, b_1), (a_2, b_3), \ldots, (a_m, b_m)\}$ where (a_i, b_i) are the locations of the symbol i in the sequence with $a_i < b_i$, thus $b_i - a_i = i$. The Skolem sequence $(1, 1, 4, 2, 3, 2, 4, 3)$ may be represented by the *Skolem pairs* $\{(1, 2), (4, 6), (5, 8), (3, 7)\}$.

We consider a variation where the arrangement is a circle rather than a sequence. A *Skolem circle* may be represented as a labelling of a cycle graph on $2m$ nodes as in Figure 1, or as a set of ordered pairs

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\},$$

where $b_i - a_i \equiv i \mod 2m$. Note that $a_i > b_i$ if and only if $b_i \leq i$. A Skolem circle may be suitable to use in applications where a finite cyclic group is useful.

In Section 2 we list some known results on Skolem sequences and the analogous results for Skolem circles. In Section 3 we investigate the structure

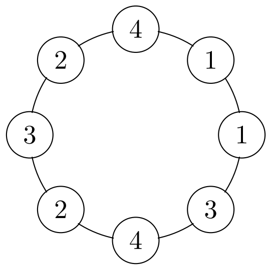

Figure 1: A weak Skolem labelling of a cycle graph on 8 nodes, also called a Skolem circle on 4 symbols. There are three different edges that could be removed to form a Skolem labeling, hence this Skolem circle contains 6 distinct Skolem sequences.

of Skolem Circles showing how they can be broken into subsequences of Skolem-type. Computational techniques are used in Section 4 to enumerate distinct Skolem circles of small orders.

2 Skolem circles

A *cycle graph* is a graph on n nodes that has n edges arranged in a cycle of length n. A *Skolem circle* of order m is a weak Skolem labelled cycle graph with $2m$ nodes. A Skolem circle is a cycle graph, G, with $2m$ nodes, each node having a label from the set $\{1, 2, \ldots, m\}$ such that

1. each label appears exactly twice
2. for any two nodes v_1, v_2 which have the same label, x, the length of the shortest path from v_1 to v_2 is x.

A Skolem circle may be represented as a labelling of a cycle graph on $2m$ nodes as in Figure 1, or as a set of ordered pairs

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\},$$

where

$$b_i - a_i \equiv i \mod 2m.$$

The inspiration for the definition of Skolem Circle comes from considering Skolem sequences using modular arithmetic, rather than integer arithmetic.

A Skolem sequence may be wrapped around a cycle to form a Skolem circle. For example the sequence (1, 1, 4, 2, 3 ,2 ,4 ,3) can be wrapped around to form the circle in Figure 1.

Skolem sequences have been studied for a few decades now, there is quite a lot known about them. Some of the properties of Skolem sequences have analogous properties of Skolem circles.

Theorem 2.1 (Skolem [19]). *A Skolem Sequence of order m exists if and only if $m \equiv 0, 1 \mod 4$.*

In [12] it is shown that Skolem labelled cycle graphs exist for every cycle on $2m$ nodes if and only if $m \geq 8$ and $m \equiv 0, 1 \mod 4$. Those Skolem circles which are weak Skolem labelled graphs (and not Skolem labelled graphs) can be constructed by wrapping a Skolem sequence around the circle graph, thus the possible orders of Skolem circles are the same as the possible orders of Skolem sequences.

Theorem 2.2. *A Skolem circle of order m exists if and only if $m \equiv 0, 1 \mod 4$.*

Some Skolem circles contain several Skolem sequences. The Skolem circle in Figure 1 contains six distinct Skolem sequences. There are three anticlockwise sequences $(1, 1, 4, 2, 3, 2, 4, 3)$, $(4, 2, 3, 2, 4, 3, 1, 1)$, $(2, 3, 2, 4, 3, 1, 1, 4)$, and three clockwise sequences $(3, 4, 2, 3, 2, 4, 1, 1)$, $(1, 1, 3, 4, 2, 3, 2, 4)$, $(4, 1, 1, 3, 4, 2, 3, 2)$.

A circle is a highly symmetric arrangement of vertices. Fixing a positional labelling on a circle can be useful to discuss aspects of the circle. A positional labelling of the circle contains the symbols $\{1, 2, \ldots, 2m\}$ in lexicographic order, beginning with any vertex, and moving either clockwise or anticlockwise. Allowing travel around a circle clockwise or anticlockwise and allowing any beginning vertex we consider Skolem sequences to be *circle equivalent* if they form the same Skolem circle allowing for cyclic shifts and reversal of direction. Skolem circles

$$C_1 = \big\{(a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m)\big\}$$

and

$$C_2 = \big\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)\big\}$$

are circle equivalent if there exists x such that $(a_i + x, b_i + x) = (\alpha_i, \beta_i) \mod 2m$ for all $1 \leq i \leq m$ or $(x - a_i, x - b_i) = (\alpha_i, \beta_i) \mod 2m$ for all $1 \leq i \leq m$.

For ease of comparing Skolem circles we consider a Skolem circle to be given the *standard positional labelling* if $(a_1, b_1) = (1, 2)$ and $3 \leq a_2 \leq m$. Graphs in this paper are drawn to represent the labelling with vertex 1 at an angle of 0 from the horizontal, and a vertex $x + 1$ at an angle of $x\pi/m$. The Skolem circles in Figures 1 and 2 are both shown in the standard positional labelling. The standard positional labelling also allows representation of a Skolem circle as a sequence. The Skolem circle of Figure 1 may be represented as the sequence $(1, 1, 4, 2, 3, 2, 4, 3)$. The six Skolem sequences:

$$\begin{array}{lll} (1,1,4,2,3,2,4,3), & (4,2,3,2,4,3,1,1), & (2,3,2,4,3,1,1,4), \\ (3,4,2,3,2,4,1,1), & (1,1,3,4,2,3,2,4), & (4,1,1,3,4,2,3,2) \end{array}$$

are circle equivalent. This notion of circle equivalence will be used in Section 4 to put a lower bound on the number of distinct Skolem circles.

There exists Skolem circles which cannot be constructed by wrapping a Skolem sequence around a cycle. Figure 2 shows a Skolem labelling of a cycle graph with 16 vertices. It is a Skolem labelling of a cycle graph, as removal of any edge will violate the Skolem property.

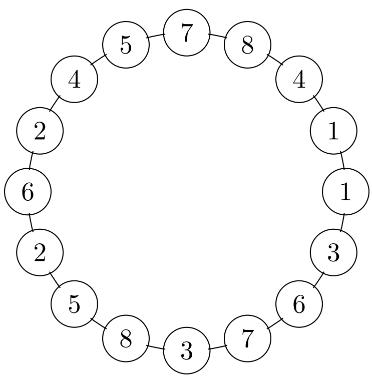

Figure 2: A Skolem labelling of a cycle graph on 16 vertices, which has no corresponding Skolem sequence.

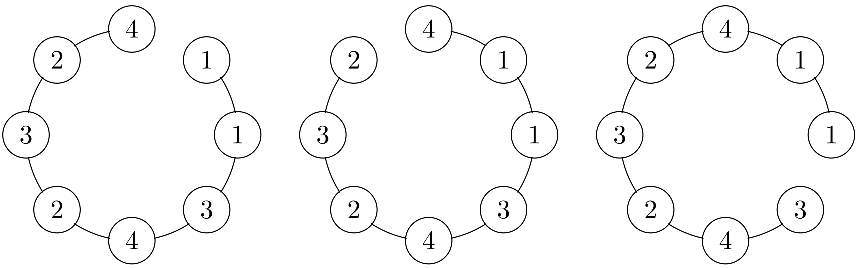

Figure 3: Examining the Skolem circle of Figure 1, there are three different edges that could be removed whilst maintaining a Skolem labelling. Hence this Skolem circle contains 6 distinct Skolem sequences.

3 Structures of Skolem circles

Skolem sequences can contain subsequences which are Skolem-type sequences or extended Skolem type sequences. This is apparent when regarding a Skolem circle as a weak Skolem labelled cycle graph. Removing appropriate edges from a weak Skolem labelled graph creates a Skolem labelled graph, although this graph may no longer be a cycle (see Figure 3), and if more than one edge is removed the Skolem labelled graph is not connected (see Figure 4).

Let C be a Skolem circle of order m to which j different edges may be removed to create $2j$ distinct Skolem sequences, then C is a *j-edge-removable* Skolem circle. A circle with 3 removable edges is shown in Figure 3. When removing all edges at once the resulting graph may be a Skolem labelled graph, or as in case of Figure 4, may form a extended Skolem Labelled graph by treating the symbol m as a null.

The edges of the Skolem labelled graph that may be removed to partition a Skolem circle into Skolem-type (or extended Skolem-type) subsequences are called *removable edges.*

At each removable edge, two Skolem sequences are constructed (clockwise and anticlockwise) see Figure 3. All Skolem sequences create a Skolem circle with at least one removable edge. The Skolem circle in Figure 1 is 3-edge-removable, the anticlockwise subsequences are $(1,1),(4),(2,3,2,4,3)$. The Skolem circle in Figure 2 has 0 removable edges making it a Skolem labelling of a cycle graph.

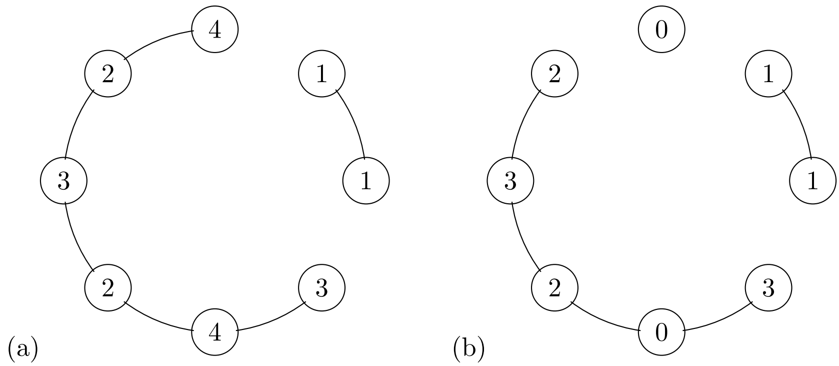

Figure 4: (a) Removing two edges from the Skolem circle of Figure 1 creates a Skolem labelled graph which is not connected. (b) Replacing the symbol 4 with a null, the Skolem circle is partitioned into 3 Skolem-type sequences.

Note that removing an edge from a Skolem circle creates two Skolem sequences, clockwise and anticlockwise. These observations lead to the following result.

Lemma 3.1. *A Skolem circle with j removable edges represents a set of $2j$ circle equivalent Skolem sequences.*

Example 3.2. Construct a 4-edge-removable Skolem circle by pasting together Langford Sequences. Begin with the smallest Langford sequence $(1, 1)$. Next a Langford sequence of defect 2, for example

$$(3, 4, 2, 3, 2, 4).$$

Then a Langford sequence of defect 5, for example

$$(13, 11, 9, 7, 5, 12, 10, 8, 6, 5, 7, 9, 11, 13, 6, 8, 10, 12).$$

Finally a Langford sequence of defect 14, for example

$$(40, 38, 36, 34, 32, 30, 28, 26, 24, 22, 20, 18, 16, 14, 39, 37, 35,\\ 33, 31, 29, 27, 25, 23, 21, 19, 17, 15, 14, 16, 18, 20, 22, 24, 26, 28,\\ 30, 32, 34, 36, 38, 40, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39).$$

These sequences can be concatenated together and wrapped around a cycle to form a 4-edge-removable Skolem circle of order 40.

Example 3.2 can be formalised into a general construction.

Lemma 3.3. *Let $\mathcal{S} = \{S_1, S_2, \ldots, S_j\}$ be a set of j Skolem-type sequences and let X_i be the set of symbols in sequence S_i for $i \in \{1, 2, \ldots, j\}$ such that*

$$\bigcup_i X_i = \{1, 2, 3, \ldots m\} \qquad \text{and} \qquad \bigcap_i X_i = \emptyset,$$

then the concatenation of the sequences in $\mathcal{S}$, wrapped around a cycle forms a j-edge-removable Skolem circle.

Next we find appropriate Skolem type sequences.

Theorem 3.4 (SIMPSON [18])**.** *A Langford Sequence of order m and defect d exists if and only if*

1. $m \geq 2d - 1$,
2. *if d is odd then $m \equiv 0, 1 \mod 4$, and if d is even then $m \equiv 0, 3 \mod 4$.*

Knowing that appropriate Langford sequences exist a general construction can be described.

Theorem 3.5. *Let $j \in \mathbb{N}$, then there exists a j-edge-removable Skolem circle of order $((3^{n+1})/2) - 1$.*

Proof. For $n \geq 0$ let $d_n = (3^n + 1)/2$ and $m_n = 3^n$. With these parameters if n is odd then d_n is even and $m_n \equiv 3 \mod 4$, and if n is even then d_n is odd and $m_n \equiv 1 \mod 4$. Thus the conditions of Theorem 3.4 are satisfied and therefore Langford sequences with appropriate parameters exist.

Let S_n be a Langford sequence of defect d_n and order m_n. Let X_n be the set of symbols contained in S_n. Note that $X_0 = \{1\}, X_1 = \{2, 3, 4\}$. Hence

$$\bigcup_{i=0}^{1} X_i = \{1, 2, 3, 4\} \quad \text{and} \quad \bigcap_{i=0}^{1} X_i = \emptyset.$$

Hence by Lemma 3.3 the concatenation of S_0 and S_1 forms a 2-edge-removable Skolem circle. Now assume for induction that the concatenation of $S_0, \ldots,$ and S_{n-1} is a n-edge-removable Skolem circle with order $d_{n-1} + m_{n-1} - 1$. We have

$$\begin{aligned} d_{n-1} + m_{n-1} - 1 &= (3^{n-1} + 1)/2 + 3^{n-1} - 1 \\ &= (3^{n-1} + 1 + 2 \times 3^{n-1})/2 - 1 = d_n - 1. \end{aligned}$$

Hence

$$\left(\bigcup_{i=0}^{n-1} X_i\right) \cup X_n = \{1,2,3,4\ldots,d_n+m_n-1\} \quad \text{and} \quad \left(\bigcap_{i=0}^{n-1} X_i\right) \cap X_n = \emptyset.$$

So by Lemma 3.3 the concatenation of $S_0,\ldots,S_{n-1},S_n$ is an $(n+1)$-edge-removable Skolem circle. The order of the Skolem circle is the largest value in S_n, and is hence $d_n+m_n-1 = (3^{n+1})/2-1$. By the principle of induction the concatenation of $S_0,\ldots,S_j$ is a $j+1$-edge-removable Skolem circle of order $(3^{j+1})/2-1$. □

We now give an algorithm for a specific construction of the required Langford sequences.

Theorem 3.6. *A Langford sequence, S_j, of order $m = 3^j$ and defect $d = (3^j+1)/2$ can be constructed using the Skolem pairs $(y+1, m+d-y)$ with $y \in \{0,1,2,\ldots,d-1\}$ and $(d+z, 2m-z+1)$ with $z \in \{1,2,\ldots,d-1\}$.*

Proof. Let x be symbols of the Skolem sequence, which also representes the distances of the Skolem pairs.

If x is the same parity as $d+m-1$, then for $y \in \{0,1,2,\ldots,d-1\}$, let $x = d+m-1-2y$ and the Skolem pairs are $(y+1, m+d-y)$. This means that the symbols with the same parity as $d+m-1$ appear in $S_j[i]$ with $i \in [1,d] \cup [m+1, 3(m+1)/2-1]$. If x is the opposite parity as $d+m-1$ then for $z \in \{1,2,d-1\}$ let $x = d+m-2z$ and the Skolem pairs are $(d+z, 2m-z+1)$. This means that the symbols with the opposite parity as $d+m-1$ appear in $S_j[i]$ with $i \in [d+1,m] \cup [3(m+1)/2, 2m]$.

When x is the same parity as $d+m-1$, the gap between the Skolem pairs is

$$m+d-y-(y+1) = d+m-1-2y = x$$

and when the x is the opposite parity, the gap betwean the Skolem pairs is

$$2m-z+1-(d-z) = 2m-d+1-2z = m+d-2z = x.$$

Hence all symbols $x \in [d, d+m-1]$ appear exactly twice at a Skolem distance of x making a Langford sequence. □

Example 3.2 is a 4-edge removable Skolem circle constructed using the algorithm of Theorem 3.6. The algorithm shows that it is possible to construct a j-edge-removable Skolem circle for any $j \in \mathbb{N}$, the Skolem circle just needs to be large enough. The next result gives a bound for how large the circle must be.

Theorem 3.7. *Let S be a Skolem circle of order m, then the maximum number of removable edges is $O(\log m)$.*

Proof. Let C be a j-edge-removable Skolem circle with $j \geq 1$. Let S_1, S_2, S_3, $\ldots, S_j$ be the j subsequences of Skolem type or extended Skolem type that when concatenated together give C. Let the symbols $\{1, 2, \ldots, m\}$ as used in the Skolem circle be treated as integers with the usual ordering. Let the sequences be ordered by their largest symbol. Note that for all symbols $x \neq m$, both copies of x are in the same sequence, thus the only ambiguity in the ordering is if the symbol m is the largest symbol in two different subsequences, in which case these two may be ordered according to the next largest symbol (as noted below at most one of the sequences can contain only the symbol m so there is no ambiguity to this ordering). Note that if a subsequence, S_i, contains symbol m once then S_i is an extended Skolem sequence with m as the null symbol. At most two sequences contain m, and they will be ordered as S_j (and S_{j-1} if needed), thus S_i for $i \leq j-2$ do not contain the symbol m, and must be Skolem type sequences.

The largest symbol appearing twice in S_1 is at least 1. Let $x \in \{2, \ldots, j-3\}$. We proceed by induction on x.

Assume that for every $i \in \{1, 2, 3, \ldots, x-1\}$ the largest symbol appearing in S_i is at least 2^{i-1}. Let t_i be the largest symbol appearing in S_i, then $t_i \geq 2^{i-1}$. Let X_i be the set of symbols which appear in the sequence S_i. Let $\mathcal{X} = \bigcup_{i=1}^{x} X_i$, then t_x is the largest symbol in $\mathcal{X}$. Thus $|\mathcal{X}| \leq t_x$. However the length of a Skolem type sequence is at least one more than the largest symbol. Therefore the number of distinct symbols appearing in S_i is at least $(t_i + 1)/2$. Hence

$$\sum_{i=1}^{x} \frac{t_i + 1}{2} \leq |\mathcal{X}| \leq t_x, \tag{1}$$

$$\left(\sum_{i=1}^{x-1} (t_i + 1) \right) + t_x + 1 \leq 2t_x, \tag{2}$$

$$\left(\sum_{i=1}^{x-1} (t_i + 1) \right) + 1 \leq t_x. \tag{3}$$

Using the inductive hypothesis

$$\left(\sum_{i=1}^{x-1}(2^{i-1}+1)\right)+1 \leq t_x, \tag{4}$$

$$2^x + x \leq t_x, \tag{5}$$

$$2^{x-1} \leq t_x. \tag{6}$$

By induction the largest symbol appearing in S_{j-2} is at least 2^{j-3}.

The sequences S_j and S_{j-1} must be non-empty. Exactly one of the Skolem pair (a_m, b_m) must lie between the Skolem pair (a_{m-1}, b_{m-1}), hence at least one of the sequences S_j or S_{j-1} must contain at least $m/2$ different symbols. Hence $|X_j \cup X_{j-1}| \geq m/2$.

Hence

$$m/2 \geq |\{1,2,\ldots,m\} \setminus (X_j \cup X_{j-1})| = |\bigcup_{i=1}^{j-2} S_i| \geq \sum_{i=1}^{j-2}(2^{i-1}+1)/2 \geq 2^{j-3}$$

which can be rearranged to $j \leq 2 + \log_2(m)$. □

4 Enumeration of Skolem circles

One of the fundamental questions asked about any combinatorial structure is how many? General lower bounds on the number of Skolem and Langford sequences have been calculated [1] some 30 years ago, some recent work improves the bounds on Langford sequences [10]. We use the ideas of circle equivalence and a known lower bound on the number of Skolem sequences to give a general lower bound on the number of distinct Skolem circles. We also compute the number of distinct Skolem circles for small orders.

Theorem 4.1 (Abrham [1]). *There are at least $2^{\lfloor m/3 \rfloor}$ distinct Skolem sequences of order m.*

Combine Theorem 4.1 with Theorem 3.7, and we have a lower bound on the number of distinct Skolem circles.

Theorem 4.2. *For $m \geq 8$ with $m \equiv 0, 1 \mod 4$, there are at least*

$$\frac{2^{\lfloor m/3 \rfloor - 1}}{2 + \log_2(m)} + 1$$

Skolem circles of order m.

Proof. From Theorem 4.1 there are at least $2^{\lfloor m/3 \rfloor}$ distinct Skolem sequences of order m. From Theorem 3.7 there are at most $2 + \log_2(m)$ removable edges in each Skolem circle, and hence each Skolem circle of order m represents at most $2(2 + \log_2(m))$ Skolem sequences. It was shown in [12] that at least one 0-edge-removable Skolem circle exists for each $m \geq 8$. $\square$

As with Theorem 4.1, the lower bound described in Theorem 4.2 is far from sharp. For $m = 8$ the bound gaurantees at least two Skolem circles, however our computations find almost 200 (see Table 2).

Next we enumerate the number of distinct Skolem circles for small orders. As with many combinatorial problems, the small orders $m = 4$ and $m = 5$ can be done by hand, then larger orders require computer based techniques.

Theorem 4.3. *There is exactly one Skolem circle of order 4.*

Proof. There are exactly 6 distinct Skolem sequences of order 4 [16]. The Skolem circle of Figure 1 contains all of these. Hence this is the only edge-removable Skolem circle. From previous results [12] we know that there are no 0-edge-removable Skolem circles of order 4. $\square$

Theorem 4.4. *There are exactly two Skolem circles of order 5.*

Proof. There are exactly 10 distinct Skolem sequences of order 5 [16]. They can be partitioned into two circle equivalence classes.

The class $\left\{ \begin{array}{ll} (4,1,1,5,4,2,3,2,5,3), & (3,5,2,3,2,4,5,1,1,4), \\ (2,3,2,5,3,4,1,1,5,4), & (4,5,1,1,4,3,5,2,3,2) \end{array} \right\}$
is a 2-edge-removable Skolem circle.

The class $\left\{ \begin{array}{ll} (1,1,5,2,4,2,3,5,4,3), & (3,4,5,3,2,4,2,5,1,1), \\ (5,2,4,2,3,5,4,3,1,1), & (1,1,3,4,5,3,2,4,2,5), \\ (2,4,2,3,5,4,3,1,1,5), & (5,1,1,3,4,5,3,2,4,2) \end{array} \right\}$
is a 3-edge-removable Skolem circle.

It was shown in [12] that there are no 0-edge-removable Skolem circles of order 5. $\square$

For larger values of m, computer based searches were used. Computations were done on the QUT High Performance Computing facilities using Matlab [11] and C++[4].

Skolem circles (and Skolem sequences) of order m can be classified according to the number of removable edges.

The following technique was used to search for Skolem circles. Begin by setting up sets of logical vectors A_i containing all the possible places for each Skolem pair (a_i, b_i). Note that we use Skolem circles in standard positional labelling, therefore there is exactly one place for placing the pair of 1s, and $(m-2)$ for placing a pair of 2s. Thus $|A_1| = 1$ and $|A_2| = m-2$. The symbols m are always antipodal, and hence after placing the pair of 1s, there are $m-2$ options for placing the ms, $|A_m| = m-2$. All other sets $|A_i| = 2m-4$, since the symbol 1 is already fixed in positions 1 and 2.

Vectors from the set $A_2, A_3, \ldots$ are progressively chosen, and checked for any clashes. In this way a complete list of Skolem circles can be created for small values of m. This algorithm was inspired by an algorithm previously used to enumerate Langford Sequences [14]. Lists of Skolem circles of small order are available on `http://www.joannelhall.com/gallery/skolem`. More computational time would obviously increase the size of the Skolem circles that can be catalogued.

The following algorithm was used to calculate the number of removable-edges in a Skolem circle.

Algorithm 4.5.

1. *Let C be a Skolem circle, with standard position labelling, represented as a vector $\vec{c}$ of length $2m$.*
2. *Let $\vec{v}$ be a vector of length $2m$ with v_i being the i^{th} component of $\vec{v}$. Initialise all entries in $\vec{v}$ as 1. For each $i \in \{2, 3, 4, \ldots, m-1\}$ set the components $v_{a_i+1}, v_{a_i+2}, \ldots, v_{b_i-1}$ to 0, where the subscripts are calculated* $\bmod 2m$. *This sets the components of $\vec{v}$ with positions between Skolem pairs to 0. All components not set to 0 remain as 1.*
3. - *If $\vec{v} = \vec{0}$, then C is an 0-edge removable Skolem Circle. The components of $\vec{v}$ set to 1 are those which are not between any Skolem pair, and hence are nodes of a removable edge.*
   - *If the symbol m has a removable edge on either side, then the position a_m (or b_m) is next to two removable edges. Thus the two removable edges around the null sequence (m) will induce three*

1s in the vector $\vec{v}$. Exactly one of a_m or b_m must be between a_{m-1} and b_{m-1}, hence two adjacent removable edges can occur at most once in any circle. All other removable edges are nonadjacent, and hence all other removable edges create a unique pair of adjacent 1s in the vector $\vec{v}$.

4. *Let $w(\vec{v})$ be the Hamming weight of $\vec{v}$. Then C is a j-edge removable Skolem circle with $j = {}^1/_2 w(\vec{v})$ if $w(\vec{v})$ is even,and $j = {}^1/_2(w(\vec{v}) + 1)$ if $w(\vec{v})$ is odd.*

The following is an example of Algorithm 4.5.

Example 4.6.

1. *Start with a Skolem circle in standard positional labelling represented as a vector.*

$$\vec{c} = (1\ 1\ 2\ 8\ 2\ 3\ 6\ 7\ 3\ 4\ 5\ 8\ 6\ 4\ 7\ 5)$$

2. *Create a logical vector of the same length as the Skolem circle with all values set to 1.*

$$\vec{v} \leftarrow (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$$

Starting with the symbol 2 in $\vec{c}$, set all positions between, but not including, the 2's of $\vec{v}$ to 0.

$$\vec{c} = (1\ 1\ \mathbf{2}\ 8\ \mathbf{2}\ 3\ 6\ 7\ 3\ 4\ 5\ 8\ 6\ 4\ 7\ 5)$$
$$\vec{v} \leftarrow (1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$$

Continue for all other symbols up to 7.

3	$\vec{c} = (1\ 1\ 2\ 8\ 2\ \mathbf{3}\ 6\ 7\ \mathbf{3}\ 4\ 5\ 8\ 6\ 4\ 7\ 5)$ $\vec{v} \leftarrow (1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$
4	$\vec{c} = (1\ 1\ 2\ 8\ 2\ 3\ 6\ 7\ 3\ \mathbf{4}\ 5\ 8\ 6\ \mathbf{4}\ 7\ 5)$ $\vec{v} \leftarrow (1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1)$
5	$\vec{c} = (1\ 1\ 2\ 8\ 2\ 3\ 6\ 7\ 3\ 4\ \mathbf{5}\ 8\ 6\ 4\ 7\ \mathbf{5})$ $\vec{v} \leftarrow (1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1)$
6	$\vec{c} = (1\ 1\ 2\ 8\ 2\ 3\ \mathbf{6}\ 7\ 3\ 4\ 5\ 8\ \mathbf{6}\ 4\ 7\ 5)$ $\vec{v} \leftarrow (1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)$
7	$\vec{c} = (1\ 1\ 2\ 8\ 2\ 3\ 6\ \mathbf{7}\ 3\ 4\ 5\ 8\ 6\ 4\ \mathbf{7}\ 5)$ $\vec{v} \leftarrow (1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)$

removables edges	0	1	2	3	4
$m = 4$	0	0	0	1	0
$m = 5$	0	0	1	1	0
$m = 8$	24	96	60	12	0
$m = 9$	280	574	284	62	0
$m = 12$	271,880	146,436	34,400	4,244	0
$m = 13$	2,742,984	1,035,186	207,756	22,810	288
$m = 16$	3,764,810,632	530,928,868	75,697,744	5,872,996	33,760
$m = 17$	46,071,353,270	4,751,383,672	620,552,462	43,754,420	184,848

Table 1: The number of distinct Skolem circles of each order, classified according to the number of breakpoints

	Skolem sequences [16]	Skolem circles
$m = 4$	6	1
$m = 5$	10	2
$m = 8$	504	192
$m = 9$	2,656	1,200
$m = 12$	455,936	456,960
$m = 13$	3,040,560	4,009,024
$m = 16$	1,400,156,768	4,377,344,000
$m = 17$	12,248,982,496	51,487,228,672

Table 2: The number of distinct Skolem sequences and Skolem circles

3. *As $\vec{v} \neq \vec{0}$, the Skolem circle has removeable edges.*

4. *Take the weight of the edge vector $w(\vec{v}) = 6$, and thus there are $6/2 = 3$ removeable edges in the Skolem circle $\vec{c}$.*

Algorithm 4.5 was used in an exhaustive search to compute the number of Skolem circles up to 17 symbols, classifyied by the number of removable edges. Table 1 tabulates the number of Skolem circles found, classifying for the number of removeable edges. Table 2 compares the number of Skolem sequences with the number of Skolem circles for upto 17 symbols.

From Table 2 note that for $m \geq 12$ the number of Skolem circles is greater than the number of Skolem sequences. A simple combinatorial argument gives an idea of why the number of Skolem circles grows faster than the number of Skolem sequences.

The growth rate of the number of Skolem seuqences may be calculated as:

1. Begin with an empty sequence with $2m$ positions to fill.
2. Then a_1 may take any value other than $2m$, this is $(2m-1)$ different possible values for a_1, with $b_1 = a_1 + 1$.
3. The symbol a_2 can take any value other that $2m, 2m-1$, thus there is a maximum of $2m-2$ different possible values for a_2, then $b_2 = a_2+2$.
4. The symbol a_3 can take any value other that $2m, 2m-1, 2m-2$, thus there is a maximum of $2m-3$ different possible values for a_3.
5. The symbol a_i with $i \geq 3$ can take any value other than $2m$, $2m-1$, $\ldots, 2m-i+1$, there is a maximum of $2m-i$ possible values for a_i.
6. For $i = m-2$ there is a maximum of 2 possible arrangement for the symbols $m, m-1, m-2$ [14].

Taking the product of these possibilities an upper bound on the maximum number of possible Skolem sequences is obtained.

$$2 \prod_{i=1}^{m-2} (2m-i) \simeq O(m!)$$

The same counting techniques is used to find an upper bound on the number of Skolem circles.

1. A Skolem circle always begins with $a_1 = 1$ and $b_1 = 2$.
2. Due to standard positional labelling, the symbol a_2 can take values from $\{3, 4, \ldots, m\}$ thus there is a maximum of $m-2$ different possible values for a_2, then $b_2 \equiv a_2 + 2 \mod 2m$.
3. The symbol a_i can take any value other that 1 and 2, thus there is a maximum of $2m-2$ different possible values for a_i, then $b_i \equiv a_i + i \mod 2m$.
4. For $i = m-2$ there is a maximum of 2 possible arrangements for the symbols $m, m-1, m-2$[14].

Taking the product of these possibilities we see that there is a maximum of

$$2(m-2)\,(2m-2)^{m-5} \simeq O(m^m)$$

possible Skolem circles.

These are both naive upper bounds, however they are calculated in the same way, so give us some intuition.

$$O(m^m) > O(m!),$$

and hence the number of Skolem circles eventually grows faster than the number of Skolem sequences.

5 Further ideas

The motivation behind investigating Skolem circles was to portray sequences with symbols taken from a finite cyclic group, which was found to correlate well with previous work on Skolem labelling of cycle graphs. Investigating Skolem structures with symbols taken from other group structures may correlate with Skolem labelling of other families of graph.

A similar investigation of Langford or Rosa labelling of cycle graphs may also lead in interesting directions and further applications in cryptography.

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