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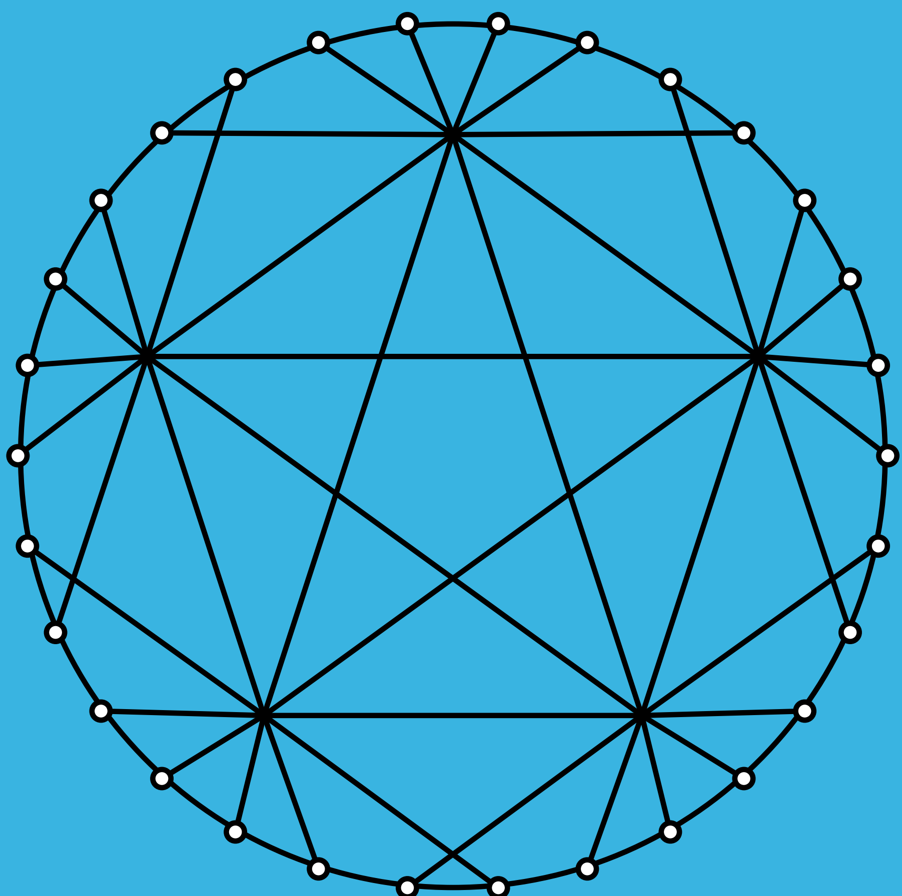

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There is but one PDS in $\mathbb{Z}^3$ inducing just square components

Italo J. Dejter and Luis R. Fuentes

University of Puerto Rico, Rio Piedras, PR 00936-8377
italo.dejter@gmail.com and luis.fuentes@upr.edu

Carlos A. Araujo

Universidad del Atlántico, Barranquilla, Colombia
carlosaraujo@mail.uniatlantico.edu.co

Abstract: It is known that in the unit distance graph of the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ there exists a dominating set S with 4-cycles as sole induced components and each vertex of $\mathbb{Z}^3 \setminus S$ having a unique neighbor in S. We show S is unique.

Keywords: perfect dominating sets; unit distance graph; integer lattice

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1 Perfect dominating sets, (PDS s)

Let $\Gamma = (V, E)$ be a graph and let $S \subset V$. The closed neighborhood of a vertex $\theta \in V$ in Γ is denoted $N[\theta]$. Let $[S]$ be the subgraph of Γ induced by S. The induced components of S, namely the connected components of $[S]$ in Γ, are said to be the *components* of S. Several definitions of perfect dominating sets in graphs are considered in the literature [18, 20]. We work

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with the following one [23] denoted with the short acronym PDS, to make a distinctive difference:

S is a PDS of $\Gamma \Leftrightarrow$ each vertex of $V \setminus S$ has a unique neighbor in S.

This definition (of PDS) differs from that of a 'perfect dominating set' as in [15, 17, 22] (that for us is a stable PDS coinciding with the perfect code of [4] or with the efficient dominating set of [3, 18]). With our not necessarily stable definition of perfect dominating set, denoted PDS, our main result, stated below as Theorem 1, has a narrowing spirit as that of Theorem 2.6 of just cited [22].

Let $0 < n \in \mathbb{Z}$. The following graphs are considered. The unit distance graph Λ_n of the n-dimensional integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ has vertex set $\mathbb{Z}^n$ and exactly one edge between each two vertices if and only if their Euclidean distance is 1. An *n-cube* is the cartesian graph product $Q_n = K_2 \square K_2 \square \cdots \square K_2$ of precisely n copies of the complete graph K_2. In particular, a 2-cube Q_2 is a square, that is a 4-cycle. A *grid graph* is the cartesian graph product of two path graphs.

Our definition of a PDS S allows components of S in Γ which are not isolated vertices. For example: **(a)** tilings with generalized Lee r-spheres, for fixed r with $1 < r \leq n$ in $\mathbb{Z}$ (e.g., crosses with arms of length one if $r = n$), furnish Λ_n with PDS s whose components are r-cubes [14], including that of our Theorem 1, below; (It is most remarkable that $r = n \Leftrightarrow n \in \{2^r - 1, 3^r - 1; 0 < r \in \mathbb{Z}\}$ [6]); **(b)** *total perfect codes* [1, 21], that is PDS s whose components are copies of $K_2 = P_2$ in the Λ_n s and grid graphs; (these appear as *diameter perfect Lee codes* [13, 19]); **(c)** PDS s in n-cubes [5, 7, 8, 10, 11, 23], where $0 < n \in \mathbb{Z}$, including the perfect codes of [12]; **(d)** PDS s in grid graphs [8, 21].

Theorem 1. *There is only one PDS in Λ_3 whose components are 4-cycles.*

This is proved as Theorem 4 once some auxiliary notions are presented.

2 Induced components

The *distance* $d(u, v)$ between two vertices u and v of Λ_n is defined as the minimum length of any path connecting u and v. The following is an elementary extension of a result of [23] for n-cubes.

Theorem 2. *Let S be a* PDS *in Λ_n. Let J_S be a set of indices j for the corresponding components S^j of S. Each S^j is a cartesian graph product of connected subgraphs of Λ_1. Thus, if such S^j is a finite subgraph Θ of Λ_n, then S^j is of the form $P_{i_1^j} \Box P_{i_2^j} \Box \cdots \Box P_{i_n^j}$, where $P_{i_k^j}$ is a path of length $i_k^j - 1 \geq 0$, for $k = 1, \ldots, n$.*

A PDS in Λ_n whose components are all isomorphic to a fixed finite graph Θ (as in Theorem 2) is called a PDS$[\Theta]$. If no confusion arises, n-tuples representing elements of $\mathbb{Z}^n$ are written with neither commas nor external parentheses. We denote $00\ldots0 = O$, $10\ldots0 = e_1$, $010\ldots0 = e_2$, $\ldots$, $00\ldots1 = e_n$.

At the end of Section 6 of [14] (in the original setting of item (a) above in Section 1), all the indices i_k^j of our Theorem 2 are shown to be less than 2.

3 Lattice-like dominating sets

Let $\Theta = (V, E)$ be a finite subgraph of Λ_n and let $z \in \mathbb{Z}^n$. Then $\Theta + z$ denotes the graph $\Theta' = (V', E')$, where $V' = V + z = \{w;$ there exists $v \in V, w = v + z\}$ and $uv \in E \Leftrightarrow (u + z)(v + z) \in E'$. Let S be a PDS$[\Theta]$ and let a copy D of Θ be a component of S. Then S is said to be *lattice-like* if there exists a lattice L (that is, a subgroup L of $\mathbb{Z}^n$) such that D' is a component of S if and only if there is $z \in L$ with $D' = D + z$. Examples above ([14, 6, 13, 19]) are lattice-like.

If S is a PDS$[\Theta]$ with $\Theta = (V, E)$, then S can be seen as a tiling of $\mathbb{Z}^n$ by the induced subgraph Θ^* of Λ_n on the set $V^* = \{v \in \mathbb{Z}^n; d(v, V) \leq 1\}$. Thus, a lattice-like tiling will be understood in the same way as a lattice-like PDS. We need the following form of Theorem 6 [19] for the proof of Theorem 4. Recall that given a graph G, the distance $d(v, H)$ between a vertex v of G and a subgraph H of G is the shortest distance between v and the vertices of H.

Theorem 3. *Let Θ be a subgraph of Λ_n. Let Θ^* be an induced supergraph of Θ in Λ_n such that a vertex v is in Θ^* if and only if $d(v, \Theta) \leq 1$. Let $D = (V, E)$ be a copy of Θ^* that contains vertices $O, e_1, \ldots, e_n$. Then, there is a lattice-like PDS$[\Theta]$ if and only if there exists an abelian group G of order $|V|$ and a group epimorphism $\Phi : \mathbb{Z}^n \to G$ such that the restriction of Φ to V is a bijection.*

4 The proof

Theorem 4. *There do not exist non-lattice-like PDS*$[Q_2]$ *s in* Λ_3*. In addition, there exists exactly one lattice-like PDS*$[Q_2]$ *in* Λ_3*.*

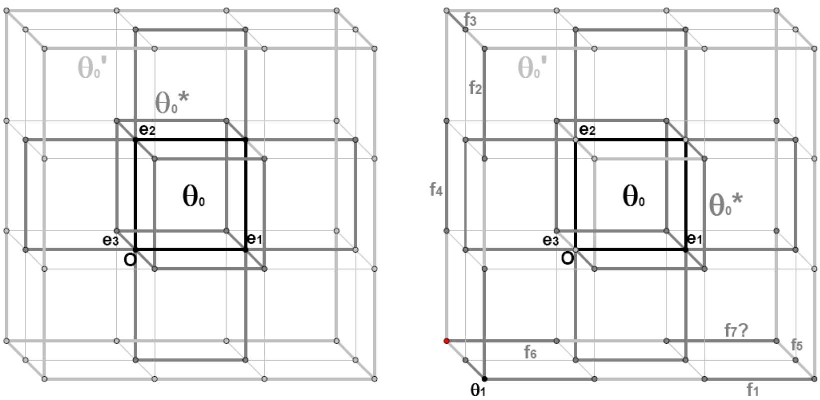

Figure 1: $\Theta_0 \subset \Theta_0^* \subset \Theta_0'$, and the case of one corner of $\Theta_0' - \Theta_0^*$ in S

Proof. Theorem 8 [14] insures the existence of a PDS$[Q_2]$ in Λ_3. In fact, the connected components of such PDS$[Q_2]$ are the generalized Lee spheres $S_{3,2,0}$ inside the corresponding generalized Lee spheres $S_{3,2,1}$ (in their inductive construction in Section 1 [14]) that form the lattice tiling $\Lambda_{3,2}$ (in the notation of [14]) insured by that Theorem 8. According to the theorem, this $\Lambda_{3,2}$ has generator matrix (as defined in Section 3 [14]):

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 2 & 5 \\ 0 & 0 & 10 \end{pmatrix} \tag{1}$$

In terms of Theorem 3, the generator matrix (1) corresponds to the group epimorphism $\Phi : \mathbb{Z}^3 \to G = \mathbb{Z}_{20}$ given by $\Phi(e_1) = 2$; $\Phi(e_2) = 5$ and $\Phi(e_3) = 6$ where Φ is obtained first by multiplying the matrix (1) by an unknown vector and then solving the corresponding system of equations mod 20. To see that this is the only PDS$[Q_2]$ in Λ_3, we note that there are only two possible abelian groups G for the epimorphism Φ, namely: $G = \mathbb{Z}_{20}$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. It can be easily checked [16] that **(a)** there are just 32 epimorphism from $\mathbb{Z}^3$ onto $G = \mathbb{Z}_{20}$ and none from $\mathbb{Z}^3$ onto $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$; **(b)** every possible assignment for $\Phi(e_1)$, $\Phi(e_2)$

and $\Phi(e_3)$ has order 10, 10 and 4, respectively, in $\mathbb{Z}_{20}$. As a result, all 4-cycles induced by each lattice-like PDS$[Q_2]$ associated (via Theorem 3) to a corresponding of these 32 epimorphisms are placed in the same way in Λ_3. Each such lattice-like PDS$[Q_2]$ in Λ_3 is equivalent to the one obtained via matrix (1).

Assume there is a non-lattice-like PDS$[Q_2]$ S in Λ_3 so that the components of $[S]$ are 4-cycles Q_2; let $\Theta = \Theta_0$ be such a component. We may assume that Θ_0 has vertices O, e_1, e_2, $e_1 + e_2$. The graph $\Theta^* = \Theta_0^*$ is contained in a graph Θ_0' isomorphic to $P_4 \Box P_4 \Box P_3$ as on the left of Figure 1, where Θ_0 has its edges thick black, the rest of Θ_0^* has them dark-gray and the rest of Θ_0' has them light-gray, thick for the paths between the eight corners (vertices of degree 3 in Θ_0': $-e_1 - e_2 \pm e_3$, $2e_1 - e_2 \pm e_3$, $-e_1 + 2e_2 \pm e_3$, $2e_1 + 2e_2 \pm e_3$) and thin for the rest. The realization of Θ_0^* in $\mathbb{R}^3$ has convex hull containing tightly Θ_0'. Different colors (black, green, red, etc.) and traces are used in the representations of Figures 1-14 in a larger exposition of this work in [9].

We recall that a 1-factor of a graph is a spanning 1-regular subgraph of it. Assume no vertex of $\Theta_0' - \Theta_0^*$ is in S. By symmetry there is a 1-factor F in $\Theta_0' - \Theta_0^*$ each of whose edges has an endvertex $\notin V(\Theta_0' - \Theta_0^*)$ dominated by a vertex in a 4-cycle induced by S. In each case we will reach a contradiction: F is either as in case (a) or (b) below, depending on the feasible dispositions of four edges of F over the four maximal paths of length 2 between the eight corners of Θ_0', namely either with their eight endvertices having convex hull tightly containing a copy of $P_4 \Box P_4 \Box P_2$ (say convex hull $[-1, 2] \times [-1, 2] \times [-1, 0]$) or not (in which case partial convex hulls $\{-1\} \times [-1, 2] \times [-1, 0]$ and $\{2\} \times [-1, 2] \times [0, 1]$ appear, not leading to a total convex hull as above), that we have respectively either as the four edges f_1, f_5, f_{12}, f_{13}, for (a), or as the four edges f_1, f_4, f_8, f_{12} for (b). These instances are: (with (a) further subdivided into subcases (a$_1$) and (a$_2$), below)

(a) The edges of F are:

$$
\begin{array}{llll}
f_1 & = (-e_1 - e_2, -e_1 - e_2 - e_3) & f_2 & = (-e_1 + e_3, -e_1 + e_2 + e_3) \\
f_3 & = (-e_1 - e_3, -e_1 + e_2 - e_3) & f_4 & = (-e_2 - e_3, e_1 - e_2 - e_3) \\
f_5 & = (2e_1 - e_2, 2e_1 - e_2 - e_3) & f_6 & = (2e_1 + e_3, 2e_1 + e_2 + e_3) \\
f_7 & = (2e_1 - e_3, 2e_1 + e_2 - e_3) & f_8 & = (2e_2 + e_3, -e_1 + 2e_2 + e_3) \\
f_9 & = (e_1 + 2e_2 + e_3, 2e_1 + 2e_2 + e_3) & f_{10} & = (e_1 - e_2 + e_3, 2e_1 - e_2 + e_3) \\
f_{11} & = (-e_2 + e_3, -e_1 - e_2 + e_3) & f_{12} & = (-e_1 + 2e_2, -e_1 + 2e_2 - e_3) \\
f_{13} & = (2e_1 + 2e_2, 2e_1 + 2e_2 - e_3) & f_{14} & = (2e_2 - e_3, e_1 + 2e_2 - e_3).
\end{array}
$$

We may take step by step either option (a_1) or option (a_2) below (where, instead of saying that a vertex v is dominated by an endvertex of an edge f, we simply say that v is dominated by f, denoted by $v \in (f)$, with (f) representing the set of vertices dominated by the endvertices of f):

(a_1) The first eight edges in (a) have each an endvertex dominated by a vertex in a 4-cycle. The involved 4-cycles contain the following edges:

- $f_1 - e_1$ (the translation of f_1 via the vector $-e_1$),
- $f_2 + e_3$ (forced, since $f_2 - e_1$ dominates $-2e_1 \in (f_1 - e_1)$),
- $f_3 - e_3$ (forced, since $f_3 - e_1$ contains $-2e_1 - e_3 \in (f_1 - e_1)$),
- $f_4 - e_2$ (forced, since $f_4 - e_3$ dominates $-e_1 - e_2 - 2e_3 \in (f_3 - e_3)$),
- $f_5 + e_1$ (forced, since $f_4 - e_2$ contains $2e_1 - 2e_2 - e_3 \in (f_4 - e_2)$),
- $f_6 + e_3$ (forced, since $f_6 + e_1$ dominates $3e_1 \in (f_5 + e_1)$),
- $f_7 - e_3$ (forced, since $f_7 + e_1$ contains $3e_1 - e_3 \in (f_5 + e_1)$) and
- $f_8 + e_2$ (forced, since $f_8 + e_3$ dominates $-e_1 + 2e_2 + 2e_3 \in (f_2 + e_3)$).

Now, there is no way for the edge f_9 to be dominated by a copy of K_2 external to $\Theta'_0 - \Theta^*_0$ (since $f_9 + e_2$ contains $e_1 + 3e_2 + e_1 \in (f_8 + e_2)$ while $f_9 + e_3$ contains $2e_1 + 2e_2 + 2e_3 \in (f_6 + e_3)$), a contradiction.

(a_2) The edges f_1, f_5 and f_4 have each one endvertex dominated by a vertex in a 4-cycle containing the respective edges $f_1 - e_2$, $f_5 - e_2$ (edge pair not contemplated in case (a_1)) and $f_4 - e_3$ (forced, since $f_4 - e_2$) containing vertex $-2e_2 \in (f_1 - e_2)$). But then only one of f_{11} and f_{10} must be dominated by $f_{11} - e_2$ or $f_{10} - e_2$, while the remaining one must be dominated by $f_{11} - e_3$ or $f_{10} - e_3$, which produces a contradiction since $f_{11} - e_2 \in (f_1 - e_2)$, and $f_{10} - e_2 \in (f_5 - e_2)$.

(b) The edges of F are:

$$\begin{array}{ll}
f_1 \,=(-e_1-e_2,-e_1-e_2-e_3) & f_2 \,=(-e_1+e_3,-e_1-e_2+e_3)\\
f_3 \,=(-e_1+e_2+e_3,-e_1+2e_2+e_3) & f_4 \,=(-e_1+2e_2,-e_1+2e_2-e_3)\\
f_5 \,=(2e_2-e_3,e_1+2e_2-e_3) & f_6 \,=(2e_2+e_3,e_1+2e_2+e_3)\\
f_7 \,=(2e_1+2e_2-e_3,2e_1+e_2-e_3) & f_8 \,=(2e_1+2e_2,2e_1+2e_2+e_3)\\
f_9 \,=(2e_1-e_3,2e_1-e_2-e_3) & f_{10}=(-e_2-e_3,e_1-e_2-e_3)\\
f_{11}=(-e_2+e_3,e_1-e_2+e_3) & f_{12}=(2e_1-e_2,2e_1-e_2+e_3)\\
f_{13}=(-e_1-e_3,-e_1+e_2-e_3) & f_{14}=(2e_1+e_3,2e_1+e_2+e_3).
\end{array}$$

We may assume step by step that the first ten edges of F have each an endvertex dominated by the copy of K_2 containing respectively:

- f_1-e_1,
- f_2+e_3 (forced, since f_2-e_1 contains $e_3-2e_1\in(f_1-e_1)$),
- f_3-e_1 (forced, since f_3+e_3 contains $e_2+2e_3-e_1\in(f_2+e_3)$),
- f_4+e_2 (forced, since f_4-e_1 contains $2e_2-2e_1\in(f_3+e_1)$),
- f_5-e_3 (forced, since f_5+e_2 contains $3e_2-e_3\in(f_4+e_2)$),
- f_6+e_3 (forced, since f_6+e_2 contains $3e_2+e_3\in(f_4+e_2)$),
- f_7+e_1 (forced, since f_7-e_3 contains $2e_1+2e_2-2_3\in(f_5-e_3)$),
- f_8+e_2 (forced, since f_8+e_1 contains $3e_1+2e_2\in(f_7+e_1)$),
- f_9-e_3 (forced, since f_9+e_1 contains $3e_1-e_3\in(f_7+e_1)$) and
- $f_{10}-e_2$ (forced, since $f_{10}-e_3$ contains $e_1-e_2-2e_3\in(f_9-e_3)$).

Now, f_{11} does not have an endvertex dominated by any copy of K_2 in the presence of the previous forced dominations of copies of K_2 (since $f_{11}-e_2$ dominates $\{-2e_2,-2e_2\}\subset\Theta(f_{10}-e_2)$ while $f_{11}+e_3$ contains $2e_3-e_2\in(f_2+e_3)$).

If just one or three corners of Θ'_0 (in this second case, for corner distance triple either $(3,3,6)$ or $(3,5,8)$) were in S, the remaining vertices of $\Theta'_0-\Theta^*_0$ forms no 1-factor F, contradicting the existence of S. In the case of one corner, let this corner be $\theta_1=-e_1-e_2-e_3$, which dominates $\theta_1+e_1,\theta_1+e_2$

and $\theta_1 + e_3$. Then F must contain:

$$\begin{array}{ll} f_1 = (e_1 - e_2 - e_3, 2e_1 - e_2 - e_3) & f_2 = (-e_1 + e_2 - e_3, -e_2 + 2e_2 - e_3) \\ f_3 = (-e_2 + 2e_2, -e_2 + 2e_2 + e_3) & f_4 = (-e_1 + e_3, -e_1 + e_2 + e_3) \\ f_5 = (2e_1 - e_2, 2e_1 - e_2 + e_3) & f_6 = (-e_1 - e_2 + e_3, -e_2 + e_3). \end{array}$$

Now, F should also contain $f_7 = (e_1 - e_2 + e_3, 2e_1 - e_2 + e_3)$, with its terminal vertex already present in f_5, a contradiction. With three corners and distance triple $(3, 3, 6)$, let these corners be θ_1, $\theta_2 = 2e_1 - e_2 - e_3$ and $\theta_3 = 2e_1 + 2e_2 - e_3$. Then F must contain $f_1 = (-e_1 + e_2 - e_3, -e_1 + 2e_2 - e_3)$ and $f_2 = (-e_1 + 2e_2 - e_3, 2e_2 - e_3)$ that have a vertex in common, a contradiction. With distance triple $(3, 5, 8)$, let the three corners be θ_1, θ_2 and $\theta'_3 = \theta_3 + 2e_3$. Then F must contain f_1 as above and $f'_2 = f_2 + 2e_3$, leaving vertex $-e_1 + 2e_2$ not in F, another contradiction.

We will rule out the cases of only two corners of Θ'_0 being in S. If the two are at distance 3, they may be taken, up to symmetry as $\theta_1 = -e_1 - e_2 - e_3$ and $\theta_2 = 2e_1 - e_2 - e_3$. In $\Theta'_0 - \Theta^*_0 - N[\theta_1] - N[\theta_2]$, we note a unique 1-factor F, formed by edges $f_1 = (-e_1 - e_2 + e_3, -e_2 + e_3)$, $f_2 = (e_1 - e_2 + e_3, 2e_1 - e_2 + e_3)$, $f_3 = (2e_1 + e_3, 2e_1 + e_2 + e_3)$, $f_4 = (2e_1 + 2e_2, 2e_1 + 2e_2 + e_3)$, $f_5 = (2e_2 + e_3, e_1 + 2e_2 + e_3)$, $f_6 = (2e_2 - e_3, e_1 + 2e_2 - e_3)$, $f_7 = (-e_1 + 2e_2 - e_3, -e_1 + e_2 - e_3)$, etc. The copies of K_2 containing $f_1, \ldots, f_6$ can be taken dominated, by symmetry and forcedly, by the copies of K_2 containing $f_1 - e_2$, $f_2 + e_3$, $f_3 + e_1$, $f_4 + e_2$, $f_5 + e_3$ and $f_6 - e_3$ respectively. The 4-cycle induced in S that contains θ_1, also contains forcedly the vertices $\theta_1 - e_1$, $\theta_1 - e_1 - e_3$ and $\theta_1 - e_3$. But then, f_7 cannot be dominated in S, a contradiction.

Now, assume that the two corners are at distance 5. They may be taken up to symmetry as $\theta_1 = -e_1 - e_2 - e_3$ and $\theta_2 = 2e_1 - e_2 + e_3$. In $\Theta'_0 - \Theta^*_0 - N[\theta_1] - N[\theta_2]$ we observe a unique 1-factor F, formed by edges $f_1 = (-e_1 + e_2 - e_3, -e_1 + 2e_2 - e_3)$, $f_2 = (e_1 - e_2 - e_3, 2e_1 - e_2 - e_3)$, $f_3 = (2e_2 + e_3, e_1 + 2e_2 + e_3)$, $f_4 = (2e_1 + 2e_2, 2e_1 + 2e_2 - e_3)$, $f_5 = (2e_1 - e_3, 2e_1 + e_2 - e_3)$, $f_6 = (-e_1 + e_3, -e_1 + e_2 + e_3)$, $f_7 = (2e_2 - e_3, e_1 + 2e_2 - e_3)$, etc. If the edge $(\theta_1, \theta_1 - e_3)$ is in S, then $f_1 - e_1$, $f_6 - e_3$, $f_3 + e_2$, $f_4 + e_1$ and $f_7 - e_3$ dominate respectively f_1, f_6 f_3, f_4 and f_7. But then f_5 cannot be dominated in S, a contradiction. So, F forces the 4-cycle with vertices θ_1, $\theta_1 - e_1$, $\theta_1 - e_1 - e_2$ and $\theta_1 - e_2$ to be in S. In this case, the copies of K_2 associated to f_1, f_2, f_7 and f_4 are dominated respectively by the copies of K_2 containing $f_1 - e_3$, $f_2 - e_3$, $f_7 + e_2$ and $f_4 + e_1$. It follows that f_5 cannot be dominated by an edge at distance 1 from it in $\Lambda_3 - \Theta'_0$, a contradiction.

It is easy to see that two corners at distance 6 or 8 do not allow even the definition of a 1-factor F in $\Theta'_0 - \Theta^*_0$ minus the two corners and their neighbors.

We pass to consider the different cases of four corners of S in $\Theta'_0 - \Theta^*_0$. The case of S having three corners on the affine plane $< e_1, e_2 > -e_3$ and one corner in the affine plane $< e_1, e_2 > +e_3$, or viceversa, is readily seen to lead to no 1-factor F in $\Theta'_0 - \Theta^*_0$ minus these corners and their neighbors. Else, either:

Instance (A): If the four corners in S are $\theta_1 = -e_1 - e_2 - e_3$, $\theta_2 = 2e_1 - e_2 - e_3$, $\theta_3 = -e_1 + 2e_2 + e_3$ and $\theta_4 = 2e_1 + 2e_2 + e_3$, then a 1-factor F of $\Theta'_0 - \Theta^*_0 - \cup_{i=1}^4 N[\theta_i]$ is formed by the edges $f_1 = (-e_1 - e_2 + e_3, -e_1 + e_3)$, $f_2 = (-e_1 + e_2 - e_3, -e_1 + 2e_2 - e_3)$, $f_3 = (-e_2 + e_3, e_1 - e_2 + e_3)$, $f_4 = (2e_1 - e_2 + e_3, 2e_1 + e_3)$, $f_5 = (2e_1 + e_2 - e_3, 2e_1 + 2e_2 - e_3)$, $f_6 = (2e_2 - e_3, e_1 + 2e_2 - e_3)$. We first rule out the case of the edges $(\theta_1, \theta_1 - e_1)$ and $(\theta_3, \theta_3 + e_3)$ being in S (or any other pair of edges in the same relative geometrical positions as these two, with respect to Θ'_0). In this case, f_1 cannot be dominated by any copy of K_2: the two candidates, $f_1 - e_1$ and $f_1 + e_3$ cannot be in S. Because of this, three cases can be distinguished here up to symmetry, for the 4-cycles corresponding respectively to the four corners above, namely:

(a) $\Theta_1 = (\theta_1, \theta_1 - e_1, \theta_1 - e_1 - e_2, \theta_1 - e_2)$, $\Theta_2 = (\theta_2, \theta_2 + e_1, \theta_2 + e_1 - e_2, \theta_2 - e_2)$, $\Theta_3 = (\theta_3, \theta_1 - e_1, \theta_3 - e_1 - e_2, \theta_3 - e_2)$, $\Theta_4 = (\theta_4, \theta_2 + e_1, \theta_4 + e_1 - e_2, \theta_4 - e_2)$. Then the following edges must be in S, dominating forcedly the edges of F: $f_1 + e_3$, $f_2 - e_3$, $f_3 - e_2$, $f_4 + e_3$, $f_5 - e_3$, $f_6 + e_2$. The following 4-cycles are induced by S: $\Theta_5 = (-e_1 + e_2 - e_3, -e_1 + 2e_2 - e_3, -e_1 + 2e_2 - 2e_3, -e_1 + e_2 - 2e_3)$ and $\Theta_6 = (2e_1 + e_2 - e_3, 2e_1 + 2e_2 - e_3, 2e_1 + 2e_2 - 2e_3, 2e_1 + e_2 - 2e_3)$. The graphs $\Theta'_5 - \Theta^*_5$ and $\Theta'_6 - \Theta^*_6$ have the respective vertices $x = -3e_3$ and $y = e_1 - 3e_3$ as non-corner vertices, so they cannot dominate $z = -2e_3$ and $w = e_1 - 2e_3$, yielding a contradiction.

(b) $\Theta_1 = (\theta_1, \theta_1 - e_2, \theta_1 - e_2 - e_3, \theta_1 - e_3)$, $\Theta_2 = (\theta_2, \theta_2 - e_2, \theta_2 - e_2 - e_3, \theta_2 - e_3)$, $\Theta_3 = (\theta_3, \theta_3 + e_2, \theta_3 + e_2 + e_3, \theta_3 + e_3)$, $\Theta_4 = (\theta_4, \theta_4 + e_2, \theta_4 + e_2 + e_3, \theta_4 + e_3)$. Then the following edges must be in S, dominating forcedly the edges of F: $f_1 - e_1$, $f_2 - e_1$, $f_4 + e_1$, $f_5 + e_1$ and possibly:

(b$_1$) $f_6 + e_2$, in which case: **(b$_{11}$)** either $\Theta_5 = \Theta_0 - 3e_3$ is in S and dominates $\Theta_0 - 2e_3$, so that $x = -e_1 + e_2 - 2e_3$ cannot be dominated by any of its neighbors; **(b$_{12}$)** or $\Theta_5 = (-3e_3, e_1 - 3e_3, e_1 - 4e_3, -4e_3)$ is in S, so the end-vertices of the edge $g = (e_2 - 2e_3, e_1 + e_2 - 2e_3)$ cannot be dominated by S;

(b$_2$) $f_6 - e_3$, in which case the end vertices of the edge $g = (e_2 - 2e_3, e_1 + e_2 - 2e_3)$ cannot be in S or dominated by S, since $h = (e_2 - 3e_3, e_1 + e_2 - 3e_3)$ cannot be in S.

(c) $\Theta_1 = (\theta_1, \theta_1 - e_1, \theta_1 - e_1 - e_2, \theta_1 - e_2)$, $\Theta_2 = (\theta_2, \theta_2 - e_2, \theta_2 - e_2 - e_3, \theta_2 - e_3)$, $\Theta_3 = (\theta_3, \theta_1 - e_1, \theta_3 - e_1 - e_2, \theta_3 - e_2)$, $\Theta_4 = (\theta_4, \theta_4 + e_2, \theta_4 + e_2 + e_3, \theta_4 + e_3)$. Then the following edges must be in S, dominating forcedly the edges of F: $f_1 + e_3$, $f_2 - e_3$, $f_3 - e - 2$, $f_4 + e_1$, $f_5 + e_1$, $f_6 + e_2$. It follows that $x = -2e_3$ cannot be dominated by S.

Or **Instance (B)**: For the rest, we need by symmetry only to consider the case in which the four corners of S in $\Theta'_0 - \Theta^*_0$ are $\theta_1 = -e_1 - e_2 + e_3$, $\theta_2 = 2e_1 - e_2 + e_3$, $\theta_3 = -e_1 + 2e_2 + e_3$ and $\theta_4 = 2e_1 + 2e_2 + e_3$. In the intersection of the affine plane $< e_1, e_2 > -e_3$ and $\Theta'_0 - \Theta^*_0$, a 1-factor F is formed by the edges of the copies of K_2 that should be dominated externally (off Θ'_0) by induced copies of K_2 in S (parts themselves of 4-cycles induced by S). We may assume that this 1-factor is formed by the edges $f_1 = (-e_2 - e_3, e_1 - e_2 - e_3)$, $f_2 = (2e_2 - e_3, e_1 + 2e_2 - e_3)$, $f_3 = (-e_1 - e_2 - e_3, -e_1 - e_3)$, $f_4 = (-e_1 + e_2 - e_3, -e_1 + 2e_2 - e_3)$, $f_5 = (2e_1 - e_2 - e_3, 2e_1 - e_3)$ and $f_6 = (2e_1 + e_2 - e_3, 2e_1 + 2e_2 - e_3)$. It is enough to consider by symmetry three cases of how F could be dominated externally, as just mentioned, These cases have in common that f_1 is dominated by $f_1 - e_2$, f_3 by $f_3 - e_3$, f_4 by $f_4 - e_1$, and differ in that:

(a) f_2 is dominated by $f_2 + e_2$, f_5 by $f_5 + e_1$, f_6 by $f_6 - e_3$;

(b) f_2 is dominated by $f_2 - e_3$, f_5 by $f_5 - e_3$, f_6 by $f_6 + e_1$;

(c) f_2 is dominated by $f_2 + e_2$, f_5 by $f_5 - e_3$, f_6 by $f_6 + e_1$.

In either case, by considering the dominating 4-cycle $\Theta_1 = (-e_1 - e_2 - 2e_3, -e_1 - 2e_3, -e_1 - 3e_3, -e_1 - e_2 - 3e_3)$, the corresponding $\Theta'_1 - \Theta^*_1$ contains two corners at distance 5, namely $x = -2e_2 - e_3$ and $y = -2e_1 + e_2 - e_3$, which was ruled out above.

We just finished showing that there do not exist non-lattice like PDS$[Q_2]$s in Λ_3. Thus, the only standing case of a PDS$[Q_2]$ in Λ_3 is the lattice-like one that remained by means of the commented programming code at the beginning of the present proof that leads to the generator matrix (1) or its associated group epimorphism $\Phi : \mathbb{Z}^3 \to \mathbb{Z}_{20}$. This establishes the statement of the theorem. $\square$

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