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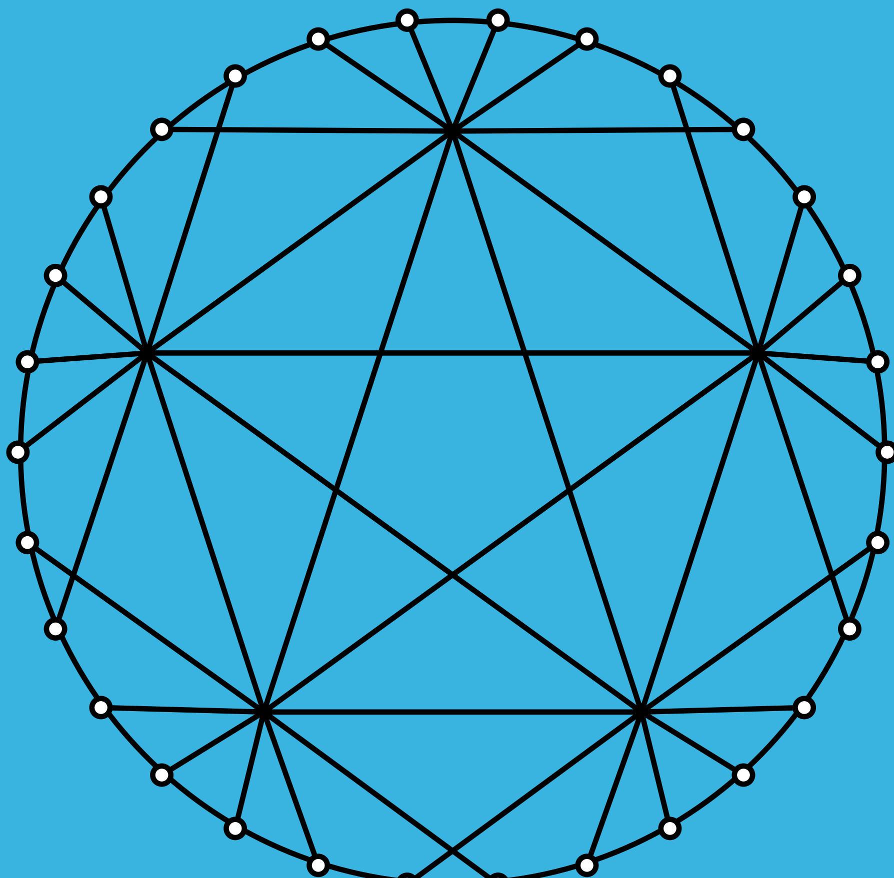

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On some new resolvable 1-rotational $(45, 5, 2)$-BIBDs

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Abstract: In this paper we classify 1-rotational $(45, 5, 2)$-BIBDs having an automorphism group isomorphic to $\mathbb{Z}_{44}$ acting on the set of blocks with the orbit lenghts distribution $(11, 11, 44, 44, 44, 44)$. New $(45, 5, 2)$-BIBDs are constructed. Among the constructed designs there are exactly 22 resolvable $(45, 5, 2)$-BIBDs. Until now, only one resolvable $(45, 5, 2)$-BIBD was known. The smallest parameter set for which the existence question of a resolvable BIBD is still open is $(45, 5, 1)$.

1 Introduction

A (v, k, λ)-*balanced incomplete block design* (in short (v, k, λ)-BIBD) is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ is a set of *points*, $\mathcal{B}$ is a multiset of *blocks* disjoint with $\mathcal{P}$, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is an incidence relation such that every block $B \in \mathcal{B}$ is incident with exactly k points and each pair of distinct points of $\mathcal{P}$ is incident with exactly λ blocks. In a (v, k, λ)-BIBD every point is incident with exactly $r = \dfrac{\lambda(v-1)}{k-1}$ blocks, and r is called *replication number* of (v, k, λ)-BIBD. The number of blocks is denoted by b. A Steiner $S(2, k, v)$ design is a $(v, k, 1)$-BIBD such that each pair of distinct points is incident with exactly one block. A BIBD is said to be *simple* if

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it contains no repeated blocks, and particularly *super-simple* if no pair of distinct blocks contains more than two points in common. A partition of the set of blocks $\mathcal{B}$ into *parallel classes*, each of which is a partition of the set of points $\mathcal{P}$, is called a *resolution* $\mathcal{R}$, and a (v, k, λ)-BIBD which has at least one resolution is called a *resolvable* (v, k, λ)-BIBD or shorter (v, k, λ)-RBIBD.

We say that a group G acts *regularly* on a set X if it acts *transitively* on X (i.e. for every pair of elements $x, y \in X$, there is an element $g \in G$ such that $xg = y$) and the stabilizer in G of every element of X is trivial, i.e. $G_x = \{1_G\}$, $\forall x \in X$.

A *subnormal series* of a group G is a sequence of subgroups, each a normal subgroup of the next one, shown in a standard notation

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_n = G.$$

In addition, if it holds that each $G_i \triangleleft G$, then the series is called *normal series* of G. The quotient groups G_{i+1}/G_i are called *factor groups* of the series. We assume that the reader is familiar with basic facts of the design theory and the group theory. For more information on the topic we refer the reader to [2] and [14].

An automorphism group G of a (v, k, λ)-BIBD is a group of permutations on its set of points leaving invariant its set of blocks. In particular, a 1-*rotational* (v, k, λ)-BIBD is a (v, k, λ)-BIBD admitting an automorphism group fixing one point and acting regulary on the others. Thus, a 1-rotational (v, k, λ)-BIBD may be assumed to have the set of points $\mathcal{P} = \mathbb{Z}_{v-1} \cup \{\infty\}$ with the 1-rotational automorphism given by $i \to i + 1 \pmod{(v-1)}$, $\infty \to \infty$. A (v, k, λ)-RBIBD is G-*invariantly resolvable* when it admits G as an automorphism group leaving invariant at least one resolution. The set of all automorphisms of a BIBD $\mathcal{D}$ forms its full automorphism group denoted by $\mathrm{Aut}(\mathcal{D})$.

Quite a deal of attention has been paid to answer the existence questions for RBIBDs from [13, Table I.1.28.] According to the table, the smallest parameter set for which the existence question of a resolvable BIBD is still open is $(45, 5, 1)$, which is a resolvable Steiner $S(2, 5, 45)$ design, and there are at least 16 non-isomorphic Steiner $S(2, 5, 45)$ designs . Later on, new $S(2, 5, 45)$ designs are constructed having an automorphism group of order two with maximum number of fixed points (see [4]), an automophism group of order five (see [12]), and admitting an automorphism group $\mathbb{Z}_6$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ or S_3 (see [7]). However, to the best of our knowledge, no survey of known results on resolvable Steiner $S(2,5,45)$ designs has been published, so it is still unkown wheather there exists a resolvable one.

If we consider the next larger $\lambda = 2$ then, according to the tables of small BIBDs in [1] and [13], the smallest unknown $(v,5,2)$-RBIBD was for $v = 45$. Recently, M. Buratti, J. Yan and C. Wang in [3] constructed 1-rotational RBIBDs in a more general context concerning *partitioned difference families* and they found, in particular, the first known example of a $(45,5,2)$-RBIBD. According to [13, Table I.1.28., Design #317], there exist at least 17 non-isomorphic $(45,5,2)$-BIBDs and they can be constructed as 2-multiples of known Steiner $S(2,5,45)$ designs. To the best of our knowledge, the first simple $(45,5,2)$-BIBD which is also super-simple BIBD, i.e. any 2 different blocks share at most two points, was constructed in [10, Lemma 2.8]. The design has the full automorphism group $\mathbb{Z}_{22}$ and it is not resolvable $(45,5,2)$-BIBD, which can be easily checked by computer using [9] and [15].

In this paper we present new (resolvable) $(45,5,2)$-BIBDs using a method of tactical decomposition. All of the constructed designs are 1-rotational $(45,5,2)$-BIBDs with repeated blocks. In this paper we prove that there cannot exist a simple 1-rotational $(45,5,2)$-BIBD admitting an action of an automorphism group $G \cong \mathbb{Z}_{44}$ with two block orbits of length 11 and four full block orbits (namely of size $|G|$). Hence, in this paper we give the classification of 1-rotational $(45,5,2)$-BIBDs with the action of the presumed automorphism group G with the block orbit lenghts distribution $(11,11,44,44,44,44)$. Among the 11606 constructed 1-rotational $(45,5,2)$-BIBDs, exactly 22 are resolvable, so there are at least 22 non-isomorphic $(45,5,2)$-RBIBDs.

This paper is organized as follows: after the brief introduction, in Section 2 we present basic information on the construction and refinement of orbit matrices for an action of a presumed solvable automorphism group on a BIBD. In Section 3 we present new 1-rotational $(45,5,2)$-BIBDs and $(45,5,2)$-RBIBDs with $\mathbb{Z}_{44}$ as an automorphism group.

1-rotational $(45,5,2)$-BIBDs with the presumed action of the automorphism group $\mathbb{Z}_{44}$ were classified within a reasonable amount of time, with the help of computers. Orbit matrices and designs are obtained using programs written for GAP ([9]). For isomorphism testing, and for the analysis of their full automorphism groups we used [9] and [15].

2 Outline of the construction

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a (v, k, λ)-BIBD and let $G \leq \mathrm{Aut}(\mathcal{D})$. Further, let the group G acts on $\mathcal{D}$ with m point orbits and n block orbits. We denote the point G-orbits by $\mathcal{P}_1, \dots, \mathcal{P}_m$ and the block G-orbits by $\mathcal{B}_1, \dots, \mathcal{B}_n$, and put $|\mathcal{P}_i| = \nu_i$ and $|\mathcal{B}_j| = \beta_j$, $i = 1, \dots, m$, $j = 1, \dots, n$. The number of blocks of $\mathcal{B}_j$ which are incident with a representative of the point orbit $\mathcal{P}_i$ we denote by a_{ij}. The number a_{ij} does not depend on the choice of a point $P \in \mathcal{P}_i$, and the following conditions hold (see [5, 8, 11]):

(c1) $0 \leq a_{ij} \leq \beta_j, \quad 1 \leq i \leq m,\ 1 \leq j \leq n,$

(c2) $\displaystyle\sum_{j=1}^{n} a_{ij} = r, \quad 1 \leq i \leq m,$

(c3) $\displaystyle\sum_{i=1}^{m} \frac{\nu_i}{\beta_j} a_{ij} = k, \quad 1 \leq j \leq n,$

(c4) $\displaystyle\sum_{j=1}^{n} \frac{\nu_t}{\beta_j} a_{sj}\ a_{tj} = \lambda \nu_t + \delta_{st}(r - \lambda), \quad 1 \leq s, t \leq m,$

where $\displaystyle\sum_{i=1}^{m} \nu_i = v,\ \sum_{j=1}^{n} \beta_j = b$ and $b = \frac{vr}{k}$.

Definition 2.1. *An $(m \times n)$-matrix (a_{ij}) with entries satisfying the conditions $(c1) - (c4)$ is called orbit matrix for the parameters (v, k, λ) and orbit lengths distributions $(\nu_1, \dots, \nu_m)$ and $(\beta_1, \dots, \beta_n)$.*

The construction of BIBDs admitting an action of an automorphism group, using orbit matrices, consists of the following two basic steps (see [5, 11]):

1. Construction of orbit matrices for the presumed automorphism group,
2. Construction of designs from the obtained orbit matrices. This step is often called an indexing of orbit matrices.

In indexing of orbit matrices, we have to determine which blocks are incident with the representative of a point orbit. That leads us to the notion of an index set.

Definition 2.2. *The set of indices of blocks of the orbit $\mathcal{B}_j$ indicating which blocks of $\mathcal{B}_j$ are incident with the representative of the point orbit $\mathcal{P}_i$ is called index set for the position (i, j) of the orbit matrix.*

In the first step of the construction of $(m \times n)$-orbit matrices that could produce a block design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ with a presumed automorphism group $G \leq Aut(\mathcal{D})$, it is very important to reduce the number of those matrices which will produce isomorphic designs, as much as possible. To conduct that reduction we imply elements of centralizer of the group G in the group $S_m \times S_n$, as described in [6].

The second step of the construction of BIBDs, called indexing, often lasts too long and can not be performed in a reasonable amount of time. Hence, to make such a construction possible, it could be very useful to use a normal series of a persumed automorphism group G acting on a BIBD, and the process is called a *refinement* of orbit matrices. An algorithm for a refinement of an orbit matrix of a BIBD using principal series of its Abelian automorphism group, and using a composition series of its solvable automorphism group G

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

is described in [5] and [6], respectively. More generally, if a presumed automorphism group G acting on a BIBD is a finite Abelian group, one can also use a normal series of G instead of a composition series of G to conduct the refinement of orbit matrices. So, the condition that each factor group G_{i+1}/G_i must be simple is not necessary.

The method for refinement of an orbit matrix for the presumed automorphism group G of a BIBD $\mathcal{D}$ is based on the fact that each G-orbit of $\mathcal{D}$ decomposes to H-orbits of the same size, where H is a normal subgroup of G (see [5, Theorem 2]). Therefore, each orbit matrix for the group G decomposes to orbit matrices for the group $H \lhd G$, and the quotient group G/H acts transitively on the set of H-orbits obtained from one G-orbit (see [6]). Very often, the algorithm for the refinement of orbit matrices consists of several steps, depending on the length of a normal series of a presumed automorphism group $G \leq Aut(\mathcal{D})$. In the first step of the algorithm we need to find all refinements for the normal subgroup $G_{n-1} \lhd G$. Secondly, we find refinements of the resulting orbit matrices and obtain orbit matrices for the group $G_{n-2} \lhd G_{n-1}$, etc., until the desired designs are constructed. Application of the groups G/G_{n-1}, G_{n-1}/G_{n-2}, $\ldots$, $G_{n-(i-1)}/G_{n-i}$ in the i-th step of the algorithm significantly speeds up the refinement (see [5], Example 1). For the elimination of orbit matrices which will produce isomorphic BIBDs, we use elements of the normalizer of a presumed automorphism group and some particular automorphisms of the orbit matrices. For more details see [5] and [8].

BIBDs constructed by the method has to be checked for isomorphism because one orbit matrix or two different orbit matrices may produce isomorphic designs. More details on the construction of designs using orbit matrices and tactical decomposition can be found in [5] and [6].

3 New (resolvable) $(45, 5, 2)$-BIBDs having $\mathbb{Z}_{44}$ as an automorphism group

In this section we give the classification of 1-rotational $(45, 5, 2)$-BIBDs admitting an action of an automorphism group $G \cong \mathbb{Z}_{44}$ with two block orbits of length 11 and four full block orbits (namely of size $|G|$). Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be the $(45, 5, 2)$-RBIBD constructed in (see [3, Example 2.9]), which was obtained using a resolvable 1-rotational difference family. Considering the method of its construction, the design $\mathcal{D}$ is 1-rotational with repeated blocks.
If $\mathcal{P} = \{1, \ldots, 45\}$, then the set of blocks $\mathcal{B}$ of the design $\mathcal{D}$ can be obtained by developing the base blocks

$$\{1, 2, 13, 24, 35\},\ \{1, 2, 13, 24, 35\}, \{2, 3, 6, 30, 34\},$$
$$\{2, 3, 23, 32, 41\}, \{2, 4, 9, 12, 38\},\ \{2, 4, 16, 23, 29\},$$

under the action of its full automorphism group $Aut(\mathcal{D}) \cong \mathbb{Z}_{44}$ generated by the permutation

$$(2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28,$$
$$29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45).$$

Hence, the first two base blocks are repeated and each of them have an orbit of size 11, whereas each of the last four base blocks have an full orbit of size 44. We wanted to construct some new 1-rotational $(45, 5, 2)$-BIBDs with the presumed action of automorphism group $\mathbb{Z}_{44}$, i.e. acting on $(45, 5, 2)$-BIBDs with the point and block orbit lengths distribution $(1, 44)$ and $(11, 11, 44, 44, 44, 44)$, respectively. Then we construct orbit matrices, as described in Section 2. Solving the system of equations (c1) - (c4), we get the following Lemma.

Lemma 3.1. *There is exactly one orbit matrix for a* 1*-rotational* $(45, 5, 2)$*-BIBD having* $G \cong \mathbb{Z}_{44}$ *as an automorphism group acting on the set of blocks with the orbit lengths distribution* $(11, 11, 44, 44, 44, 44)$*. The orbit matrix* A *is given in Table 1.*

A	11	11	44	44	44	44
1	11	11	0	0	0	0
44	1	1	5	5	5	5

Table 1: Orbit matrix for $(45, 5, 2)$-BIBDs under the particular action of $\mathbb{Z}_{44}$

From Table 1, the fixed point is incident with all blocks from the two short orbits, but each non-fixed point is incident with exactly one block from the two short orbits and with exactly five blocks from each of the full orbits. The result of indexing the first row (fixed point) and the first two columns (short block orbits) of the orbit matrix A is unique, since normalizers of the group $\mathbb{Z}_{11}$ in S_{11} are applied, as it is shown with reverse lexicographical order in Table 2 (see [5], [8]). Hence, the first 22 columns in Table 2, which

$$
\begin{array}{rr|cccccc}
 & & \overbrace{11111111111}^{\times 11} & \overbrace{11111111111}^{\times 11} & \overbrace{1\cdots 1}^{\times 44} & \overbrace{1\cdots 1}^{\times 44} & \overbrace{1\cdots 1}^{\times 44} & \overbrace{1\cdots 1}^{\times 44} \\
\hline
 & 1 & 11111111111 & 11111111111 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\
\times 44 \left\{ \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right. &
\begin{array}{c} 1\\1\\1\\1\\1\\1\\1\\1\\ \vdots \\1\\1\\1\\1 \end{array} &
\begin{array}{c} 10000000000\\10000000000\\10000000000\\10000000000\\01000000000\\01000000000\\01000000000\\01000000000\\ \vdots \\00000000001\\00000000001\\00000000001\\00000000001 \end{array} &
\begin{array}{c} 10000000000\\10000000000\\10000000000\\10000000000\\01000000000\\01000000000\\01000000000\\01000000000\\ \vdots \\00000000001\\00000000001\\00000000001\\00000000001 \end{array} &
? & ? & ? & ?
\end{array}
$$

Table 2: Incidence matrix of a $(45, 5, 2)$-BIBD obtained by indexing of A

correspond to the blocks from two orbits of length 11, are repeated blocks of a 1-rotational $(45, 5, 2)$-BIBD, and the following lemma holds.

Lemma 3.2. *A* 1*-rotational* $(45,5,2)$*-BIBD admitting an action of an automorphism group* $G \cong \mathbb{Z}_{44}$ *with two block orbits of length* 11 *and four full block orbits of length* 44 *is not a simple design, and its repeated blocks form two block orbits of length* 11*.*

However, it would be very difficult to proceed with indexing of the remaining part of A, i.e. rows and columns of A corresponding to the full point and block orbits respectively, since there are $\binom{44}{5}$ possibilities for index sets for the positions $(2,3)-(2,6)$ in the orbit matrix A. Hence, we applied the method of refinement using a normal series of the automorphism group $G \cong \mathbb{Z}_{44}$, as described in Section 2. Here, the refinement of the orbit matrix A can be conducted on two different ways. We used the following two different normal series of the group $G \cong \mathbb{Z}_{44}$

$$\{1\} \lhd \mathbb{Z}_4 \lhd \mathbb{Z}_4 \times \mathbb{Z}_{11} \quad \text{and} \quad \{1\} \lhd \mathbb{Z}_{11} \lhd \mathbb{Z}_4 \times \mathbb{Z}_{11},$$

since every finite Abelian group is a direct product of cyclic groups. Hence, we made the refinement of A in two different ways in order to check the correctness of the obtained results.

Firstly, we made the refinement of the orbit matrix A using the first normal series of the group G, and decomposed it to orbit matrices for the subgroup $\mathbb{Z}_4 \lhd \mathbb{Z}_4 \times \mathbb{Z}_{11}$, considering the action of the quotient group $\mathbb{Z}_{44}/\mathbb{Z}_4 \cong \mathbb{Z}_{11}$ on a $(45,5,2)$-BIBD, as described in Section 2. As a result of that refinement, 562 orbit matrices are constructed. Further, in the next refinement (called indexing) 228246 $(45,5,2)$-BIBDs are constructed, and among them exactly 11606 are non-isomorphic. For the final isomorphism testing of constructed designs and determining structures of their full automorphism groups we used [9] and [15].

Furthermore, to check the correctness of the obtained results, we also made the refinement of A using the second normal series of G and we constructed 17 orbit matrices for the subgroup $\mathbb{Z}_{11} \lhd \mathbb{Z}_4 \times \mathbb{Z}_{11}$, considering the action of the quotient group $\mathbb{Z}_{44}/\mathbb{Z}_{11} \cong \mathbb{Z}_4$ on a $(45,5,2)$-BIBD. As a result of indexing of the 17 orbit matrices, 150860 $(45,5,2)$-BIBDs are constructed. After additional checking for isomorphisms, the same result is obtained. Information about the number of orbit matrices and designs constructed by the application of the described method is given in Table 3.

Therefore, we proved the following theorem.

Theorem 3.3. *There are exactly* 11606 *non-isomorphic* 1*-rotational* $(45, 5, 2)$*-BIBDs having an automorphism group* $G \cong \mathbb{Z}_{44}$ *acting with the block orbit lengths distribution* $(11, 11, 44, 44, 44, 44)$*. They have repeated blocks from the orbits of length* 11*. Among them there are* 11594 *designs with the full automorphism group of order* 44 *isomorphic to the group* $\mathbb{Z}_{44}$*,* 4 *of them having the full automorphism group of order* 88 *isomorphic to the group* $\mathbb{Z}_{11} \times D_8$*, and* 8 *designs having the full automorphism group of order* 88 *isomorphic to the group* $\mathbb{Z}_4 \times D_{22}$*.*

# orbit matrices in the first step	1
# orbit matrices after refinement for $\mathbb{Z}_4 \lhd \mathbb{Z}_4 \times \mathbb{Z}_{11}$	562
# constructed designs	228246
# non-isomorphic designs	11606
# orbit matrices after refinement for $\mathbb{Z}_{11} \lhd \mathbb{Z}_4 \times \mathbb{Z}_{11}$	17
# constructed designs	150860
# non-isomorphic designs	11606

Table 3: The results of the construction of 1-rotational $(45, 5, 2)$-BIBDs with the particular action of an automorphism group $G \cong \mathbb{Z}_{44}$

Among the 11606 constructed $(45, 5, 2)$-BIBDs there are exactly 22 resolvable $(45, 5, 2)$-BIBDs with $\mathbb{Z}_{44}$ as an automorphism group, checked by computer using [9] and [15]. In Table 4 we give the base blocks of the 22 constructed $(45, 5, 2)$-RBIBDs, denoted by $\mathcal{D}_1$, $\mathcal{D}_2$, $\ldots$, $\mathcal{D}_{22}$.
The group $G_1 = \langle g_1, g_2 \rangle$ is the full automorphism group of the designs $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_9, \mathcal{D}_{15}$ and $\mathcal{D}_{17}$, the group $G_2 = \langle g_1, g_3 \rangle$ is the full automorphism group of $\mathcal{D}_3, \mathcal{D}_5$, and $\mathcal{D}_{13}$, the group $G_3 = \langle g_1, g_4 \rangle$ is the full automorphism group of $\mathcal{D}_4$, $\mathcal{D}_7$, $\mathcal{D}_8$, $\mathcal{D}_{11}$, $\mathcal{D}_{12}$, $\mathcal{D}_{14}$, $\mathcal{D}_{16}$, $\mathcal{D}_{19}$, $\mathcal{D}_{21}$ and $\mathcal{D}_{22}$, and the group $G_4 = \langle g_1, g_5 \rangle$ is the full automorphism group of $\mathcal{D}_6, \mathcal{D}_{10}, \mathcal{D}_{18}$ and $\mathcal{D}_{20}$. The generators of the groups are

$g_1 = (2, 3, 4, 5)(6, 7, 8, 9)(10, 11, 12, 13)(14, 15, 16, 17)(18, 19, 20, 21)(22, 23, 24, 25)$
$(26, 27, 28, 29)(30, 31, 32, 33)(34, 35, 36, 37)(38, 39, 40, 41)(42, 43, 44, 45)$
$g_2 = (2, 8, 10, 16, 18, 24, 26, 32, 34, 40, 42, 4, 6, 12, 14, 20, 22, 28, 30, 36, 38, 44)$
$(3, 9, 11, 17, 19, 25, 27, 33, 35, 41, 43, 5, 7, 13, 15, 21, 23, 29, 31, 37, 39, 45)$
$g_3 = (2, 9, 12, 15, 18, 25, 28, 31, 34, 41, 44, 3, 6, 13, 16, 19, 22, 29, 32, 35, 38, 45, 4, 7, 10, 17,$
$20, 23, 26, 33, 36, 39, 42, 5, 8, 11, 14, 21, 24, 27, 30, 37, 40, 43)$
$g_4 = (2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42)(3, 7, 11, 15, 19, 23, 27, 31, 35, 39, 43)$
$(4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44)(5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45)$
$g_5 = (2, 7, 12, 17, 18, 23, 28, 33, 34, 39, 44, 5, 6, 11, 16, 21, 22, 27, 32, 37, 38, 43, 4, 9, 10, 15,$
$20, 25, 26, 31, 36, 41, 42, 3, 8, 13, 14, 19, 24, 29, 30, 35, 40, 45).$

	Base blocks		Base blocks
$\mathcal{D}_1$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,10,17,23\},\{2,7,21,23,39\},$ $\{2,7,25,27,36\},\{2,9,14,22,40\}$	$\mathcal{D}_2$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,13,30,40\},\{2,6,21,29,45\},$ $\{2,7,24,37,39\},\{2,8,16,34,41\}$
$\mathcal{D}_3$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,16,25,39\},\{2,6,24,33,44\},$ $\{2,7,10,15,34\},\{2,8,24,36,39\}$	$\mathcal{D}_4$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,10,31,45\},\{2,7,14,33,35\},$ $\{2,8,17,20,26\},\{2,9,16,22,30\}$
$\mathcal{D}_5$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,16,25,39\},\{2,6,22,34,41\},$ $\{2,7,12,33,43\},\{2,8,10,28,35\}$	$\mathcal{D}_6$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,11,24,26\},\{2,6,18,29,37\},$ $\{2,7,29,35,41\},\{2,9,12,20,39\}$
$\mathcal{D}_7$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,16,21,38\},\{2,6,24,31,44\},$ $\{2,7,13,34,41\},\{2,9,20,29,37\}$	$\mathcal{D}_8$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,13,29,44\},\{2,6,20,23,38\},$ $\{2,7,14,24,38\},\{2,7,26,35,41\}$
$\mathcal{D}_9$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,21,29,39\},\{2,6,33,34,41\},$ $\{2,7,13,19,37\},\{2,8,18,28,43\}$	$\mathcal{D}_{10}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,17,32,34\},\{2,6,21,27,30\},$ $\{2,7,10,34,44\},\{2,7,15,32,41\}$
$\mathcal{D}_{11}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,12,20,43\},\{2,6,13,25,33\},$ $\{2,7,16,32,34\},\{2,7,17,22,35\}$	$\mathcal{D}_{12}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,23,32,44\},\{2,6,30,35,43\},$ $\{2,7,26,36,38\},\{2,9,20,27,36\}$
$\mathcal{D}_{13}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,13,30,39\},\{2,6,14,19,44\},$ $\{2,7,17,33,36\},\{2,8,16,28,37\}$	$\mathcal{D}_{14}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,17,33,37\},\{2,7,13,16,35\},$ $\{2,7,15,21,27\},\{2,9,14,24,32\}$
$\mathcal{D}_{15}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,14,20,23\},\{2,6,23,30,45\},$ $\{2,7,18,26,36\},\{2,8,14,21,39\}$	$\mathcal{D}_{16}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,15,30,36\},\{2,6,16,35,43\},$ $\{2,7,10,23,32\},\{2,7,14,34,40\}$
$\mathcal{D}_{17}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,16,18,25\},\{2,6,23,32,43\},$ $\{2,7,20,28,36\},\{2,7,27,34,40\}$	$\mathcal{D}_{18}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,15,20,26\},\{2,6,27,39,44\},$ $\{2,8,21,24,33\},\{2,9,16,24,32\}$
$\mathcal{D}_{19}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,11,22,30\},\{2,6,15,37,40\},$ $\{2,7,29,36,43\},\{2,8,14,20,35\}$	$\mathcal{D}_{20}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,12,20,41\},\{2,6,24,30,43\},$ $\{2,7,18,24,39\},\{2,7,26,34,41\}$
$\mathcal{D}_{21}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,12,37,44\},\{2,6,13,28,36\},$ $\{2,7,18,27,36\},\{2,7,19,30,43\}$	$\mathcal{D}_{22}$	$\{1,2,3,4,5\},\{1,2,3,4,5\},$ $\{2,6,16,18,31\},\{2,6,16,19,39\},$ $\{2,7,10,19,28\},\{2,7,13,27,35\}$

Table 4: The base blocks of the $(45,5,2)$-RBIBDs $\mathcal{D}_1$, $\mathcal{D}_2$, $\ldots$, $\mathcal{D}_{22}$ under the action of their full automorphism groups

Thus, the following theorem holds.

Theorem 3.4. *There are exactly* 22 *non-isomorphic* 1*-rotational* $(45,5,2)$*-RBIBDs having the automorphism group* $G \cong \mathbb{Z}_{44}$ *acting with the block orbit lengths distribution* $(11,11,44,44,44,44)$*. All of them have repeated blocks from the orbits of size* 11 *and their full automorphism groups are isomorphic to the group* $\mathbb{Z}_{44}$*.*

All 11606 non-isomorphic 1-rotational $(45,5,2)$-BIBDs constructed in this paper can be found at http://www.math.uniri.hr/~ddumicic/results/

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