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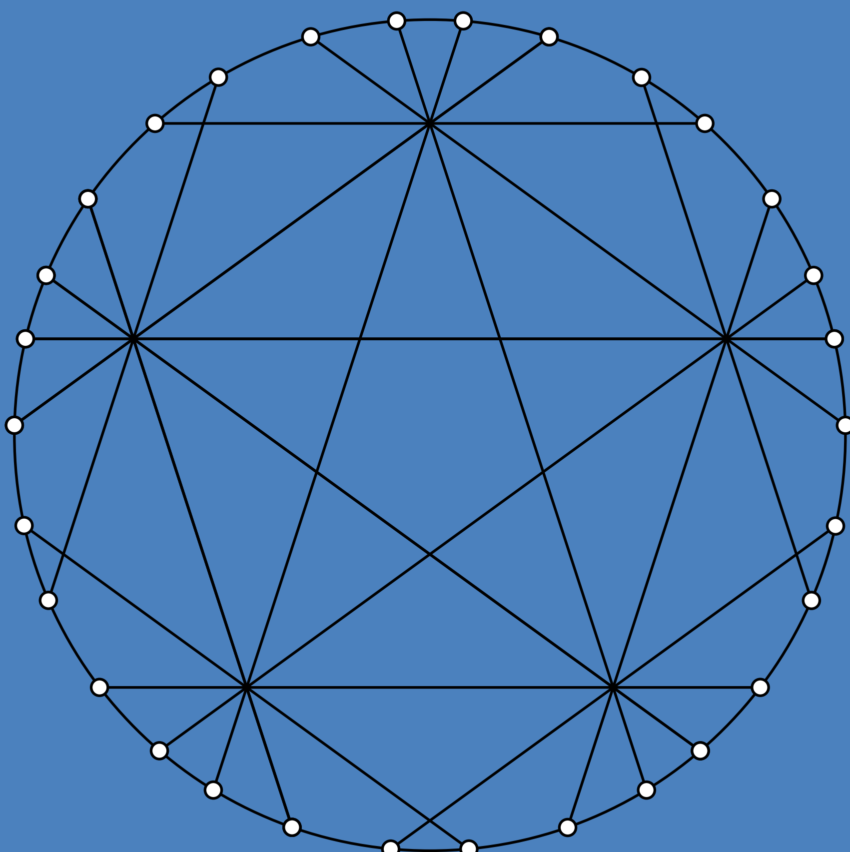

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Horadam Sequences: A Survey Update and Extension

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Abstract

We give an update on work relating to Horadam sequences that are generated by a general linear recurrence formula of order two. This article extends a first ever survey published in early 2013 in this *Bulletin*, and includes coverage of a new research area opened up in recent times.

1 Introduction

1.1 Preamble

Looking back over time we can see that, within the vast and ill-defined field of applied mathematics, the advent of serious computing power in the 1970s began to accelerate a then already discernible breach between classic analytical mathematics (concentrating on problems and theories involving continuous phenomena) and that which tackled discrete concepts and entities. The latter has, over the last sixty years or so, developed its own characteristics and subject personality—aided *en route* by increased exposure to software tools/languages—and it is in this broad sphere of what

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is nowadays known as discrete mathematics that the theory of integer sequences lies, with many applications now found in number theory and combinatorics to name but two areas. Away from these, difference equations and discrete dynamical systems together define a further fundamental and wide-ranging field of research promoting discrete models of importance in the natural sciences, engineering, economics and finance; topics of particular interest to mathematicians include iteration theory, complex dynamics, chaos theory, topological and combinatorial dynamics, stability theory, boundary value problems, symmetries and integrable systems, q-difference equations, ergodic theory, numerical analysis, dynamic equations on time-scale, and difference-differential equations. The umbrella covering linear difference equations in particular forms part of this domain of interest, under which resides those of order two whose most general form is the essence of a Horadam sequence. Difference equations appear, of course, through strong connections with a variety of observed evolution and other natural phenomena (continuous time systems are measured in discrete time), and as such they constitute important modelling descriptors. They also arise when differential equations are discretised in readiness for numerical interrogation.

Certain sequences have had more impact than others in grabbing the attention of researchers—some because they display interesting mathematical features or else possess an impressive array of enumerative interpretations (or both), others for different reasons. Sitting venerably at the head of the large class of second order recurrence sequences, and with an accessible general term closed form (for each characteristic root case), the Horadam sequence engenders a natural desire to examine its properties and occupies a position in which breadth and depth are to be found in results cultivated from its basic definition and from its close connections with 'near neighbour' sequences (*i.e.*, part specialisations, or others recoverable from it such as Fibonacci, Pell, Lucas, Jacobsthal, Tagiuri, Fermat, and so on). A major element of its appeal surely lies with the fact that it offers scope for analysis which retains, for the most part, a necessary level of algebraic manageability even in its full generality, in addition to which, as we shall see, still today the sequence has new theoretical and applications aspects of itself to disclose after half a century of existence and concomitant intellectual scrutiny.

1.2 The Horadam Sequence

We recall that, given (arbitrary) initial values $w_0 = a$, $w_1 = b$, a Horadam sequence $\{w_n\}_{n=0}^{\infty} = \{w_n\}_0^{\infty} = \{w_n(a, b; p, q)\}_0^{\infty}$ is defined by the linear recurrence

$$w_n = pw_{n-1} - qw_{n-2} \tag{1}$$

of order 2; it accommodates many well documented sequences of interest which have initial values a, b and/or parameters p, q fully or part specialised (they include, as alluded to above, so called 'primordial' and 'fundamental' ones of particular interest for historical reasons, see Section 2.1 later). Roots of the characteristic polynomial $\lambda^2 - p\lambda + q$ for (1) give rise to separate degenerate ($p^2 = 4q$) and non-degenerate ($p^2 \neq 4q$) case closed forms for w_n which are standard undergraduate exercises to construct. For $p^2 \neq 4q$ ($p, q \neq 0$), there are two distinct characteristic roots $\alpha(p, q) = (p + \sqrt{p^2 - 4q})/2$, $\beta(p, q) = (p - \sqrt{p^2 - 4q})/2$, with $\alpha + \beta = p$, $\alpha\beta = q$ and, for $n \geq 0$, a closed form

$$\begin{aligned} w_n(a, b; p, q) &= w_n(\alpha(p, q), \beta(p, q), a, b) \\ &= \frac{(b - a\beta)\alpha^n - (b - a\alpha)\beta^n}{\alpha - \beta}. \end{aligned} \tag{2}$$

For $p^2 = 4q$, on the other hand, the characteristic roots co-incide as simply $\alpha(p) = \beta(p) = p/2$ and, for $n \geq 0$,

$$\begin{aligned} w_n(a, b; p, p^2/4) &= w_n(\alpha(p), a, b) \\ &= bn\alpha^{n-1} - a(n - 1)\alpha^n. \end{aligned} \tag{3}$$

1.3 Remit

A survey article [37] was published in the early part of 2013, attempting to set down a good portion of research carried out on Horadam sequences, and highlight those behind it, from the point in time when the sequence was first properly announced as a mathematical construct in the 1960s—Horadam and one or two contemporaries began the process at the beginning of the decade but it was two now very familiar 1965 papers of his that are considered instrumental in formalising the start of a sustained period of study yet to end, being thus works of major provenance in the genealogy of the sequence. The nature of investigations reported in [37] displayed a diverse collection of ideas, analysis and results, since which new papers have appeared and, not surprisingly, some missed first time around have been brought to our attention. In view of these, and some positive responses we received (including that from Professor Horadam himself, to whom the article was dedicated), it seems appropriate to add to that survey which was designed to serve as both a timeline of research and as a resource for the discrete mathematics community; in doing so, the visibility of the sequence is maintained and its relative standing underscored.

Mathematical fashions come into vogue, and then become somewhat *passé*, at an ever increasing pace—in this they mimic other areas of science and art. As well as being an update to work on the Horadam sequence this article is offered in the same vein as its forerunner—as an acknowledgement of a longevity to the profile of the Horadam sequence within the body of published mathematical writings and, by association, as a recognition of its creator Alwyn F. Horadam. In framing both contextually and technically this area of study, they may combine to act as a useful prospectus for someone planning to enter the great adventure of reseach within it and so assist any beginner in choosing a starting point which is both meaningful and credible.

2 Research Update and Extension

This comprises three subsections—one concerning general topics of interest, another describing works on the subject of (both real and complex) Horadam sequence periodicity in which the author has been involved over a recent period of time, and a further one discussing other selected articles and emergent applications of the sequence; a summary section completes matters. It should be mentioned that during the review process a not inconsiderable number of papers looked at were observed to begin by introducing the notion of a general Horadam sequence but to then immediately assign values to some parameters and analyse the resulting sequence(s)—accordingly, many are not suitable for inclusion in any survey article, particularly those that focus on second order recurrence sequences with little or no generality to them. There is also an issue surrounding the nature/level of some works produced in this area, but its discussion would best form part of a wider one—outwith the article in hand here, certainly—on what constitutes research.

2.1 General Topics of Interest

As a first addendum to [37] we note that Bunder [12], as well as giving a variety of results on Horadam terms (closed forms, recurrences, ratio limits, and connections between $w_n(a, b; p, q)$ and terms $w_n(0, 1; p, q), w_n(2, 1; p, q)$ from sequences with particular initial values), generalised a number theoretic result of D. Gerdemann to give necessary and sufficient conditions under which—for two consecutive values of n—a finite linear combination of Horadam terms $w_{n+h}, w_{n+h+1}, \ldots, w_k$ combines as a single Horadam term w_n possessing a multiplicative constant which is itself a combination of characteristic root powers. Bunder also features in a 1975 omission [11] from the previous survey in which, given $z_0 = a$, $z_1 = b$, on defining a sequence $\{z_n\}_0^\infty$ through the power product recurrence $z_n = (z_{n-1})^p(z_{n-2})^q$ $(n \geq 2)$ it is possible to identify an explicit closed form $z_n = a^{w_n(1,0;p,-q)}b^{w_n(0,1;p,-q)}$ for the general $(n+1)$th term of $\{z_n\}_0^\infty$ valid for $n \geq 0$. A short inductive proof (and a generalised version) of Bunder's original observation is seen in [36], and [38] offers a new and concise proof from first principles; note that a three deep version of his non-linear recurrence has a closed form solution also related to Horadam type sequences (this work is in press).

The works [11, 36, 38] have echoes in a paper by A.G. Shannon. Let $r \geq 1$ and consider a set of r recurrence sequences, each of order r, the sth one of which we denote as

$$\begin{aligned}\{u_{s,n}^{(r)}\}_{n=1}^\infty &= \{u_{s,n}^{(r)}\}_1^\infty \\ &= \{u_{s,n}^{(r)}(u_{s,1}^{(r)}, u_{s,2}^{(r)}, \ldots, u_{s,r}^{(r)}; P_{r,1}, P_{r,2}, \ldots, P_{r,r})\}_1^\infty. \qquad (4)\end{aligned}$$

For any $s = 1, \ldots, r$, it is defined by the alternating sign linear recurrence

$$u_{s,n}^{(r)} = \sum_{j=1}^{r}(-1)^{j+1}P_{r,j}u_{s,n-j}^{(r)}, \qquad n \geq r+1, \qquad (5)$$

and characterised by the r initial values $u_{s,1}^{(r)}, u_{s,2}^{(r)}, \ldots, u_{s,r}^{(r)}$ and the r recurrence variables $P_{r,1}, P_{r,2}, \ldots, P_{r,r}$. These r-sets of recurrence sequences have been discussed by Shannon in a useful overview [57] where he inter-relates some of the major results by, and generalised sequences associated with, A.F. Horadam, H.C. Williams, A.N. Philippou and E. Lucas. Shannon introduces such sequence sets in the context of generalised Horadam type sequences. Interestingly, the imposition of exclusive 0,1 initial sequence values by him, according to the rule $u_{s,n}^{(r)} = \delta_{s,n}$ $(n = 1, \ldots, r)$ involving the Kroneker Delta function, immediately links them to aforementioned works. In the $r = 2$ case, for example, then a pair of sequences $\{u_{s,n}^{(2)}\}_1^\infty$ are defined, where $s = 1, 2$; these are, specifically,

$$\begin{aligned}
\{u^{(2)}_{1,n}\}_1^\infty &= \{u^{(2)}_{1,n}(u^{(2)}_{1,1}, u^{(2)}_{1,2}; P_{2,1}, P_{2,2})\}_1^\infty \\
&= \{w_n(1, 0; P_{2,1}, P_{2,2})\}_0^\infty, \\
\{u^{(2)}_{2,n}\}_1^\infty &= \{u^{(2)}_{2,n}(u^{(2)}_{2,1}, u^{(2)}_{2,2}; P_{2,1}, P_{2,2})\}_1^\infty \\
&= \{w_n(0, 1; P_{2,1}, P_{2,2})\}_0^\infty, \qquad (6)
\end{aligned}$$

with sequences $\{w_n(1, 0; P_{2,1}, -P_{2,2})\}_0^\infty$ and $\{w_n(0, 1; P_{2,1}, -P_{2,2})\}_0^\infty$ central to [11, 36, 38].

On a different topic, Yazlik and Taskara obtained in 2012 the spectral norm and eigenvalues of a circulant matrix comprising so called generalized k-Horadam numbers [62], followed by publication of a paper [64] which gave the determinant and inverse of such a matrix (the articles each contain at least one evaluated sum involving these k-Horadam numbers); related earlier work by the authors and by Kocer *et al.* was mentioned in [37], and this work was extended also in [65]. Referring to the Horadam number as a generalised Fibonacci number, in 2009 Čerin [17] gave formulas for sums of products of two Horadam terms (from distinct sequence sources) differing in both the initial conditions producing them and position within their respective sequence. Extensions of these sums—involving the stand alone inclusion, or added combinations of, multiplier term(s) in the summand (that is, summing index binomial coefficients/linear polynomials/integer powers)—were also examined, and closed forms found by him. The interplay between Horadam sequence elements and tri-diagonal matrix determinants was moved forward [60] by Taskara *et al.*, who related matrix entries to Horadam numbers and characteristic roots, going on to show that sequence terms can be represented as the determinant of a tri-diagonal matrix comprising entries taken from the four defining parameters a, b, p, q of a Horadam sequence. Yazlik and Taskara's k-Horadam sequence $\{H_{k,n}\}_{n\geq 0}$ satisfies the recurrence $H_{k,n+2} = f(k)H_{k,n+1} + g(k)H_{k,n}$ (with initial values $H_{k,0} = a$, $H_{k,1} = b$), where $f(k), g(k)$ are scalar valued polynomials. In a follow on paper [63] to a 2012 publication (cited in [37]), they defined a negatively subscripted generalised k-Horadam sequence and derived some relationships between positively and negatively subscripted sequence terms and both permanents and determinants of tri-diagonal matrices whose entries contain these numbers.

In 2011 Kiliç *et al.* [24] extended work by Melham [48] to investigate certain sums consisting of products of at most two terms of $\{w_n(a, b; p, -1)\}_0^\infty$, the results necessarily involving other derivative sequences. The main features of these articles are (i) the types of alternating and non-alternating sums chosen for examination, and (ii) a variability in the lower values of summing indices; we note that Melham was influenced strongly by a 1981 paper of D.L. Russell [56] which delivered evaluations of sums of single Horadam terms, and products and squares of terms, with certain restrictions on the p, q parameters of (1) (see also the 1969 paper of Iyer [21] which is relevant here).

Associating a 2×2 matrix $W(p, q)$ with the basic Horadam recurrence equation (1), G. Cerda [15] gave results for properties of powers of W, making connections with the Binet closed forms of the well known initial values specific sequences $w_n(0, 1; p, q)$ and $w_n(2, p; p, q)$ (termed the respective nth generalised Fibonacci and Fibonacci-Lucas sequence, although known classically as the (fundamental) generalised Fibonacci and (primordial) generalised Lucas sequences. A subsequent publication [16] gave a host of further identities for the same sequences through the introduction of a so called generalised Lucas matrix $V(p, q)$ for which he also formulated results and used in combination with $W(p, q)$.

Bunder has developed the work in [11, 36, 38] by considering the natural order of operations (addition, multiplication, exponentiation, tetration, and so on) in the context of recurrence sequences, see [13]. Recalling that the standard linear Horadam recurrence for w_n is based on the sum of two products (pw_{n-1} and $-qw_{n-2}$), and that for z_n is based on the product of two exponentiated terms ($(z_{n-1})^p$ and $(z_{n-2})^q$), he develops the idea of operationally higher order recursions effecting the exponentiation of exponents and incorporated in a Horadam-style recurrence relation deploying a function associated with W. Ackermann from the 1920s as a tool to achieve it. Noting that the closed forms (2),(3) for w_n are representable via simple arithmetic functions, and z_n has a form involving particular initial values instances of w_n, he shows that in this respect no further sequence generalisations exist other than in a small number of special cases.

A different area of work, which eluded the authors of [37], is now brought to attention. In 1993 Terracini [61] investigated the convergence, under a variety of conditions, of quotients w_{n+1}/w_n of neighbouring Horadam sequence terms, with all four recurrence parameters a, b, p, q said to belong to a normed field $\mathcal{K}$. Among the topics examined were circumstances under which the sequence of quotient terms exists and, assuming $p \neq 0$, its analytic convergence properties for $\mathcal{K} = \mathbf{R}$ (with $p^2 - 4q > 0$) and for $\mathcal{K} = \mathbf{Z}$ (with $p^2 - 4q \neq 0$). He also showed that in the case $q = \pm 1$, $\mathcal{K} = \mathbf{R}$, it is possible to choose suitable initial values a, b such that the quotients w_{n+1}/w_n are convergents of the simple continued fraction for the characteristic root with largest magnitude. This latter result extended one by P. Kiss [27] (see also [28, 31]) who published a number of papers on these themes and related problems of interest. For example, a 1991 paper [29] surveyed results concerning the diophantine approximative property of second order linear recurrences and discussed the dependence of the characteristic root estimate on characteristic polynomial discriminant (also [33]). Based on some results for Lucas numbers, Shannon and Horadam studied the kth convergent p_k/q_k of the continued fraction $CF(w_n) = w_n - Q_n/[w_n - Q_n/(w_n - Q_n/\{w_n - \ldots\})]$ (where $Q_n = (b - a\beta)(a\alpha - b)(\alpha\beta)^n/(\alpha - \beta)^2$), and showed that the sequence of convergent numerators $\{p_k\}_0^\infty$ forms a generalised Fibonacci (*i.e*, Horadam type) sequence $\{p_k(1, w_n; w_n, Q_n)\}_0^\infty$ [58].

Other theory by Kiss and others is to be found on the distribution of term ratios [32] (building on Mátyás [47]), and on the geometric properties of both 2D and 3D points whose co-ordinates are successive Horadam terms [23] (in doing so, furthering earlier work from the 1980s by Horadam [20] (who, referring to a related 1974 paper by Jaiswal [22], considered the loci of such points in the x, y plane for some particular p, q values and examined briefly higher dimensional cases) and by Bergum [10] (who extended the analysis of Horadam)). For initial values 0,1, Kiss and Mátyás—with certain constraints placed on recurrence parameters—gave an approximation to π in terms of Horadam sequence elements which for large N is correct to $O(1/\log(N))$ [30].

In a 1979 paper on yet another aspect of the Horadam sequence, Kiss [26] studied zero terms which enabled him to give upper and lower bounds for the general term of a particular sequence type—this improved on a similar (lower) bound by Mignotte [49] and generalised a result by Stewart [59]; a connected paper from the mid 1960s of relevance, and referenced by Kiss, is that of Mahler [46].

At this juncture we mention one further paper by Horadam himself, not featuring in the survey [37]. It is a short 1979 offering [19] in which he extended separate results by Berzsenyi and Zeilberger from earlier in the decade so as to evaluate sums of products of Horadam sequence terms. The work was based on a single recurrence identity, taken from one of his two seminal 1965 papers, relating elements of the Horadam sequence with those of the (Fibonacci generalisation) sequence $\{w_n(1, p; p, q)\}_0^\infty$.

Early progress was slow in discrete mathematics for a considerable time, during which it enjoyed but a relatively small number of active participants. One reason put forward for this was the inverse relationship between an abundance of inherently enumerative problems and the number of standardised theories available for application—this reduced many of the former to solution by bespoke methods and feats of technical acrobatics, so to speak, formed necessarily in an *ad hoc* manner. Those days are long gone, and as the general field has developed so have areas such as the theory of sequences which now boasts a solid foundation of fundamental results to underpin it (supported by the wonderful On-Line Encyclopaedia of Integer Sequences web platform)—specific work on the Horadam sequence forms only a part, of course, but it is not an inconsequential one.

The study of this sequence has, from tentative awakenings in the late 1950s, passed through the usual phases replicated by no end of other mathematical spheres, subjects and smaller clusters of individual topics of interest: beginning with a natural exuberance and creative energy for it (individual solutions to problems sometimes acting as facilitators of, or pointers to, a broader analysis/theory waiting to be uncovered), momentum gained from early results was followed by periods of consolidation and in turn maturity. Some new, and in many ways surprising, aspects of the Horadam sequence have, however, even now become manifest at this relatively advanced point in time, one striking case in point involving the basic notion of cyclicity—it is this to which we turn next.

2.2 Periodicity

In a deviation from classic types of research, we report on recent attempts to blaze a combined theoretic-computational trail through what is a zone of rich and varied periodic properties exhibited by some sequences along the real line and in the complex plane.

The author has been part of a new and absorbing series of investigations into Horadam sequence periodicity which is proving to be both fruitful and informative. In [5] periodic (and some non-periodic) behaviour has been classified using properties of fundamental governing variables called generators that absorb the characteristc roots of a recurrence sequence and, together with its initial conditions, allow a convenient representation of the sequence for analysis. It is possible to determine all essential types of orbits in the complex plane, with resulting maps visually pleasing and technically instructive. Further examples of closed path geometric configurations—such as polygons and bipartite graphs—are presented in [8], and together these papers demonstrate a surprising variety in the nature of cyclic sequence patterns. The question of precisely how many paths of fixed period exist is one that has also been addressed [6], where the number—denoted for period k as the enumerative function $H_P(k)$—is found to increase with

$k \geq 1$ as $1, 1, 3, 5, 10, 11, 21, 22, 33, 34, 55, 46, 78, \ldots$, producing a first enumerative context for the O.E.I.S. Sequence No. A102309 registered as long ago as 2005 as a mere abstract mathematical construct. Two equivalent formulas are proposed for the sequence $H_P(k)$ which each involve Euler's totient function (one counts relatively prime primitive generators to identify all possible orbits, while the other counts divisors of the period), and an evaluation of the lower and upper bounds of sequence terms is also made.

We should point out that all of the theory developed around Horadam periodicity was in fact motivated by three non-complex sequences of type $\{w_n(1, \sqrt{s}; \sqrt{s}, 1)\}_0^\infty$ which, for $s = 1, 2, 3$, have respective period $6, 8, 12$ and are shown to arise from evaluations of so called Catalan polynomials [39]. These polynomials feature in another, and somewhat different, study [41] where the idea of Horadam periodicity is pursued using a matrix based approach through which conditions for cyclicity are formulated and verified. Both non-degenerate and degenerate characteristic root cases are discussed as the methodology is brought to bear on the problem, with periodic behaviour shown to be centred around the notion of an identity triplet $[p, q, \delta]$ (p, q are the characterising parameters of (1), and δ the sequence period). The underlying ideas are developed, and lead to a hitherto unseen phenomenon termed 'masked' periodicity discussed separately in relation to the non-degenerate roots case [40]. What is meant by this is that a fully general (arbitrary initial values) Horadam sequence can mask, or hide, one or two special case (specific initial values) sequence(s) of smaller period. The salient factors enabling this to occur are determined, through which it becomes evident that the underlying causes are dictated by the defining property of a primitive root of unity; this is confirmed using the generator approach [7], where the notion of masking is shown to extend to higher order linear recurrence sequences and an order three example given accordingly (it might also be mentioned that using a new generating function method—which also addresses both characteristic root cases—we are able to obtain additional results which inform further the occurrences of masking and recover other previously noted periodic sequence behaviours; the work will hopefully be published in the not too distant future). An observation made in [41] reports on a simple procedure to generate sequences with any chosen period, the article [43] detailing how indeed this works in practice and giving some illustrative instances; Catalan polynomials are once more intrinsic to what is a simple algorithm to produce self-repeating sequences that is rather unique in kind.

It is well known that the ability to pick out, *a priori*, those critical influences underpinning the behaviour/dynamics of a system or model is a real talent in mathematics and other scientific disciplines. Sometimes, however, they emerge as a product of the particular approach taken towards analysis, and this applies here to an extent—from the combined papers [5, 6, 7, 8, 39, 40, 41, 43] we are in a position to understand the inherent cyclic potential of a Horadam sequence from more than one mathematical stance. It remains, though, an open question as to how the generator driven perspective of periodicity might be fully unified with that of its alternative conceptualisation through the theory of matrices, and is one for which more study beckons.

2.3 Other Works and Applications

In Section 3 of [40] the authors have given an overview of appearances by the 2×2 matrix $\mathbf{A}(p,q) = \left(\begin{smallmatrix} p & -q \\ 1 & 0 \end{smallmatrix}\right)$ (characterised by the Horadam recurrence parameters p, q) that frames the forerunner study [41]. While the majority of references cited therein deal with the aforementioned fundamental and primordial sequences, one is pertinent to us here: Rosenbaum [55] used a matrix approach in which he employed $\mathbf{A}(p,-q)$ to formulate the degenerate and non-degenerate characteristic root case closed forms of $w_n(a,b;p,-q)$ described by (2),(3) with characteristic roots $\alpha(p,-q)$, $\beta(p,-q)$ modified accordingly. We cannot omit to mention, too, as a point of completeness, that the closed forms (2),(3)—on which so much work has been based over the years—were derived in a novel, and apparently little known, fashion by Niven and Zuckerman in 1960 [50]; the methodology adopted is alluded to in [38], and it seems that the only formal reference to their technique before then was made by R.G. Buschman [14] as long ago as 1963 (this is surprising, as the technique might have application to the solution of linear recursions of order three or more if it lends itself to extension). For those with interest, we note that using the terminology 'generalised Fibonacci sequence' as a synonym for Horadam sequence, Austin and Austin—writing at a level they describe as educational mathematics—showed how, based on the essential Binet type expression for a sequence general term closed form, one can generate many order two integer sequences and find the associated recurrence formulas describing them [1].

Non-linear difference equations have been the subject of much attention over the last couple of decades or so, with solution analysis of low order rational difference equations used as prototypes for the study of higher order systems (whose forms are many and varied). Characteristics such as attractivity, boundedness, stability and periodicity have caught the interest of many researchers, noting that the role of a linear order two Horadam type recursion as the linearised version of a difference equation

$x_{n+1} = F(x_n, x_{n-1})$ for a rational non-linear function F is not new, and facilitates the study of local solution stability about an equilibrium point. On this theme Halim and Bayram [18] have, for instance, considered the difference equation $x_{n+1} = q/(p + x_{n-k})$, determining its stability properties and asymptotic behaviours based on the fact that its solution can be expressed in terms of elements from the (fundamental) Horadam sequence $\{w_n(0,1;p,-q)\}_0^\infty$; among the results given, it is shown that the equation has a unique convergent equilibruim point $E = \frac{1}{2}(-p + \sqrt{p^2 + 4q}) \in \mathbf{R}^+ = (0,\infty)$ which is asymptotically stable both locally and (since it is a global attractor) globally. A related paper is that of Bacani and Rabago [3] who have studied the pair of difference equations $x_{n+1} = q/[p + (x_n)^v]$, $y_{n+1} = q/[-p + (y_n)^v]$, giving interesting results on their solutions in the case $v = 1$ and some observations on solutions for $v > 1$—all framed by the same sequence $\{w_n(0,1;p,-q)\}_0^\infty$. It is noted (Theorem 15 therein) that for $v > q = p + 1$ the difference equation for x_n has a prime period 2 solution of form $\{\ldots, q/p, q/[p + (q/p)^v], \ldots\}$, with similar results existing (Theorems 22 and 23) for the y_n equation.

In a deviation away from this area, the sequence $\{w_n(0,1;p,-q)\}_0^\infty$ underpins much of the work of Rabago [53] in his article on homogeneous second order linear recurrence differential equations with period k (see also his results on periodic functions [54], where this specific sequence occurs once more). In a rather unusual 2012 offering, Rabago [52] gave a formula to generate intermediate (or 'missing') terms from a finite string of Horadam terms $w_n(a,c;p,-q)$ in which only the first element $w_0 = a$ and the last in the string are specified; denoting the final prescribed term as b, the author shows that the key result is the relation $w_1 = c = [b + w_n(0,1;p,-q)aq]/w_{n+1}(0,1;p,-q)$. Another nice quirk of mathematics is that any sequence $\{x_n\}_0^\infty$ satisfying the arithmetic-geometric recurrence relation $x_{n+1} = ax_n + (a+d)rx_{n-1} + (a+2d)r^2x_{n-2} + \cdots + (a+nd)r^nx_0$ (with d, r the common difference and common ratio of the usual arithmetic and geometric progressions) can be shown to satisfy a reduced order two Horadam type recursion [2].

Two essential properties of the Horadam sequence—being the linearity of its governing recurrence equation, and the resulting availability of closed form general term formulas—are pleasing ones to motivate theoretical study. Application is a topic whose potential itself shows much promise, and an example can be seen in [4]. Uniformly distributed pseudo-random number generators are commonly employed in numerical algorithms and simulations. In the article a pseudo-random number generation algorithm is based on the geometric properties of a complex Horadam sequence, and for certain model parameters the sequence exhibits uniformity in the distribution of arguments. This feature was exploited to design a pseudo-random number generator which was evaluated using Monte Carlo π estimations and found to perform comparatively favourably with ones in standard usage such as the Multiplicative Lagged Fibonacci and the 'twister' Mersenne generators (see also [9] which mentions this work and consolidates that of [8]).

In addition to the creation of pseudo-random number generators, recurrence sequences of given length have found applications in the study of multi-phase signals for different radio electronic systems. It is anticipated that Horadam sequences with long periods will provide insights into the problem of how best to produce an area covering (in some defined sense) that consists of a multitude of Horadam sequence terms; in this context, the computational complexity of efficiently generating terms in high volume becomes interesting, with tractability potentially influenced by the use of parallel processors. The ability to create a large number of recurrence sequences over finite fields might also offer up applications in cryptography. Geometric patterns related to the Fibonacci numbers have been linked to optimal solutions for the layout of mirrors in a concentrated solar power plant, so that, given variability allowed in the four characterising parameters of (1), the study of Horadam sequences may provide a deeper understanding of how plants such as the chamomile or sunflower optimise the patterns of their flowers—this in turn could lead to new data distribution algorithms, the development of novel data search techniques, or the design of structures with certain properties optimised.

Applications (and the potential for them) aside, new theoretical features of Horadam sequences continue to emerge. The classic geometric mean sequence has—drawing on the result of Bunder in [11] referred to earlier (Section 2.1)—delivered new identities which connect Jacobsthal numbers with parameterised familes of Horadam numbers [45]; they are established first by inference, and then proven independently. In the balanced power (that is, $p + q = 1$, $q \neq -1$ (or $p \neq 2$)) case of Bunder's recurrence, the functional exponent of a scalar multiplier introduced to the defining recurrence equation shows dependency on a Horadam sequence (as well as that exhibited in the powers of both initial sequence values a, b) [42]. In [44] sequence based closed form entries of an exponentiated 2-square matrix offer the general term of a particular family type of polynomials (each family part of a larger class) in terms of two separate, but equivalent, generalised Fibonacci polynomials that are each seen to be derived from functional versions of a Horadam recursion.

We repeat again that little attention has been paid to part specialised Horadam sequences, other than to mention some works involving the (fundamental) generalised Fibonacci and (primordial) generalised Lucas sequences (these being recognised as important ones), for to do so would have made the task much bigger than it already is—even as this article goes to press, other papers are manifesting themselves as relevant to its remit (such as that on computing sums of products of binary sequences [25] and on the solutions nature of a certain diophantine equation [51]).

3 Summary

One of the defining features of mathematics, as a subject, is that some areas have managed to successfully steer a path between the pragmatists and abstracters who in extreme cases reside at opposite ends of the research spectrum. This is due, in the main, to their appeal to both types of mindset, and while falling under a broad applied classification it would seem that interest in Horadam sequences has come mostly from those slightly on the purer side of the great mathematical divide.[1] An undeniably important factor in the advance of science is the discerning of patterns and regularities in nature, so that more and more phenomena can be subsumed into general categories and laws. The same can be said of similar observations made in many mathematical fields, and there must be plenty of examples in which analysis of particular second order linear recurrence sequences has fuelled deeper theory formulated for the more general Horadam sequence. There is, of course, clear evidence to show that a reverse path is possible whereby results found for the Horadam sequence have immediate consequences for, and application to, a plethora of special case instances; through this two-way process one can easily appreciate its attraction.

We have seen built up a body of literature that has evolved steadily, and collectively offers an impressive array of results based on decades of endeavour exploiting the rather giving nature of the sequence (there is some variability in quality across the totality of outputs, as suggested at the beginning of Section 2, but this is to be expected). That contributions show no sign of stalling merits their continued documentation here as a source of encouragement to study and develop the theory and applications of Horadam sequences still further. As this article is drawn to a close there is,

[1] We should mention that the notion of modulo periodicity has also been examined but—as this seemed to move away from the original ideas of Horadam and associates in the early 1960s, and so the profile of work which followed in consequence—it was not included in the survey [37], nor is it addressed here.

perhaps, a general point to be made. Academic surveys and expositions which offer technical appraisal—even when comprehensive in scope and well written—too often attract little regard for some reason and are used blithely for convenience as part of other types of publication held as simply more worthwhile; this is both unfortunate and unfair (the author has written on the issue at some length in [34]). If not combining as something which is close to a definitive account, it is nevertheless hoped that together this work and its precursor [37] will at least stand as an informative and panoptic record of activity to be owned by the current community of mathematicians, and in turn bequeathed to a future generation of analysts for whom a productive time still awaits as they tease out more insights into the acclaimed Horadam sequence—certainly, one would hope that once verdant pastures of discovery are not yet exhausted, and remain fertile *terra incognita* for the observant and willing mathematician journeying through the land of linear recurrence sequences.

Dedication

This article is dedicated to the memory of Alwyn F. Horadam, who passed away in July 2016. A personal tribute to the man and his sequence was published in this journal by the present author in [35]; note that the aforementioned (Section 2.3) closed form Horadam sequence term formulations of Niven and Zuckerman [50] are set out in an appendix therein, to which the interested reader is directed.

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