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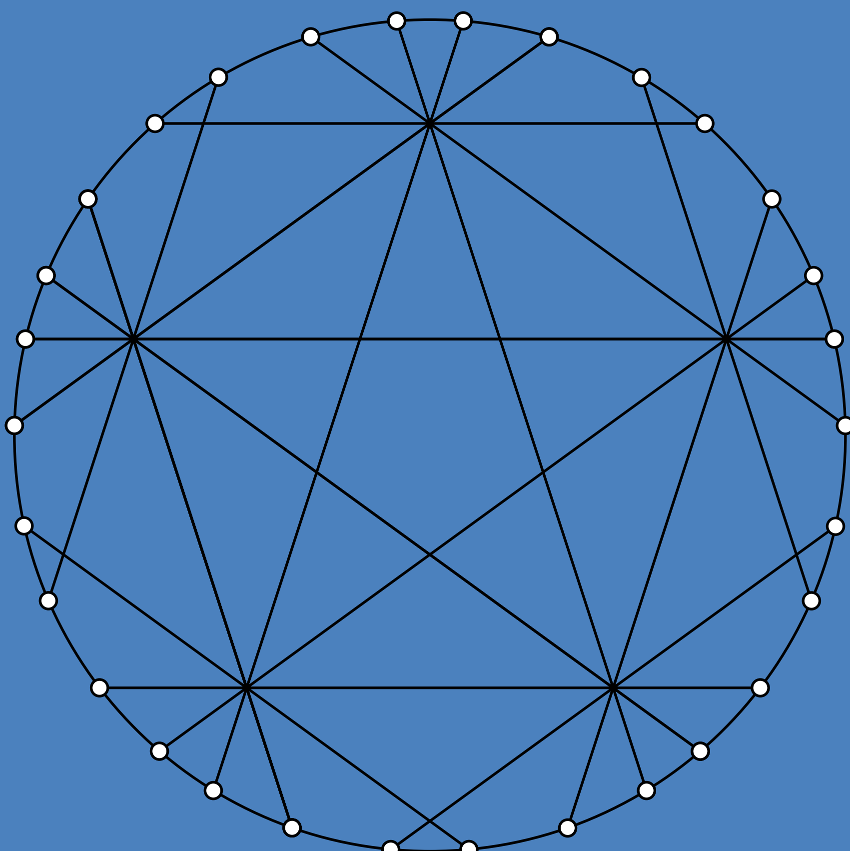

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A New Method for Constructing Circuit Codes

Kevin M. Byrnes *

Abstract

Circuit codes are cycles in the graph of the n dimensional hypercube. They are theoretically and practically important, as circuit codes can be used as error correcting codes. A circuit code is characterized by three parameters: its dimension, its spread (which determines how many errors it can detect), and its length (which determines its accuracy). We present a new method for constructing a circuit code of spread $k+1$ from a circuit code of spread k. This method leads to record code lengths for 18 circuit codes of spread $k = 7$ and 8 in dimension $22 \leq n \leq 30$. We also derive a new lower bound on the length of circuit codes of spread 4, which improves upon bound suggested by Singleton for dimension $n \geq 86$.

Keywords: Circuit Code, Snake in the Box, Coil in the Box, k-Coil, Error Correcting Code

1 Introduction

Let $I(n)$ denote the graph of the n dimensional hypercube, that is the graph on 2^n vertices where each vertex corresponds to a binary vector of length n, and two vertices x and x' are adjacent if their binary vectors differ in exactly one position. For any subgraph G of $I(n)$ and any two vertices $x, x' \in G$ we define the distance $d_G(x, x')$ as the minimum number of edges in G needed to travel from x to x'. If there is no path in G from x to x' then $d_G(x, x') = \infty$. Observe that $d_{I(n)}(x, x')$ equals the number of positions where the binary vectors corresponding to x and x' differ.

*E-mail: dr.kevin.byrnes@gmail.com

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A *circuit* C is a graph consisting of a sequence of distinct vertices $(x_1, \ldots, x_N)$ where each pair of cyclically consecutive vertices is adjacent, and the edges between these consecutive vertices. For brevity we will often say that $C = (x_1, \ldots, x_N)$ is a circuit, in which case the edges are implied. For any pair of vertices x_i, x_j in a circuit $C = (x_1, \ldots, x_N)$ with $i < j$ there are exactly two paths between x_i and x_j in C, traversing the edges: $x_i x_{i+1}, \ldots, x_{j-1} x_j$ and $x_j x_{j+1}, \ldots, x_{N-1} x_N, x_N x_1, \ldots, x_{i-1} x_i$ respectively. An *n-dimensional code* is a subgraph of $I(n)$.

Definition 1.1. *A subgraph C of $I(n)$ is a circuit code of spread k (an (n, k) circuit code) if:*

1. *C is a circuit.*
2. *If x and x' are vertices of C with $d_{I(n)}(x, x') < k$ then $d_C(x, x') = d_{I(n)}(x, x')$.*

An equivalent characterization of circuit codes was proven by Klee.

Lemma 1.2 (Klee [14] Lemma 2)**.** *An n-dimensional circuit code C of length $N \geq 2k$ has spread k if and only if for all vertices $x, x' \in C$, $d_C(x, x') \geq k \Rightarrow d_{I(n)}(x, x') \geq k$.*

Finding long circuit codes is practically and theoretically important, since circuit codes can be used as error-correcting codes [12]. Circuit codes of spread 1 are known as Gray codes [8], and circuit codes of spread 2 are known as coils or snakes in the box (however, current terminology uses "snake" to refer to an open path) [12]. Both of these types of circuit codes have been extensively studied. Let $K(n, k)$ denote the maximum length of an (n, k) circuit code, it is well-known that $K(n, 1) = 2^n$ and $K(n, 2) \geq \frac{77}{256} 2^n$ [1]. In contrast, circuit codes of spread $k \geq 3$ are less-well understood and exact values for $K(n, k)$ are generally only known for $n \leq 17$ and $k \leq 7$ and some special (n, k) pairs.

In this note we present a simple new construction for generating a circuit code of spread $k + 1$ from a circuit code of spread k. This allows the better studied codes of smaller spreads to be leveraged to create codes of larger spreads, and results in 18 new records for codes of spread 7 and 8, and in dimension $22 \leq n \leq 30$. Specifically, we prove the following theorem.

Theorem 1.3. *Let C be an (n, k) circuit code with length $N \geq 2(k + 1)$. Then there exists an $(n + r, k + 1)$ circuit code C' with length $N' = N + 2q$, where $q = \lceil \frac{N}{2(k+1)} \rceil$ and $r = \lceil \log_2 q \rceil + 1$.*

A useful application of Theorem 1.3 is a new lower bound on $K(n,4)$ which improves upon the lower bound suggested by Singleton [20] when $n \geq 86$.

Theorem 1.4. *For $n \geq 6, K(\lfloor 1.53n \rfloor, 4) \geq 40 \cdot 3^{(n-8)/3}$, and hence $K(n,4) \geq 40 \cdot 3^{(\lfloor .6535n \rfloor - 8)/3}$.*

2 Previous Constructions and Bounds

We begin by surveying the theoretical lower bounds for $K(n,k)$ and some of the most important constructions used in their derivation. Exact values for $K(n,k)$ are known for only a few special cases, given in Table 1.

Table 1: Exact values for $K(n,k)$.

$K(n,k) = 2n$	for $n < \lfloor \frac{3k}{2} \rfloor + 2$	(See [20])
$K(\lfloor \frac{3k}{2} \rfloor + 2, k) = 4k+6$	for k even	(See [7])
$K(\lfloor \frac{3k}{2} \rfloor + 2, k) = 4k+4$	for k odd	(See [7])
$K(\lfloor \frac{3k}{2} \rfloor + 3, k) = 4k+8$	for k odd ≥ 9	(See [7])

The following constructions apply for a wide variety of (n,k) combinations. Here we state the "result" of each construction and refer the reader to the original paper for the precise construction details.

Construction 2.1 (Singleton [20]). *Let C be an (n,k) circuit code with length N. Then there exists an $(n+1,k)$ circuit code C' with length $N' = N + 2\lfloor \frac{N}{2k} \rfloor$.*

Construction 2.2 (Singleton [20]). *Let C be an (n,k) circuit code with length N, and $k \geq 3$. Then there exists an $(n+2,k)$ circuit code C' with length $N' = N + 4\lfloor \frac{N}{2(k-1)} \rfloor$.*

Construction 2.3 (Singleton [20]). *Let C be an (n,k) circuit code with length N for $k \geq 3$ and k odd. Then there exists an $(n + \frac{k+1}{2}, k)$ circuit code C' with length $N' = N + (k+1)\lfloor \frac{N}{k+1} \rfloor$.*

Construction 2.4 (Singleton [20]). *Let C be an (n,k) circuit code with length N for $k \geq 2$ and k even. Then there exists an $(n + \frac{k+2}{2}, k)$ circuit code C' with length $N' = N + (k+2)\lfloor \frac{N}{k+1} \rfloor$.*

Construction 2.5 (Deimer [5]). *Let C be an $(n+1, k+1)$ circuit code with length N. Then there exists an (n,k) circuit code C' with length $N' \geq N - \lfloor \frac{N}{n+1} \rfloor$.*

Construction 2.6 (Klee [14]). *Let k be even and let $2 \leq n_1 \leq n_2$. Suppose C_1 is an $(n_1, k-1)$ circuit code of length $N_1 \geq 2k$ where N_1 is divisible by k, and suppose C_2 is an (n_2, k) circuit code with length $N_2 \geq 2k$. If $k = 2$ there exists an $(n_1 + n_2, k)$ circuit code C' of length $N' = \frac{N_1 N_2}{k}$. If $k \geq 4$ there exists an $(n_1 + n_2 + 1, k)$ circuit code C' of length $N' = \frac{N_1(N_2+2)}{k}$.*

These constructions result in the following lower bounds for $K(n, k), k \geq 3$.

Table 2: Lower bounds for $K(n, k)$.

$K(n,2) \geq \frac{77}{256} 2^n$		(See [1])
$K(n,3) \geq 32 \cdot 3^{(n-8)/3}$	for $n \geq 6$	(See [20])
$K(n,k) \geq (k+1) 2^{\lfloor 2n/(k+1) \rfloor - 1}$	for k odd and $\lfloor \frac{2n}{k+1} \rfloor \geq 2$	(See [20])
$K(n,4) \succ \delta^n$	for $0 < \delta < 3^{1/3}$	(See [14])
$K(n,k) \succ \delta^n$	for k even and $0 < \delta < 4^{1/k}$	(See [14])
$K(n,k) \gtrsim 4^{n/(k+1)}$	for odd $k > 3$	(See [14])

The last three inequalities in Table 2 are asymptotic bounds, where $f(n) \lesssim g(n)$ means $\liminf_{n \to \infty} g(n)/f(n) > 0$, and $f(n) \prec g(n)$ means $\lim_{n \to \infty} g(n)/f(n) = \infty$.

In addition to the previous constructions, the "necklace" construction of Paterson and Tuliani has been particularly important, leading to many new records for $K(n, k)$ [18]. However, identifying arrangements of necklaces satisfying the conditions of that construction required a backtrack search, limiting the dimensions examined to $n \leq 17$. The conditions placed upon the arrangement of necklaces also become more restrictive as k increases, and for the range of dimensions n examined, no suitable arrangements for codes of spread $k \geq 7$ were found [18].

For $n \leq 17$ and $k \leq 7$ many of the current records for $K(n, k)$ (reported in Table 3) have been set by computational methods, e.g. exhaustive search [15, 11], pruning based approaches [21, 16], genetic algorithms [19, 3, 6, 13], or other computational approaches [4, 22, 2].

3 Generating an $(n + r, k + 1)$ Circuit Code from an (n, k) Circuit Code

3.1 Transition Sequences

Each vertex of $I(n)$ corresponds to a binary vector of length n, so for every circuit $C = (x_1, \ldots, x_N)$ of $I(n)$ we can define a transition sequence $T = (\tau_1, \ldots, \tau_N)$ where τ_i denotes the position in which x_i and x_{i+1} (or x_N and x_1) differ. Using the convention that $x_1 = \vec{0}$ for any circuit, we see that the transition sequence corresponds uniquely to the edges in C. Since $I(n)$ is bipartite this implies $|T|$ is even [10].

Define a *segment* of a sequence $T = (\tau_1, \ldots, \tau_N)$ as a subsequence of cyclically consecutive elements. For any $x_i, x_j \in C = (x_1, \ldots, x_N)$ with $i < j$ there are exactly two segments in T between x_i and x_j, corresponding to the two paths in C traversing the edges: $x_i x_{i+1}, \ldots, x_{j-1} x_j$ and $x_j x_{j+1}, \ldots, x_{N-1} x_N, x_N x_1, \ldots, x_{i-1} x_i$. These segments are $(\tau_i, \tau_{i+1}, \ldots, \tau_{j-1})$ and $(\tau_j, \tau_{j+1}, \ldots, \tau_N, \tau_1, \ldots, \tau_{i-1})$. If $i = j$ then the two segments are $\varnothing$ and T. These segments are called complements because they partition T. If $\hat{T}$ is a segment in T, its complement is denoted $\hat{T}^{\mathsf{C}}$, and $(\hat{T}^{\mathsf{C}})^{\mathsf{C}} = \hat{T}$.

The set of *transition elements* $\{t_1, \ldots, t_m\}$ $(m \leq n)$ of T are the unique elements of T. When T is the transition sequence of a circuit each $t_i \in \{t_1, \ldots, t_m\}$ must appear in T an even number of times. A useful result to which we shall refer is the following.

Lemma 3.1 (Singleton [20]). *Let C be a circuit code of spread k and length $N \geq 2(k + 1)$ with corresponding transition sequence T. Then any $k + 1$ cyclically consecutive elements of T are all distinct.*

3.2 A New Circuit Code Construction

The idea behind proving Theorem 1.3 is to strategically insert members of a new set of transition elements $\{s_1, \ldots, s_r\}$ into T, the transition sequence of an (n, k) circuit code, so that the resulting sequence T' is the transition sequence of an $(n + r, k + 1)$ circuit code. An $(n + r, k + 1)$ circuit code can then be constructed by setting the first vertex to $\vec{0}$ and defining subsequent vertices from T'. As Example 1 illustrates, the straightforward approach of inserting all r new transition elements after each complete segment of T of length $k+1$ can fail to increase the spread. Thus a more careful approach (the following Construction 3.2, which is illustrated concretely in Example 2) is needed.

Example 1. *The following transition sequence from [14] results in a* $(6,2)$ *circuit code of length* 24*:*

$$T = (1,2,6,4,5,6,1,3,5,4,6,5,1,2,6,4,5,6,1,3,5,4,6,5).$$

For any $r > 0$ *there are three possible new transition sequences formed by inserting the sequence* $X = 7,\ldots,6+r$ *after the end of every segment of* T *of length 3, these are (temporarily ignoring overbraces):*

$$T' = (\overbrace{X,1,2,6,X,4,5,6,X,1,3,5,X,4},6,5,X,1,2,6,X,4,5,6,X,1,3,5,X,4,6,5)$$
$$T'' = (\overbrace{1,X,2,6,4,X,5,6,1,X,3,5,4,X},6,5,1,X,2,6,4,X,5,6,1,X,3,5,4,X,6,5)$$
$$T''' = (1,2,\overbrace{X,6,4,5,X,6},1,3,X,5,4,6,X,5,1,2,X,6,4,5,X,6,1,3,X,5,4,6,X,5)$$

Each of these sequences has length $N' = 24+8r$*. If* T', T''*, or* T''' *is the transition sequence of a spread* 3 *circuit code it follows from Lemma 1.2 that in every segment of length* ≥ 3 *corresponding to a shortest path in the circuit between two vertices, i.e. every segment with length between* 3 *and* $\frac{N'}{2}(= 12+4r)$*, at least* 3 *transition elements must appear an odd number of times. This condition is violated in* T', T''*, and* T''' *by the overbraced segments. Thus* T *cannot be extended to a* $(6+r,3)$ *transition sequence by inserting* X *after each segment of* T *of length* 3*.*

Unlike the simple method of Example 1, we will prove the following construction is guaranteed to result in the transition sequence of a circuit code of increased spread.

Construction 3.2.

Split T in half into $T^1 = (\tau_1, \ldots, \tau_{N/2})$ and $T^2 = (\tau_{N/2+1}, \ldots, \tau_N)$

$q \leftarrow \lceil \frac{N}{2(k+1)} \rceil$

Split T^1 into q segments:

$T^1_j = (\tau_{(k+1)\cdot(j-1)+1}, \ldots, \tau_{(k+1)\cdot j})$ for $j = 1, \ldots, q-1$

$T^1_q = (\tau_{(k+1)\cdot(q-1)+1}, \ldots, \tau_{N/2})$

Split T^2 into q segments:

$T^2_j = (\tau_{(k+1)\cdot(j-1)+N/2+1}, \ldots, \tau_{(k+1)\cdot j+N/2})$ for $j = 1, \ldots, q-1$

$T^2_q = (\tau_{(k+1)\cdot(q-1)+N/2+1}, \ldots, \tau_N)$

$r \leftarrow \lceil \log_2 q \rceil + 1$

Define new transition elements $\{s_1, \ldots, s_r\}$ with $T \cap \{s_1, \ldots, s_r\} = \varnothing$

for $j = 1$ to $q-1$ **do**

$i \leftarrow$ largest value in $\{1, \ldots, r-1\}$ such that 2^{i-1} divides j

$T'^1_j \leftarrow (T^1_j, s_i)$

$T'^2_j \leftarrow (T^2_j, s_i)$

$T'^1_q \leftarrow (T^1_q, s_r)$

$T'^2_q \leftarrow (T^2_q, s_r)$

return $T' = (T'^1_1, T'^1_2, \ldots, T'^1_q, T'^2_1, T'^2_2, \ldots, T'^2_q)$

Example 2 demonstrates how Construction 3.2 is applied to the transition sequence T of a $(10,3)$ circuit code. There (and elsewhere) we use T'^i to denote the segment $(T'^i_1, \ldots, T'^i_q)$ of T'.

Example 2. *A transition sequence* $T = (\tau_1, \ldots, \tau_N)$ *is* symmetric *if* $T^1 = (\tau_1, \ldots, \tau_{N/2}) = (\tau_{N/2+1}, \ldots, \tau_N) = T^2$. *Consider the transition sequence* T *of a symmetric* $(10,3)$ *circuit code of length* $N = 72$ *(from [20]) with*

$$T^1 = T^2 = (\underbrace{5,8,1,9}_{T^i_1}, \underbrace{6,10,1,8}_{T^i_2}, \underbrace{2,9,1,10}_{T^i_3}, \underbrace{7,8,1,9}_{T^i_4}, \underbrace{5,10,1,8}_{T^i_5}, \underbrace{3,9,1,10}_{T^i_6}, \underbrace{6,8,1,9}_{T^i_7}, \underbrace{7,10,1,8}_{T^i_8}, \underbrace{4,9,1,10}_{T^i_9}).$$

Here $q = \lceil \frac{72}{2(3+1)} \rceil = 9$, $r = \lceil \log_2 9 \rceil + 1 = 5$, $\{t_1, \ldots, t_m\} = \{1, \ldots, 10\}$, *and* $\{s_1, \ldots, s_5\} = \{11, \ldots, 15\}$. *Apply Construction 3.2 to* T *by splitting* T *into* T^1 *and* T^2 *and subdividing* T^i *into* $q = 9$ *segments as indicated. Then insert one of* $\{11, \ldots, 15\}$ *at the end of each* T^i_j *to get* T'^i_j *as follows:*

$$T'^i = (\underbrace{5,8,1,9,11}_{T'^i_1}, \underbrace{6,10,1,8,12}_{T'^i_2}, \underbrace{2,9,1,10,11}_{T'^i_3}, \underbrace{7,8,1,9,13}_{T'^i_4}, \underbrace{5,10,1,8,11}_{T'^i_5}, \underbrace{3,9,1,10,12}_{T'^i_6}, \underbrace{6,8,1,9,11}_{T'^i_7}, \underbrace{7,10,1,8,14}_{T'^i_8}, \underbrace{4,9,1,10,15}_{T'^i_9})$$

The sequence $T' = (T'^1, T'^2)$ will be the transition sequence for a $(15, 4)$ circuit code of length 90.

An important property of Construction 3.2 is that any segment of T' of length $\geq k + 2$ contains at least one member of $\{s_1, \ldots, s_r\}$. This is easily shown as follows. Since $N(= |T|)$ is even we have $|T^1| = |T^2| = N/2$, and therefore $q = \lceil \frac{N/2}{k+1} \rceil = \lceil \frac{|T^1|}{k+1} \rceil = \lceil \frac{|T^2|}{k+1} \rceil$. Because $T_1^1, \ldots, T_{q-1}^1$ and $T_1^2, \ldots, T_{q-1}^2$ all contain $k + 1$ elements, this means $|T_q^1| = |T_q^2| \in \{1, \ldots, k + 1\}$. Finally, since the segments $T_i'^1 (T_i'^2)$ of T' are formed by appending an element of $\{s_1, \ldots, s_r\}$ to the end of $T_i^1 (T_i^2)$ for $i = 1, \ldots, q$ we see that any segment of T' with length $\geq k + 2$ must contain the end of a segment $T_i'^1$ or $T_i'^2$ and therefore contains an element of $\{s_1, \ldots, s_r\}$.

The sequence $T' = (\tau_1', \ldots, \tau_{N'}')$ generated by Construction 3.2 naturally defines a sequence of vertices $(x_1', \ldots, x_{N'}')$ in $I(n + r)$ as follows. Fix $x_1' = \vec{0}$ and define x_{i+1}' as the vertex equal to x_i' in all positions except τ_i', for $1 \leq i \leq N' - 1$. Clearly x_i' is adjacent to x_{i+1}' for $1 \leq i \leq N' - 1$. The next two results establish that all the x_i' are distinct and that $x_{N'}'$ is adjacent to x_1'. Hence $C' = (x_1', \ldots, x_{N'}')$ is a circuit.

Lemma 3.3. *Let C be an (n, k) circuit code of length $N \geq 2(k + 1)$ and transition sequence T. Let $T' = (\tau_1', \ldots, \tau_{N'}')$ be the transition sequence resulting from applying Construction 3.2 to T. For $1 \leq i < j \leq N'$ let $\hat{T}$ be the segment $(\tau_i', \ldots, \tau_{j-1}')$ of T'. Then some transition element of $\hat{T}$ appears an odd number of times. Furthermore, if $\hat{T}$ contains one of the transition elements $\{s_1, \ldots, s_r\}$, then some $s_p \in \{s_1, \ldots, s_r\}$ appears in $\hat{T}$ exactly once.*

Proof. Let $\{t_1, \ldots, t_m\}$ be the transition elements of T, then the transition elements of T' are $\{t_1, \ldots, t_m\} \cup \{s_1, \ldots, s_r\}$. Let $A = \hat{T} \cap \{t_1, \ldots, t_m\}$ and let $B = \hat{T} \cap \{s_1, \ldots, s_r\}$. If $|B| = 0$ then $|A| \leq k + 1$, so $\hat{T}$ is a segment of T of length $\leq k + 1$. By Lemma 3.1 this means that every element of $\hat{T}$ is distinct, appearing exactly once.

Now suppose $|B| > 0$, we will show some $s_p \in \{s_1, \ldots, s_r\}$ appears in $\hat{T}$ exactly once. Either τ_i' or τ_{j-1}' are both in T'^1 or both in T'^2, or $\tau_i' \in T'^1$ and $\tau_{j-1}' \in T'^2$. Suppose τ_i' and τ_{j-1}' are both in T'^1 and let s_p denote the maximum index member of B. Then s_p appears in $\hat{T}$ exactly once, otherwise (by construction) s_w appears in $\hat{T}$ between two appearances of s_p for some $w > p$. But this contradicts the definition of s_p. The argument for when τ_i' and $\tau_{j-1}' \in T'^2$ is identical.

Now suppose that $\tau'_i \in T'^1$ and $\tau'_{j-1} \in T'^2$, then $s_r \in \hat{T}$. The transition element s_r appears in T' only in position $\frac{N'}{2} = (\frac{N}{2} + q)$ and $N'(= N + 2q)$. Since $j \leq N'$ and $\hat{T}$ ends with element τ'_{j-1}, we see that $\tau'_{N'} \notin \hat{T}$. Thus s_r occurs exactly once in $\hat{T}$. □

Corollary 3.4. *Let C be an (n,k) circuit code of length $N \geq 2(k+1)$ and transition sequence T. Let $T' = (\tau'_1, \ldots, \tau'_{N'})$ be the transition sequence resulting from applying Construction 3.2 to T, and let $(x'_1, \ldots, x'_{N'})$ be the vertex sequence defined by T'. Then $(x'_1, \ldots, x'_{N'})$ defines a circuit.*

Proof. Define $x'_{N'+1}$ as being equal to $x'_{N'}$ in all positions except $\tau'_{N'}$. Then travelling from x'_1 to $x'_{N'+1}$ requires using all of the transitions in T'. The transition elements of T' are $\{t_1, \ldots, t_m\} \cup \{s_1, \ldots, s_r\}$. Each t_i appears in T' the same number of times that it appears in T, an even number. By construction, each s_j appears an equal number of times in T'^1 and T'^2, so s_j appears an even number of times in T'. Since every transition element of T' appears an even number of times, we conclude that $x'_1 = x'_{N'+1}$. Thus in $(x'_1, \ldots, x'_{N'})$ every pair of cyclically consecutive vertices is adjacent. Now let $x'_i, x'_j \in (x'_1, \ldots, x'_{N'})$ with $i < j$, then $\hat{T} = (\tau'_i, \ldots, \tau'_{j-1})$ is a transition sequence between x'_i and x'_j in T'. By Lemma 3.3 some transition element of $\hat{T}$ appears an odd number of times and hence x'_i and x'_j are distinct. Hence $(x'_1, \ldots, x'_{N'})$ are all distinct and $(x'_1, \ldots, x'_{N'})$ defines a circuit. □

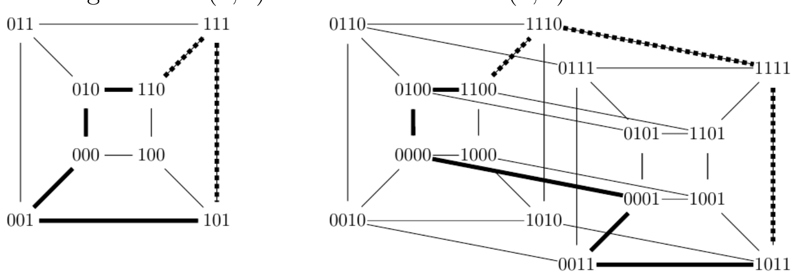
From Corollary 3.4 we see that T' defines a circuit $C' = (x'_1, \ldots, x'_{N'})$ in $I(n+r)$, and by construction, $N' = N + 2q$. Thus to prove Theorem 1.3 we only need to show that C' has spread $k+1$. To do so we require a technical result. If x is a vertex of $I(n)$ and $\tilde{n} < n$, we denote by x^* the "natural" projection of x onto $I(\tilde{n})$ formed by taking the first $\tilde{n}$ elements of the binary vector x. There is an important relationship between the transition sequence T' from Construction 3.2 and the transition sequence T of the underlying (n,k) circuit code C.

Lemma 3.5. *Let C be an (n,k) circuit code of length $N \geq 2(k+1)$ with transition sequence T. Let $T' = (\tau'_1, \ldots, \tau'_{N'})$ and $C' = (x'_1, \ldots, x'_{N'})$ be the transition sequence and circuit code (in dimension $n+r$) resulting from applying Construction 3.2 to T. Let $x'_i, x'_j \in C'$ with $i < j$ and let $\hat{T}$ be a shortest transition sequence in T' between x'_i to x'_j. Then $\hat{T} \cap \{t_1, \ldots, t_m\}$ is a shortest transition sequence in T between x'^*_i and $x'^*_j \in C$.*

Proof. Let $x'_i, x'_j \in C'$ with $i < j$, then there are two segments in T' between x'_i and x'_j. Let $\hat{T}$ denote the shorter of these (chosen arbitrarily if both segments have the same length) and let $\hat{T}^{\mathsf{C}}$ denote its complement. Then $\hat{T}^{\mathsf{C}}$ is also a segment between x'_i and x'_j in T'. Also note that x'^*_i and $x'^*_j \in C$. It is necessary that the subsequence $\hat{T} \cap \{t_1, \ldots, t_m\}$ is a segment between x'^*_i and x'^*_j in T. Since there are only two segments between x'^*_i and x'^*_j in T, and they partition T, we conclude that $\hat{T}^{\mathsf{C}} \cap \{t_1, \ldots, t_m\}$ is the other segment. Because $|\hat{T}| \leq |\hat{T}^{\mathsf{C}}|$, $|\hat{T}| \leq \frac{N}{2} + q$ and $\hat{T}$ contains no transitions spaced $\frac{N}{2} + q$ apart in T' (e.g. τ'_1 and $\tau'_{N/2+q+1}$ are spaced $\frac{N}{2} + q$ apart in T', as are τ'_{N+2q} and $\tau'_{N/2+q}$). For any $\tau'_\alpha, \tau'_\beta \in T'$ spaced $\frac{N}{2} + q$ apart, if $\tau'_\alpha \in \hat{T}$ then $\tau'_\beta \in \hat{T}^{\mathsf{C}}$. Also, if τ'_α is the υth element of T'^1 then τ'_β is the υth element of T'^2 (and similarly if $\tau'_\alpha \in T'^2$ and $\tau'_\beta \in T'^1$). Since elements of $\{s_1, \ldots, s_r\}$ are located in the same relative positions of T'^1 and T'^2, $\tau'_\alpha \in \{t_1, \ldots, t_m\} \iff \tau'_\beta \in \{t_1, \ldots, t_m\}$ (even if $\tau'_\alpha \neq \tau'_\beta$). So for every $\tau'_\alpha \in \hat{T} \cap \{t_1, \ldots, t_m\}$ there is a corresponding $\tau'_\beta \in \hat{T}^{\mathsf{C}} \cap \{t_1, \ldots, t_m\}$ (and no other $\tau'_\gamma \in \hat{T} \cap \{t_1, \ldots, t_m\}$ corresponds to this τ'_β). Thus $|\hat{T} \cap \{t_1, \ldots, t_m\}| \leq |\hat{T}^{\mathsf{C}} \cap \{t_1, \ldots, t_m\}|$. Hence $\hat{T} \cap \{t_1, \ldots, t_m\}$ is a shortest segment between x'^*_i and x'^*_j in T. □

Figure 1 illustrates this, showing a $(3, 2)$ circuit code C with transition sequence $T = (2, 1, 3, 2, 1, 3)$ (on the left) and the $(4, 3)$ circuit code C' (on the right) with transition sequence $T' = (2, 1, 3, 4, 2, 1, 3, 4)$ resulting from Construction 3.2. E.g. for $x'_i = 1100$ and $x'_j = 1011$ the shortest path in C' between x'_i and x'_j, indicated by dashed lines, "contains as a subpath" the shortest path in C between $x'^*_i = 110$ and $x'^*_j = 101$.

Figure 1: A $(3, 2)$ Circuit Code and a $(4, 3)$ Circuit Code.



We now have everything we need to proceed to the main proof.

Proof of Theorem 1.3. Let C be an (n,k) circuit code with length $N \geq 2(k+1)$ and transition sequence T. Apply Construction 3.2 to T to get a new transition sequence $T' = (\tau'_1, \ldots, \tau'_{N'})$ and vertex sequence $(x'_1, \ldots, x'_{N'})$. By Corollary 3.4, $C' = (x'_1, \ldots, x'_{N'})$ is a circuit and by construction $N' = N + 2q$, so it only remains to be shown that C' has spread $k+1$. By Lemma 1.2 it suffices to show for all vertices $x'_i, x'_j \in C'$ with $i < j$ that $d_{C'}(x'_i, x'_j) \geq k+1 \Rightarrow d_{I(n+r)}(x'_i, x'_j) \geq k+1$.

Suppose that x'_i and x'_j are vertices of C' with $d_{C'}(x'_i, x'_j) \geq k+1$. Let $\hat{T}$ denote the segment of T' that is the shorter transition sequence between x'_i and x'_j, and let $\hat{T}^{\complement}$ denote its complement. If $|\hat{T}| = |\hat{T}^{\complement}|$ either segment may be chosen. Note that $\hat{T}$ may "start" in T'^1 and end in T'^2, or the reverse, or may be entirely contained in T'^1 or T'^2. Finally, let $A = \hat{T} \cap \{t_1, \ldots, t_m\}$ and $B = \hat{T} \cap \{s_1, \ldots, s_r\}$, so $d_{C'}(x'_i, x'_j) = |A| + |B|$.

If $|B| = 0$ then $|A| = k+1$. In this case $\hat{T}$ is a segment of T of length $k+1$, and by Lemma 3.1 these transition elements are all distinct. So $d_{I(n)}(x'^*_i, x'^*_j) = k+1$ and $d_{I(n+r)}(x'_i, x'_j) = k+1$, and we are done.

Now suppose that $|B| > 0$. First we will show that some $s_p \in \{s_1, \ldots, s_r\}$ occurs an odd number of times in $\hat{T}$. If $\hat{T} = (\tau'_i, \ldots, \tau'_{j-1})$ then this follows from Lemma 3.3. Otherwise, then we have $\hat{T}^{\complement} = (\tau'_i, \ldots, \tau'_{j-1})$ and $|\hat{T}^{\complement}| \geq \frac{1}{2}N' = \frac{1}{2}(N+2q) \geq \frac{1}{2}(2(k+2)) = k+2$. By design of Construction 3.2 this means that $\hat{T}^{\complement} \cap \{s_1, \ldots, s_r\} \neq \varnothing$, so by Lemma 3.3 some $s_p \in \{s_1, \ldots, s_r\}$ occurs exactly once in $\hat{T}^{\complement}$. Because s_p occurs an even number of times in T', and since $\hat{T}$ and $\hat{T}^{\complement}$ are complements in T', s_p occurs an odd number of times in $\hat{T}$. In both cases, some $s_p \in \{s_1, \ldots, s_r\}$ appears an odd number of times in $\hat{T}$.

Now $d_{I(n+r)}(x'_i, x'_j) = d_{I(n)}(x'^*_i, x'^*_j) +$the number of members of $\{s_1, \ldots, s_r\}$ occuring an odd number of times in $\hat{T}$. If $d_{I(n)}(x'^*_i, x'^*_j) \geq k$ this is $\geq k+1$. Suppose $d_{I(n)}(x'^*_i, x'^*_j) < k$. By Lemma 3.5 A is a shortest transition sequence between x'^*_i and x'^*_j in T. Thus $|A| = d_C(x'^*_i, x'^*_j) = d_{I(n)}(x'^*_i, x'^*_j)$ since C has spread k. Furthermore, since $|A| < k$ we have $|B| \leq 2$, and since consecutive elements of B differ when $|\hat{T}| \leq \frac{N}{2} + q$ all elements of B must occur exactly once. Thus $d_{I(n+r)}(x'_i, x'_j) = |A| + |B| = d_{C'}(x'_i, x'_j) \geq k+1$. $\square$

4 A New Lower Bound for $K(n,4)$

Singleton [20] remarks that for $k \geq 4$ and even, the best lower bound available for $K(n,k)$ seems to be applying the third lower bound given in Table 2 to $K(n,k+1)$ (as every circuit code of spread $k+1$ is also a circuit code of spread k). In particular, for $k=4$ this gives $K(n,4) \geq 6 \cdot 2^{\lfloor 2n/6 \rfloor -1}$. Subsequently, Klee [14] established the much stronger asymptotic result: $K(n,4) \succ \delta^n$ for $0 < \delta < 3^{1/3}$, suggesting that non-asymptotic lower bounds stronger than $K(n,4) \geq 6 \cdot 2^{\lfloor 2n/6 \rfloor -1}$ may be possible. We will now prove that Theorem 1.3 gives a non-asymptotic lower bound that is stronger than $K(n,4) \geq 6 \cdot 2^{\lfloor 2n/6 \rfloor -1}$ for $n \geq 86$.

First we establish the following claim, our argument is a minor modification of the one given in Chapter 17 of [9].

Lemma 4.1. *For $n \geq 6$ there exists an $(n,3)$ circuit code C with length N divisible by 8 and satisfying $32 \cdot 3^{(n-8)/3} \leq N \leq \frac{24}{22} 32 \cdot 3^{(n-8)/3}$.*

Proof. Let C be an $(n,3)$ circuit code with transition sequence T. Suppose that t_i occurs m times in T. Construction $S5$ of [9] states that there is an $(n+3,3)$ circuit code C' with length $N' = N + 8m$, and t_i occurs $3m$ times in the new transition sequence T'. Note that if N is divisible by 4 and t_i appears $\frac{N}{4}$ times in T, then $N' = 3N$ and t_i appears $3m = \frac{N'}{4}$ times in T'.

For $n = 6,7,8$ consider the following transition sequences for $(n,3)$ circuit codes. Note that $|T_6| = 16$, $|T_7| = 24$, and $|T_8| = 32$. Also, 5 occurs 4 times in T_6, 2 occurs 6 times in T_7, and 8 occurs 8 times in T_8.

$T_6 = (1,5,2,6,3,5,4,6,1,5,2,6,3,5,4,6)$
$T_7 = (5,2,6,1,7,2,5,3,6,2,7,4,5,2,6,1,7,2,5,3,6,2,7,4)$
$T_8 = (5,2,6,8,1,7,2,8,5,3,6,8,2,7,4,8,5,2,6,8,1,7,2,8,5,3,6,8,2,7,4,8)$

Therefore by Construction S5 we see that for any $p \in \mathbb{N}$, in dimension $n = 6+3p$ there exists an $(n,3)$ circuit code with length $N = 16 \cdot 3^{(n-6)/3} \in (32 \cdot 3^{(n-8)/3}, \frac{16}{15} 32 \cdot 3^{(n-8)/3})$, in dimension $n = 7+3p$ there exists an $(n,3)$ circuit code with length $N = 24 \cdot 3^{(n-7)/3} \in (32 \cdot 3^{(n-8)/3}, \frac{24}{22} 32 \cdot 3^{(n-8)/3})$, and in dimension $n = 8+3p$ there exists an $(n,3)$ circuit code with length $N = 32 \cdot 3^{(n-8)/3}$. □

Proof of Theorem 1.4. Theorem 1.3 implies $K(n+r,4) \geq N + 2\lceil \frac{N}{2\cdot 4} \rceil \geq \frac{5}{4} N$, where $N \geq 2 \cdot 4$ is the length of an $(n,3)$ circuit code, $q = \lceil \frac{N}{2\cdot 4} \rceil$, and

$r = \lceil \log_2 q \rceil + 1$. From Lemma 4.1 we know that for $n \geq 6$ there exists an $(n, 3)$ circuit code C of length N divisible by 8, and $32 \cdot 3^{(n-8)/3} \leq N \leq \frac{24}{22} 32 \cdot 3^{(n-8)/3}$. Using this code we have $K(n+r, 4) \geq 40 \cdot 3^{(n-8)/3}$, $q = \frac{N}{2 \cdot 4}$ (by divisibility), and $r = \lceil \log_2 \frac{N}{2 \cdot 4} \rceil + 1 \leq \lfloor \log_2 \frac{N}{2 \cdot 4} \rfloor + 2$.

Now $2^{.53} > 3^{1/3}$ so $r \leq 2 + \lfloor \log_2 \frac{24}{22} 4 \cdot 3^{-8/3} \cdot 2^{.53n} \rfloor \leq .53n$. Hence $K(\lfloor 1.53n \rfloor, 4) \geq 40 \cdot 3^{(n-8)/3}$ for $n \geq 6$. And making the change of variables $u = 1.53n$ we get $K(\lfloor u \rfloor, 4) \geq 40 \cdot 3^{(\lfloor .6535u \rfloor - 8)/3}$. □

A simple analysis shows that the lower bound of Theorem 1.4 exceeds $6 \cdot 2^{\lfloor 2n/6 \rfloor - 1}$ for $n \geq 86$.

5 Computational Results

5.1 Methodology

The efficacy of Construction 3.2 was tested by applying it to circuit codes of spreads 2-9 in dimensions 3-30. Table 3 lists the greatest lower bound found for each (n, k) combination. The table was constructed as follows. For spreads 2-7 and dimensions 3-30 we seeded the table with empirical results from [20, 5, 11, 17, 2] which collectively survey all empirical records of which we are aware, for spreads 8 and 9 we seeded the table by using the exact bounds of Table 1 and the non-asymptotic lower bounds of Table 2.

Next, we applied Constructions 2.1 - 2.4 (collectively the “Singleton” constructions), the construction of Deimer (Construction 2.5), and the construction of Klee (Construction 2.6). Because these constructions were applied sequentially we iterated applying the constructions until there was no improvement in any entry of the table. To this “initial” table we then applied Construction 3.2 to the column corresponding to codes of spread k, replacing the appropriate entry in the neighboring column of the table (for codes of spread $k + 1$) if a larger lower bound was found. Each time after applying Construction 3.2 to codes of spread k we repeated the iterative application of the constructions of Singleton, Deimer, and Klee to propagate any further improvements in the lower bounds before applying the construction to codes of spread $k + 1$. Finally, after applying the construction to codes of all spreads we iteratively applied the constructions from Singleton, Deimer, and Klee once more.

Construction 2.6 was applied to our table as follows. Let C be an (n, k) circuit code with length $N > 2(k+1)^2$, and let $T = (\tau_1, \ldots, \tau_N)$ be its transition sequence with transition elements $\{t_1, \ldots, t_m\}$. Split T into $T^1 = (\tau_1, \ldots, \tau_{N/2})$, $T^2 = (\tau_{N/2+1}, \ldots, \tau_N)$ and subdivide T^i into $q = \lceil \frac{N}{2(k+1)} \rceil$ segments $T_1^i, \ldots, T_q^i$ of length $\leq k+1$ as in Construction 3.2 (where only segment T_q^i may have length $< k+1$). Note that $q > k+1$. For $i = 1, 2$ define new transition sequences $T'^1 = (T_1'^1, \ldots, T_q'^1)$ and $T'^2 = (T_1'^2, \ldots, T_q'^2)$ where $T_j'^i = (T_j^i, t_{m+1})$ for $j \leq p = (k+1)\lceil \frac{N}{2(k+1)} \rceil - \frac{N}{2}$, and $T_j'^i = T_j^i$ otherwise. Observe that $0 \leq p \leq k+1 < q$, so the $T_j'^i$ are well-defined. Finally combine T'^1, T'^2 into $T' = (T'^1, T'^2)$. Observe that t_{m+1} occurs an even number of times in T', and any two occurences of t_{m+1} are separated by a segment of T' which contains as a subsegment a segment of T of length $\geq k+1$. From this it can be shown that T' defines an $n+1$ dimensional circuit code C' of spread k (but not necessarily of spread $k+1$) and length $N' = N + 2p = 2(k+1)\lceil \frac{N}{2(k+1)} \rceil$. Thus C' satisfies the divisibility criterion of Construction 2.6 (for C_1). Because this method does not generate all $(n+1, k)$ circuit codes with length divisible by $k+1$, we also indicate in Table 3 when an entry exceeds the asymptotic lower bounds from Table 2 which are derived from Construction 2.6.

5.2 Discussion of Computational Results

Our construction found several new circuit codes for spreads of 7 and 8. Because codes of spreads 2-7 and dimensions 3-30 have been well-studied (see [11, 17] for surveys) the improvements noted in Table 3 for codes of spread 7 are perhaps the most significant. All of our new circuit codes of spread 7 and 8 are generated from the $(17, 6, 204)$ circuit code of [18], the $(15, 7, 60)$ and $(17, 7, 102)$ circuit codes of [11], and the $(18, 7, 116)$ circuit code resulting from applying Construction 2.1 to the $(17, 7, 102)$ circuit code. Applying Construction 3.2 to these 4 circuit codes, we have: $(17, 6, 204) \rightarrow (22, 7, 234)$, $(15, 7, 60) \rightarrow (18, 8, 68)$, $(17, 7, 102) \rightarrow (21, 8, 116)$, and $(18, 7, 116) \rightarrow (22, 8, 132)$. From these 4 new circuit codes, all of which are of record length, we generate the remaining circuit codes as follows.

Iteratively apply Construction 2.1 and Construction 2.3 to the $(22, 7, 234)$ circuit code (and the new circuit codes these constructions generate) to get the $(23, 7, 266)$, $(24, 7, 310)$, $(26, 7, 466)$, $(27, 7, 532)$, $(28, 7, 618)$, and $(30, 7, 930)$ circuit codes. Iteratively apply Construction 2.2 and Construction 2.4 to the $(21, 8, 116)$ and $(22, 8, 132)$ circuit codes (and the new

Table 3: Lower Bounds for $K(n,k)$ (Prior Best Bound in Parentheses).

n/k	2	3	4	5	6	7	8	9
3	6c	6c	6c	6c	6c	6c	6c	6c
4	8c	8c	8c	8c	8c	8c	8c	8c
5	14c	10c	10c	10c	10c	10c	10c	10c
6	26c	16c	12c	12c	12c	12c	12c	12c
7	48c	24c	14c	14c	14c	14c	14c	14c
8	96c	36c	22c	16c	16c	16c	16c	16c
9	188	64	30c	24c	18c	18c	18c	18c
10	362	102	46c	28c	20c	20c	20c	20c
11	668	160	70	40c	30c	22c	22c	22c
12	1340	288	102	60	36c	32c	24c	24c
13	2584	494	182	80	50c	36c	26c	26c
14	4934	812	280	106	68	48c	38c	28c
15	9868	1380	480	210	88	60	42	40c
16	19740	2240	768	288	118	76	46	44c
17	39840	3910	1224	476	204	102	54	48
18	78848	5212	1530	570	238	116	68(60)ab	52
19	157696	7818	2040	712	284	134	78	60
20	315392	10424	2688	950	330	152	86	80
21	630784	15634	3400	1140	436	198	116(98)ab	88
22	1261568	20848	4488	1422	510	234(228)ab	132(114)ab	100
23	2523136	31266	5910	1898	608	266(262)b	148(128)b	110
24	5046272	41696	7480	2280	714	310(304)b	168(158)b	124
25	10092544	62530	9870	2846	932	390	188(176)b	160
26	20185088	83392	13248	3794	1086	466(452)b	236(202)ab	176
27	40370176	125058	20304	4560	1304	532(518)b	272(234)ab	200
28	80740352	166784	34704	5690	1530	618(608)b	308(268)b	222
29	161480704	250114	57246	7586	1996	774	348(328)b	248
30	322961408	333568	97846	9120	2328	930(900)b	396(368)b	320

a = prior record also exceeded directly by applying Construction 3.2
b = record exceeds Klee's asymptotic lower bound
c = value known to be optimal

circuit codes these constructions generate) to get the $(23, 8, 148)$, $(24, 8, 168)$, $(25, 8, 188)$, $(26, 8, 236)$, $(27, 8, 272)$, $(28, 8, 308)$, $(29, 8, 348)$, and $(30, 8, 396)$ circuit codes.

Using this approach 4 out of the 18 new circuit codes result directly from applying Construction 3.2. Construction 3.2 also directly results in circuit codes that are longer than the previous record $(26, 8, 202)$ and $(27, 8, 234)$ circuit codes, but these circuit codes are shorter than the ones resulting from iteratively applying Constructions 2.1-2.4 to the $(22, 7, 234)$, $(18, 8, 68)$, $(21, 8, 116)$, and $(22, 8, 132)$ circuit codes.

The chief advantage of our construction is that it is very easy to implement, allowing the better studied codes of smaller spreads to be leveraged to generate codes of larger spreads, where the spread is too large for computer search. This adds another construction (in addition to Constructions 2.1 - 2.6) to generate non-trivial codes for large spreads. As the results for spreads $k = 7, 8$ indicate, the construction is additive to Constructions 2.1-2.6. However the results for spread $k + 1 = 9$ indicate that the success of this approach relies on good starting codes for spread k.

6 Conclusions

In this note we presented a simple method for constructing a circuit code of spread $k + 1$ from a circuit code of spread k. This construction leads to 18 new record code lengths for circuit codes of spread $k = 7, 8$ and in dimensions $22 \leq n \leq 30$ by leveraging the record length circuit codes of spread 6 and 7 from [18] and [11]. We also derived a new lower bound on the length of circuit codes of spread 4, which improves upon the bound suggested by Singleton for $n \geq 86$.

Some of the records in Table 3 stood for at least 32 years before being broken by the method described here, however we believe that further improvements of the lower bounds on $K(n, k)$ are still possible. In particular, Construction 5 from [20] describes how to extend an $(n, 7)$ circuit code under certain conditions on how close a specific pair of transition elements appear in the transition sequence. While applying that construction directly does not improve the lower bounds in the table (we tried!) the transition sequences arising from combining Construction 3.2 with the construction method of [18] are highly structured, suggesting that a modification of that approach may succeed.

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A Transition Sequences for New Record Circuit Codes

The following codes are the transition sequences for the new record length circuit codes reported in Table 3. We follow the convention of [18], [11], and others in reporting transition sequences, which assigns the labels $0, \ldots, 9$ to dimensions 1 through 10, and the characters a,...,z to dimensions 11 through 36. To maintain consistency with the rest of this note (where many of our arguments rely on the even parity of the transition sequence) we report all N transitions in the code. As [11] observes, the final transition is not technically necessary to reconstruct the circuit code since it is a cycle defined to start to $\vec{0}$. When using these transition sequences, the reader should carefully distinguish between the number "1" and the letter "l", e.g. as in the transition sequences for the $(22, 7, 234), (23, 7, 266)$, and $(24, 7, 310)$ codes.

(22,7,234)
5b32f78hgc3bef4idc80195hd478e65j1ab2f6eh017gfb3i4c8g7abh0
984de5k19034a2h1e67fb2iade3cb7hg084c36j7d5409ah1e5dg06if
ea2l3b7f69ahg873cd4i08g2391h0d56ea1j9cd2ba6hfg73b25i6c43g
89h0d4cfg5ked912a6he589f76i2bc3g7fh1280gc4j5d908bch1a95ef
6i2a14l

(23,7,266)
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(24,7,310)
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(26,7,466)
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(23,8,148) 0123l456m7h08l192ma3bil041mc253ldh0m6172le48mj031lf25m94h0l617m2a3bli05m1kg789ldc0mbh32l4edm109ilab7m6de2lch0m153ble89mja01l24cmefhdl3b0m72e9li5dmgk

(24,8,168) 0123m456nhi70m819n2ahjm3b0n41c2mhi5n3d06m17hnk2e4m803n1hifm259n406hmj17n2a3bm0hin51lg789mdc0nhib3m24end1hjm09anb76dmhien2c01m53hnkbe8m9a0n1hi2m4cenfd3hmjb0n72e9m5hindgl

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(29,8,348) 0o1p2q3rms4o5p6qnrshoip7q0rms8o1p9qnrs2oaphqjrms3obp0qnrs4o1pcq2rmshoip5qnrs3odp0q6rms1o7phqnrsko2peq4rms8o0p3qnrs1ohpiqfrms2o5p9qnrs4o0p6qhrmsjo1p7qnrs2oap3qbrms0ohpiqnrs51lgo7p8q9rmsdocp0qnrshoipbq3rms2o4peqnrsdo1phqjrms0o9paqnrsbo7p6qdrmshoipeqnrs2ocp0q1rms5o3phqnrskobpeq8rms9oap0qnrs1ohpiq2rms4ocpeqnrsfodp3qhrmsjobp0qnrs7o2peq9rms5ohpiqnrsdgl

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