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Some results on the signless Laplacian permanental polynomial and star degree of balanced trees

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Abstract. Let $Q(G)$ be the signless Laplacian matrix of a simple connected graph G with n vertices. Let $\mathrm{per}(M)$ denote the permanent of a matrix M and let I_n be the identity matrix of order n. The polynomial $\mathrm{per}\big(xI_n - Q(G)\big)$ is defined as the signless Laplacian permanental polynomial of G. The multiplicity of root 1 of the polynomial $\mathrm{per}\big(xI_n - Q(G)\big)$ is equal to the star degree of G. In this paper, we consider a balanced tree $T(h,d)$ of height h, in which all the non-pendant vertices have degree $d(>2)$ and all the leaves have the same depth. We prove that, among all possible graphs with the same degree sequence, $T(h,d)$ has the maximum star degree. Further, we show that $T(2,d)$ is determined by its signless Laplacian permanental polynomial. By applying the known fact that the Laplacian permanental polynomial and signless Laplacian permanental polynomial are the same for a bipartite graph G; one gets that $T(2,d)$ is also determined by its Laplacian permanental polynomial.

1 Introduction

We consider finite, simple, and undirected graphs throughout the paper. The order and size of a graph G are $n = |V(G)|$ and $m = |E(G)|$, where $V(G)$ and $E(G)$ refer to the vertex set and edge set of G, respectively. The diagonal matrix $D(G)$ of order $n \times n$, with the $(i,i)^{\text{th}}$ entry as the degree of vertex i in G, is called the *degree diagonal matrix* of G. The matrix of order $n \times n$, in which the $(i,j)^{\text{th}}$ entry is 1 if vertex i is adjacent to vertex j in G, and 0 otherwise, is known as the *adjacency matrix* of

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G, denoted by $A(G)$. The *signless Laplacian matrix* of G is defined as $Q(G) = D(G) + A(G)$.

Let M be a matrix of order $k \times k$ with m_{ij} as the $(i, j)^{\text{th}}$ entry. The *permanent* of M is defined by $\text{per}(M) = \sum_\sigma \prod_{i=1}^{k} m_{i\sigma(i)}$, where the sum is taken over all permutations σ of the set $\{1, 2, \ldots, k\}$. Although the definition of permanent seems similar to the determinant of a matrix, the computation of permanent is much harder than computing the determinant. The reason for this difficulty is the lack of an efficient algorithm for the computation of permanent, unlike the known algorithms, e.g., Gaussian elimination, for computing the determinant. Various extremal and enumeration problems concern the permanent of a square matrix in the fields of graph theory and combinatorics.

Let I_n be the identity matrix of order n. The *signless Laplacian permanental polynomial* of a graph G is defined as $\psi\big(Q(G); x\big) = \text{per}\big(xI_n - Q(G)\big)$. A graph G is said to be *determined by its signless Laplacian permanental polynomial* if for any graph H that has the same signless Laplacian permanental polynomial as $\psi\big(Q(G); x\big)$, G is isomorphic to H.

Recall that a vertex of degree one in a graph G is known as a *pendant vertex*. An edge whose one end-vertex is pendant is said to be a *pendant edge*. A *pendant star* in graph G is a maximal connected subgraph of G induced by pendant edges. The vertex incident to all the pendant edges is called the *center* of the pendant star. The *degree of a pendant star* is defined to be the number of its pendant edges minus one. Finally, the *star degree* of G is defined as the sum of the degrees of all pendant stars in G. We denote the star degree of G by $\text{SD}(G)$. If there are no pendant stars in G, then $\text{SD}(G) = 0$.

An illustration of the above definitions is given in Example 1.1.

Example 1.1. Consider the graph G shown in Figure 1.1. We observe that G has two pendant stars. The centers of these pendant stars are vertices v_1 and v_7. The pendant star centered at v_1 is induced by five pendant edges, so the degree of this pendant star is $5 - 1 = 4$. Similarly, the pendant star centered at v_7 is induced by three pendant edges, so the degree of this pendant star is $3 - 1 = 2$. Thus, the star degree of G is $\text{SD}(G) = 4 + 2 = 6$.

Faria [1] was the first to study the roots of the signless Laplacian permanental polynomial of a graph, where it was proved that the multiplicity of

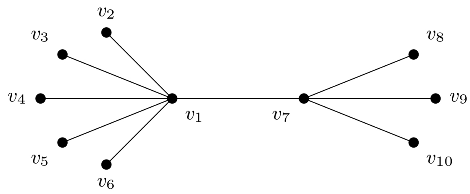

Figure 1.1: Graph G.

root 1 of $\psi\big(Q(G);x\big)$ is equal to SD(G). Later, Liu [2] proved that the multiplicity of root 0 of $\psi\big(Q(G);x\big)$ is the same as the number of isolated vertices in G. Recently, Wu et al. [3] studied the star degree of graphs in more detail.

The characterization of graphs by related polynomials is an interesting and important problem in graph theory. This problem has been solved for various polynomials, including the signless Laplacian permanental polynomial. Here, we list some of the graphs that have been proven to be determined by their signless Laplacian permanental polynomials so far:

(i) Complete, regular complete multipartite and star graphs [2].;

(ii) Path, cycle, and lollipop graphs [4];

(iii) n-vertex graphs with star degrees $n-i$, for $i \in \{2,3,4,5\}$ [3];

(iv) Some caterpillars, whose central paths have the same degree [5];

(v) Some unicyclic graphs, with two types of degrees [6].

In this paper, we compute the signless Laplacian permanental polynomial and star degree of a balanced tree, denoted as $T(h,d)$. We show that $T(h,d)$ has the maximum star degree among all possible graphs with the same degree sequence as that of $T(h,d)$. Further, we prove that $T(2,d)$ is determined by its (signless) Laplacian permanental polynomial.

2 Preliminaries

In this section, we define a balanced tree $T(h,d)$ and include some known results that will be used in the sequel.

Definition 2.1. For integers $h(>1)$ and $d(>2)$, the balanced tree $T(h,d)$ is a rooted tree of height h, where all the vertices of depth $h-i$, with $1 \le i \le h$, have degree d and the remaining vertices are leaves, i.e., all the leaves are of depth h.

An example of $T(h,d)$ for $h=d=3$ is shown in Figure 2.1.

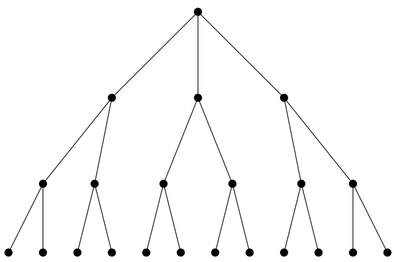

Figure 2.1: Balanced tree $T(3,3)$.

Lemma 2.2. [6] *The constant term of* $\psi\big(Q(G);x\big)$ *is zero if and only if* G *has an isolated vertex.*

Theorem 2.3. [1] *The star degree of a graph* G *is equal to the multiplicity of root* 1 *of* $\operatorname{per}\big(xI-Q(G)\big)$.

Theorem 2.4. [3] *Let* G *be a graph on* n *vertices. If* $\mathrm{SD}(G)=n-i$, *where* $i\in\{2,3,4,5\}$, *then* G *is determined by its signless Laplacian permanental spectrum.*

Next, we present the general form of the signless Laplacian permanental polynomial of a triangle-free graph having only two types of vertex degrees, $d(>1)$ and 1, as discussed in [6].

Notation 2.5. [6] For any graph G, we denote the remaining of the polynomial $\psi\big(Q(G);x\big)$ immediately after fourth term by $R\big(Q(G);x\big)$.

Theorem 2.6. [6] *A graph* G *is a triangle-free,* n*-vertex graph with* r *vertices of degree* d *and the remaining vertices being of degree 1 if and only if the signless Laplacian permanental polynomial of* G *is*

$$\begin{aligned}\psi\big(Q(G);x\big) = x^n &- (n+dr-r)x^{n-1} \\ &+ \tfrac{1}{2}[(2n+dr-2r-d+1)dr+(n-r)^2]x^{n-2} \\ &- \tfrac{1}{6}[(n+dr-r)^3-3rd(d-1)(n+dr-r) \\ &+ 2r(d-1)(d^2-2d-2)-4n]x^{n-3}+R\big(Q(G);x\big)\end{aligned}$$

with non-zero constant term.

Corollary 2.7. [6] *A graph G with n vertices and m edges, where $\frac{2m-n}{d-1}$ vertices are of degree d and the remaining are pendant vertices, is triangle free if and only if the signless Laplacian permanental polynomial of G is*

$$\begin{aligned}\psi\big(Q(G);x\big) = x^n - 2mx^{n-1} + \tfrac{1}{2}[4m^2 - (2m-n)d]x^{n-2} \\ - \tfrac{1}{3}[4m^3 - 4m - (2m-n)(3m-d+2)d]x^{n-3} \\ + R\big(Q(G);x\big)\end{aligned}$$

with non-zero constant term.

Since, in a tree with n vertices, the number of edges is $m = n-1$, the result below follows from Corollary 2.7.

Corollary 2.8. *Let T be a tree on n vertices. Then $\frac{n-2}{d-1}$ vertices of T have degree d while the remaining are pendant vertices if and only if the signless Laplacian permanental polynomial of T is*

$$\begin{aligned}\psi\big(Q(T);x\big) = x^n - 2(n-1)x^{n-1} + \tfrac{1}{2}[4n^2 - (d+8)n + 2(d+2)]x^{n-2} \\ - \tfrac{1}{3}[4n^3 - 3(d+4)n^2 + (d^2+7d+8)n - 2d(d+1)]x^{n-3} \\ + R\big(Q(T);x\big)\end{aligned}$$

with non-zero constant term.

Recall that the *corona* of two graphs G_1 and G_2 is formed with one copy of G_1 and $|V(G_1)|$ copies of G_2 by adding edges between a vertex v of G_1 and each vertex in the corresponding copy of G_2, for all $v \in V(G_1)$. Here we mention the definition of the graph $UC(r,d)$ as discussed in [6].

Definition 2.9. [6] Let $r, d \geq 3$ be integers. A unicyclic graph with two types of vertex-degrees d and 1, where each vertex of the unique cycle C_r is adjacent to $d-2$ pendant vertices, is denoted by $UC(r,d)$. It may be noted that $UC(r,d)$ is the corona of C_r and complement of K_{d-2}, i.e., $UC(r,d) \cong C_r \circ K^C_{d-2}$.

The following two lemmas discuss the graphs on the same degree sequence as that of $UC(r,d)$.

Lemma 2.10. [6] *For $r > 3$, let G be a connected graph, non-isomorphic to $UC(r,d)$, and with the same degree sequence as that of $UC(r,d)$. Then G is also unicyclic with a cycle C_l, where $l < r$.*

Lemma 2.11. [6] *Let G be a disconnected graph with the same degree sequence as that of $UC(r,d)$. Then G satisfies that, corresponding to every non-tree component G_i with n_i vertices and n_i+t_i edges, there are t_i tree components in G, where $t_i \geq 0$.*

The Laplacian permanental polynomial agrees with its signless Laplacian permanental polynomial for a bipartite graph G, as stated in the following result.

Theorem 2.12. [1] *For a bipartite graph G, $\psi\big(L(G);x\big) = \psi\big(Q(G);x\big)$.*

3 Signless Laplacian permanental polynomial of $T(h,d)$

In this section, we discuss some basic properties of $T(h,d)$. Also, we compute the signless Laplacian permanental polynomial of $T(h,d)$. The following lemma is immediate from Definition 2.1.

Lemma 3.1. *Each of the following is true in the graph $T(h,d)$:*

(i) *The number of vertices is $1+d+d(d-1)+\cdots+d(d-1)^{h-1} = \frac{d(d-1)^h-2}{d-2}$.*

(ii) *The number of pendant vertices is $d(d-1)^{h-1}$.*

(iii) *The number of non-pendant vertices is $\frac{d(d-1)^{h-1}-2}{d-2}$.*

(iv) *The number of edges is $\frac{d\{(d-1)^h-1\}}{d-2}$.*

Theorem 3.2. *The signless Laplacian permanental polynomial of $T(h,d)$ is given by*

$$\psi\big(Q\big(T(h,d)\big);x\big) = x^n - \frac{2d[(d-1)^h-1]}{d-2}x^{n-1} + \frac{d}{2(d-2)^2} \times \Big[4d(d-1)^{2h} - d(d+6)(d-1)^h + 2(d^2-d+2)\Big]x^{n-2} - \frac{d(d-1)[d(d-1)^{h-1}-2]}{3(d-2)^3} \times \Big[4d(d-1)^{2h} - (3d^2-2d+8)(d-1)^h + d(d^2-3d+6)\Big]x^{n-3} + R\big(Q\big(T(h,d)\big);x\big)$$

where $n = \frac{d(d-1)^h-2}{d-2}$, and the constant term of $\psi\big(Q\big(T(h,d)\big);x\big)$ is nonzero.

Proof. The proof directly follows from Corollary 2.8 by substituting $n = \frac{d(d-1)^h-2}{d-2}$. ☐

4 Star degree of $T(h,d)$ and determination of $T(2,d)$ by its signless Laplacian permanental polynomial

In this section, we give a general expression for the star degree of $T(h,d)$. Furthermore, we prove that $T(h,d)$ has the maximum star degree when compared to all possible graphs on the same degree sequence as that of $T(h,d)$. Finally, by applying these results, we prove that $T(2,d)$ is determined by its signless Laplacian permanental polynomial.

Theorem 4.1. *The star degree of* $T(h,d)$ *is given by*

$$\mathrm{SD}\big(T(h,d)\big) = d(d-1)^{h-2}(d-2).$$

Proof. By the definition of $T(h,d)$, it is clear that all the pendant stars have vertices of depth $h-1$ as their centers. Moreover, all of them are of the same degree, $d-2$. The number of vertices of $T(h,d)$ of depth $h-1$ is $d(d-1)^{h-2}$, which proves the result. ☐

Corollary 4.2. *If* $\mathrm{SD}\big(T(h,d)\big) = n-i$ *where* n *is the number of vertices in* $T(h,d)$*, then* $i = \frac{d(d-1)^{h-2}(2d-3)-2}{d-2}$ *and* $i > 6$.

Proof. Since $n = \frac{d(d-1)^h-2}{d-2}$, the proof directly follows from Theorem 4.1. Now, the smallest example of $T(h,d)$ is when $d = 3$ and $h = 2$. In that case, $i = 7$, which proves the second part. ☐

Remark 4.3. In Theorem 2.4, it is proved that all graphs with $\mathrm{SD}(G) = n-i$, where $i \in \{2,3,4,5\}$, are determined by their signless Laplacian permanental spectrum. From Corollary 4.2, one can find that $T(h,d)$ does not belong to these classes of graphs.

The next two lemmas provide the construction of all possible graphs that have the same degree sequence as that of $T(h,d)$. While proving these lemmas, we apply an obvious fact that two graphs with the same degree sequence must have the same number of edges.

Lemma 4.4. *Any connected graph G with the same degree sequence as that of $T(h, d)$ and non-isomorphic with $T(h, d)$ can be obtained from $T(h, d)$ by making $l > 0$ number of non-pendant vertices pendant and l number of pendant vertices non-pendant.*

Proof. Since G is connected and has the same number of edges with $T(h, d)$, we get that G is also a tree. If all the leaves in G are at level h, then G will be isomorphic with $T(h, d)$. So, some leaves in G have level strictly less than h, and some have level strictly greater than h. Also, G may have some leaves at the level h. Then to meet the degree sequence of G, we can construct it in the only way mentioned in the hypothesis. □

Lemma 4.5. *Any disconnected graph G with the same degree sequence as that of $T(h, d)$ has the following properties:*

(i) *At least one component of G is a tree with two types of vertex-degrees, d and 1. In this component, the number of non-pendant vertices is $t(> 2)$ less than that in $T(h, d)$.*

(ii) *The other components form a graph on the same degree sequence as that of $UC(t, d)$.*

Proof. Let $T(h, d)$ have n vertices. Then, the number of edges in $T(h, d)$ is $n - 1$. If G is a disconnected graph on the same degree sequence as that of $T(h, d)$, with components $G_1, \ldots, G_k$, then not all G_i's can be trees (in that case, G would have an $n - k$ number of edges, which is a contradiction). Similarly, not all G_i's can be non-trees (otherwise the number of edges in G will be greater than or equal to n, which leads to a contradiction). Therefore, at least one component is a tree, and at least one component is a non-tree.

Without loss of generality, suppose G_1 is a tree component. Clearly, G_1 has two types of vertex-degrees: d and 1. If there is only one more component, G_2, then the number of edges in G_2 must be the same as the number of vertices in it (in order to maintain the total number of edges in G). That is, G_2 is a unicyclic graph on two types of degrees, d and 1. Since at least three non-pendant vertices are needed to form a cycle, we must remove $t(> 2)$ such vertices from $T(h, d)$ to form the component G_1 and use them in G_2. In other words, G_2 is either $UC(t, d)$ or a connected graph on the same degree sequence as that of $UC(t, d)$.

Moreover, if we fix G_1 as a tree component with t less number of non-pendant vertices than that of $T(h, d)$ and if there are $k-1$ more components in G, then $G_2 \cup \cdots \cup G_k$ must have the same degree sequence as that of $UC(t, d)$. □

Now, we present our main results as Theorem 4.6 and Theorem 4.9, which prove that $T(h, d)$ has the maximum star degree among all possible graphs with the same degree sequence as that of $T(h, d)$.

Theorem 4.6. *For a connected graph G, non-isomorphic with $T(h, d)$, on the same degree sequence as that of $T(h, d)$, we have* $\mathrm{SD}(G) \leq \mathrm{SD}\big(T(h, d)\big)$.

Proof. Let G be a connected graph with the same degree sequence as that of $T(h, d)$, and $G \ncong T(h, d)$. From the definition of $T(h, d)$, it is clear that there are two types of vertices: non-pendant vertices of degree d and pendant vertices. Let us denote the non-pendant vertices and pendant vertices of $T(h, d)$ by u_i and v_j respectively, where $i \in \Big\{1, 2, \ldots, \frac{d(d-1)^{h-1}-2}{d-2}\Big\}$ and $j \in \big\{1, 2, \ldots, d(d-1)^{h-1}\big\}$ (by Lemma 3.1). Thus, G will also contain the same two types of vertices, and there will be at least one pendant vertex on level less than h (respectively, one non-pendant vertex on level h or more than h). These non-pendant and pendant vertices of G are denoted as v'_i and u'_j respectively. Suppose there are $l (> 0)$ such pendant vertices u'_i (respectively non-pendant vertices v'_i) of G, for $i \in \{1, 2, \ldots, l\}$, that are on different levels as compared to the vertices of $T(h, d)$. For convenience, the position of u_i (respectively v_i) in $T(h, d)$ is considered to be the same as that of u'_i (respectively v'_i) in G, for each $i \in \{1, 2, \ldots, l\}$. The difference in the star degrees of $T(h, d)$ and G directly depends upon the degrees of pendant stars with centers as parents of the vertices, which differ in their types. Note that v_i does not belong to a subtree with root u_i for all i.

To clarify the above notations, an example with values $h = d = l = 3$ is given in Figure 4.1.

Now, according to the choice of u_i's, there are the following three cases:

Case 1: All u_i's have the same parent. Let $P(u)$ be the parent of all u_i's.

Sub-case 1a: All v_i's have the same parent, and $l = d - 1$. If all v_i's have the same parent, say $P(v)$, then the degree of the pendant star with center $P(v)$ is $d - 2$, and the degree of pendant star with center $P(u)$ is 0. Now, if $P(v')$ is the parent of all v'_i and $P(u')$ is the parent of all u'_i, then the degree of pendant star with center $P(v')$ is 0, and the degree of pendant star with center $P(u')$ is $d - 2$. Thus, we have

$$\mathrm{SD}\big(T(h, d)\big) - \mathrm{SD}(G) = (d-2) + 0 - 0 - (d-2) = 0,$$

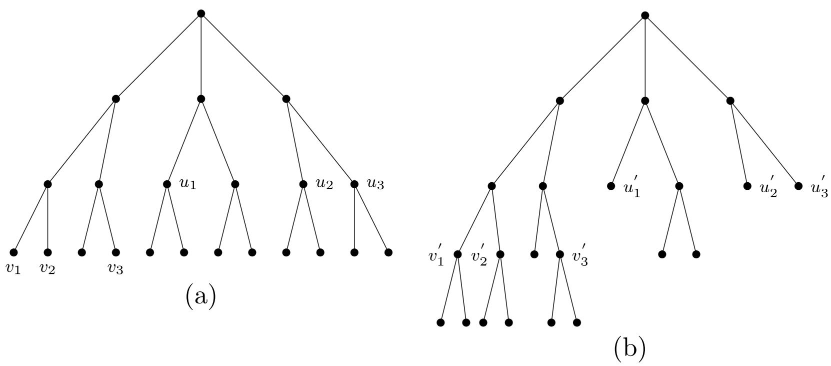

Figure 4.1: (a) Balanced tree $T(3,3)$ with $l = 3$ and (b) a graph G with the same degree sequence as that of $T(3,3)$.

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big)$.

Sub-case 1b: All v_i's have the same parent, and $l \neq d-1$. If all v_i's have the same parent, say $P(v)$, then the degree of pendant star with center $P(v)$ is $d-2$, and the degree of pendant star with center $P(u)$ is 0. Now, if $P(v')$ is the parent of all v_i' and $P(u')$ is the parent of all u_i', then the degree of pendant star with center $P(v')$ is $d-2-l$, and the degree of pendant star with center $P(u')$ is $l-1$. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = (d-2) + 0 - (d-2-l) - (l-1) = 1,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - 1$.

Sub-case 1c: All v_i's have distinct parents. Now, if all v_i's have distinct parents, say $P(v_i)$ for all $i \in \{1,2,\ldots,l\}$ respectively, then the degree of pendant star with center $P(v_i) = d-2$ for all $i \in \{1,2,\ldots,l\}$, and the degree of pendant star with center $P(u)$ is 0. Now, if $P(v_i')$ denotes the parent of v_i' for all $i \in \{1,2,\ldots,l\}$ respectively, then the degree of pendant star with center $P(v_i') = d-3$ for all $i \in \{1,2,\ldots,l\}$, and the degree of pendant star with center $P(u')$ is $l-1$. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = l(d-2) + 0 - l(d-3) - (l-1) = 1,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - 1$.

Sub-case 1d: Some of the v_i's have the same parent, and some have distinct parents. Consider a partition of the set $V = \{v_1, v_2, \ldots, v_l\}$ with p parts, where $1 < p < l$. Suppose the i^{th} part of this partition contains t_i elements of V, which have the same parent, for $i \in \{1,2,\ldots,p\}$. Here, $1 \leq t_i \leq l-1$ and $\sum_{i=1}^{p} t_i = l$. Now, if $P(v_{t_i})$ denotes the parent of all v_k belonging to the i^{th} part, for all $i \in \{1, 2, \dots, p\}$ respectively, then the degree of pendant star with center $P(v_{t_i}) = d - 2$ for all $i \in \{1, 2, \dots, p\}$, and the degree of pendant star with center $P(u)$ is 0. Now, if $P(v'_{t_i})$ denotes the parent of all v'_k belonging to the i^{th} part, for all $i \in \{1, 2, \dots, p\}$ respectively, then the degree of pendant star with center $P(v'_{t_i}) = d - 2 - t_i$ for $i \in \{1, 2, \dots, p\}$ respectively, and the degree of pendant star with center $P(u')$ is $l - 1$. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = p(d-2) + 0 - \sum_{i=1}^{p}(d - 2 - t_i) - (l-1) = 1,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - 1$.

Therefore, when all u_i's have the same parent, $\mathrm{SD}(G) \leq \mathrm{SD}\big(T(h,d)\big)$.

Case 2: All $\boldsymbol{u_i}$'s have distinct parents. Let $P(u_i)$ be the parent of u_i for all $i \in \{1, 2, \dots, l\}$ respectively.

Sub-case 2a: All $\boldsymbol{v_i}$'s have the same parent. If all v_i's have the same parent, say $P(v)$, then the degree of pendant star with center $P(v)$ is $d - 2$, and the degree of pendant star with center $P(u_i) = 0$ for all $i \in \{1, 2, \dots, l\}$. Now, if $P(v')$ is the parent of all v'_i and $P(u'_i)$ be the parent of u'_i for all $i \in \{1, 2, \dots, l\}$ respectively, then the degree of pendant star with center $P(v')$ is $d - 2 - l$, and the degree of pendant star with center $P(u'_i) = 0$ for all $i \in \{1, 2, \dots, l\}$. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = (d-2) + l \times 0 - (d - 2 - l) - l \times 0 = l,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - l$.

Sub-case 2b: All $\boldsymbol{v_i}$'s have distinct parents. Now, if all v_i's have distinct parents, say $P(v_i)$ for all $i \in \{1, 2, \dots, l\}$ respectively, then the degree of pendant star with center $P(v_i) = d - 2$ for all $i \in \{1, 2, \dots, l\}$, and the degree of pendant star with center $P(u_i) = 0$ for all $i \in \{1, 2, \dots, l\}$. Now, if $P(v'_i)$ denotes the parent of v'_i for all $i \in \{1, 2, \dots, l\}$ respectively, then the degree of pendant star with center $P(v'_i) = d - 3$ for all $i \in \{1, 2, \dots, l\}$, and the degree of pendant star with center $P(u'_i) = 0$ for all $i \in \{1, 2, \dots, l\}$. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = l(d-2) + l \times 0 - l(d-3) - l \times 0 = l,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - l$.

Sub-case 2c: Some of the $\boldsymbol{v_i}$'s have the same parent, and some have distinct parents. Consider a partition of the set $V = \{v_1, v_2, \dots, v_l\}$ with p parts, where $1 < p < l$. Suppose the i^{th} part of this partition contains t_i elements of V which have the same parent, for $i \in \{1, 2, \dots, p\}$.

Here, $1 \leq t_i \leq l-1$ and $\sum_{i=1}^{p} t_i = l$. Now, if $P(v_{t_i})$ denotes the parent of all v_k belonging to the i^{th} part, for all $i \in \{1, 2, \ldots, p\}$ respectively, then the degree of pendant star with center $P(v_{t_i}) = d-2$ for all $i \in \{1, 2, \ldots, p\}$, and the degree of pendant star with center $P(u_i) = 0$ for all $i \in \{1, 2, \ldots, l\}$. Now, if $P(v'_{t_i})$ denotes the parent of all v'_k belonging to the i^{th} part, for all $i \in \{1, 2, \ldots, p\}$ respectively, then the degree of pendant star with center $P(v'_{t_i}) = d-2-t_i$ for $i \in \{1, 2, \ldots, p\}$ respectively, and the degree of pendant star with center $P(u'_i) = 0$ for all $i \in \{1, 2, \ldots, l\}$. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = p(d-2) + l \times 0 - \sum_{i=1}^{p}(d-2-t_i) - l \times 0 = l,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - l$.

Therefore, when all u_i's have distinct parents, $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - l$, where l describes the number of pendant vertices of G that are on levels $1, 2, \ldots, h-1$.

Case 3: Some of the u_i's have the same parent, and some have distinct parents. Consider a partition of the set $U = \{u_1, u_2, \ldots, u_l\}$ with q parts, where $1 < q < l$. Suppose the i^{th} part of this partition contains s_i elements of U, which have the same parent, for $i \in \{1, 2, \ldots, q\}$. Here, $1 \leq s_i \leq l-1$ and $\sum_{i=1}^{q} s_i = l$. Let $P(u_{s_i})$ denote the parent of all u_k belonging to the i^{th} part, for all $i \in \{1, 2, \ldots, q\}$ respectively.

Sub-case 3a: All v_i's have the same parent. If all v_i's have the same parent, say $P(v)$, then the degree of pendant star with center $P(v)$ is $d-2$, and the degree of pendant star with center $P(u_{s_i}) = 0$ for all $i \in \{1, 2, \ldots, q\}$. Now, if $P(v')$ is the parent of all v'_i and $P(u'_{s_i})$ denotes the parent of all u'_k belonging to the i^{th} part, for all $i \in \{1, 2, \ldots, q\}$ respectively, then the degree of pendant star with center $P(v')$ is $d-2-l$, and the degree of pendant star with center $P(u'_{s_i}) = s_i - 1$ for all $i \in \{1, 2, \ldots, q\}$ respectively. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = (d-2) + q \times 0 - (d-2-l) - \sum_{i=1}^{q}(s_i - 1) = q,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - q$.

Sub-case 3b: All v_i's have distinct parents. Now, if all v_i's have distinct parents, say $P(v_i)$ for all $i \in \{1, 2, \ldots, l\}$ respectively, then the degree of pendant star with center $P(v_i) = d-2$ for all $i \in \{1, 2, \ldots, l\}$, and the degree of pendant star with center $P(u_{s_i}) = 0$ for all $i \in \{1, 2, \ldots, q\}$. Now, if $P(v'_i)$ is the parent of v'_i for $i \in \{1, 2, \ldots, l\}$ respectively and

$P(u'_{s_i})$ denotes the parent of all u'_k belonging to the i^{th} part, for all $i \in \{1, 2, \dots, q\}$ respectively, then the degree of pendant star with center $P(v'_i) = d-3$ for all $i \in \{1, 2, \dots, l\}$, and the degree of pendant star with center $P(u'_{s_i}) = s_i - 1$ for all $i \in \{1, 2, \dots, q\}$ respectively. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = l \times (d-2) + q \times 0 - l \times (d-3) - \sum_{i=1}^{q} (s_i - 1) = q,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - q$.

Sub-case 3c: Some of the v_i's have the same parent, and some have distinct parents. Consider a partition of the set $V = \{v_1, v_2, \dots, v_l\}$ with p parts, where $1 < p < l$. Suppose the i^{th} part of this partition contains t_i elements of V, which have the same parent, for $i \in \{1, 2, \dots, p\}$. Here, $1 \leq t_i \leq l-1$ and $\sum_{i=1}^{p} t_i = l$. Now, if $P(v_{t_i})$ denotes the parent of all v_k belonging to the i^{th} part, for all $i \in \{1, 2, \dots, p\}$ respectively, then the degree of pendant star with center $P(v_{t_i}) = d-2$ for all $i \in \{1, 2, \dots, p\}$, and the degree of pendant star with center $P(u_{s_i}) = 0$ for all $i \in \{1, 2, \dots, q\}$. Now, if $P(v'_{t_i})$ denotes the parent of all v'_k belonging to the i^{th} part, for all $i \in \{1, 2, \dots, p\}$ respectively, then the degree of pendant star with center $P(v'_{t_i}) = d - 2 - t_i$ for $i \in \{1, 2, \dots, p\}$ respectively, and the degree of pendant star with center $P(u'_{s_i}) = s_i - 1$ for all $i \in \{1, 2, \dots, q\}$ respectively. Thus, we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = p \times (d-2) + q \times 0 - \sum_{i=1}^{p} (d-2-t_i) - \sum_{i=1}^{q} (s_i - 1) = q,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - q$.

Therefore, when some of the u_i's have the same parent and some have distinct parents, we have $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - q$, where q describes the number of distinct parents of the pendant vertices of G which are on level $1, 2, \dots, h-1$.

From the above three cases, we conclude that $\mathrm{SD}(G) \leq \mathrm{SD}\big(T(h,d)\big)$ holds true for any connected graph G having the same degree sequence as that of $T(h,d)$. □

Note that $\mathrm{SD}(G) < \mathrm{SD}\big(T(h,d)\big)$ holds true in all the cases except for Subcase 1a of Case 1. So, Theorem 4.6 leads to the following consequence:

Corollary 4.7. *For a connected graph G on the same degree sequence as that of $T(2,d)$, we have $SD(G) < SD(T(2,d))$.*

Proof. For $h = 2$, in Sub-case 1a from the proof of Theorem 4.6, G is also a tree of height 2. That is, $G \cong T(2, d)$, which makes the hypothesis of Theorem 4.6 false. Thus, Sub-case 1a from the proof of Theorem 4.6 becomes invalid for $h = 2$. In the other cases, $\mathrm{SD}(G) < \mathrm{SD}\big(T(2, d)\big)$ follows from the proof of Theorem 4.6. □

Corollary 4.7 shows that the smallest example where $\mathrm{SD}(G) = \mathrm{SD}\big(T(h, d)\big)$ occurs for $h = 3$. An example of this case is given below:

Example 4.8. One of the smallest examples of a pair of non-isomorphic graphs depicting Sub-case 1a from the proof of Theorem 4.6 is shown in Figure 4.2. Here, $h = d = 3$ and $l = d - 1 = 2$. We compute the difference

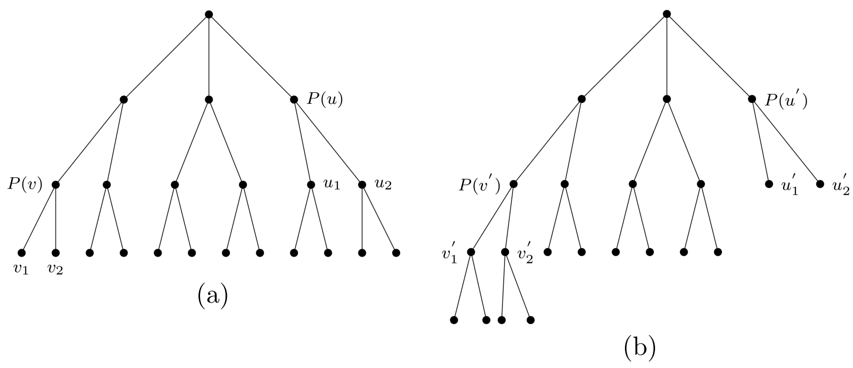

Figure 4.2: (a) Balanced tree $T(3, 3)$ with $l = 2$ and (b) a graph G belonging to Sub-case 1a in our proof for Theorem 4.6 for $T(3, 3)$.

in star degrees of $T(3, 3)$ and G by following the same method as in the proof of Theorem 4.6 (Sub-case 1a). The degree of pendant star with center $P(v)$ is 1, and the degree of pendant star with center $P(u)$ is 0. Also, the degree of pendant star with center $P(v')$ is 0, and the degree of pendant star with center $P(u')$ is 1. Thus, we have

$$\mathrm{SD}\big(T(3, 3)\big) - \mathrm{SD}(G) = 1 + 0 - 0 - 1 = 0,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(3, 3)\big)$. Since the height of G is four, $G \ncong T(3, 3)$.

Theorem 4.9. *For a disconnected graph G on the same degree sequence as that of $T(h, d)$, we have $\mathrm{SD}(G) < \mathrm{SD}\big(T(h, d)\big)$.*

Proof. Let G be a disconnected graph with the same degree sequence as that of $T(h,d)$. Suppose the components of G are $G_1,\dots,G_k$. Without loss of generality, suppose G_1 is a tree component with $l(>2)$ less number of non-pendant vertices than that of $T(h,d)$. Now, based on the nature of components $G_2,\dots,G_k$ as discussed in Lemma 4.5, there are the following two cases:

Case 1: $G_2,\dots,G_k$ all are unicyclic with two types of degrees, d and 1. That is, there are no other tree components in G (by Lemma 2.11).

Again, similar to the proof of Theorem 4.6, the difference in star degrees depends on the change in degrees of pendant stars with centers as the parents of vertices u_i and v_i for $i \in \{1,2,\dots,l\}$. Now, we compare the star degrees of $T(h,d)$ and G depending upon the following three subcases:

Sub-case 1a: All u_i's have the same parent. Following similar notations as used in the proof of Theorem 4.6 (Case 1), we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = l(d-2) + 0 - l(d-3) - (l-1) = 1,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - 1$.

Sub-case 1b: All u_i's have distinct parents. Following similar notations as used in the proof of Theorem 4.6 (Case 2), we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = l(d-2) + 0 - l(d-3) - 0 = l,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - l$.

Sub-case 1c: Some of the u_i's have the same parent, and some have distinct parents. Following similar notations as used in the proof of Theorem 4.6 (Case 3), we have

$$\mathrm{SD}\big(T(h,d)\big) - \mathrm{SD}(G) = (d-2) + q \times 0 - (d-2-l) - \sum_{i=1}^{q}(s_i - 1) = q,$$

which implies that $\mathrm{SD}(G) = \mathrm{SD}\big(T(h,d)\big) - q$.

Thus, for a graph G with components $G_1,\dots,G_k$, where G_1 is a tree and G_j (for $j \in \{2,\dots,k\}$) is a unicyclic graph with l_j non-pendant vertices of degree d and remaining vertices of G_j are pendant, such that $3 \le l_j \le l$ for all j and $\sum_{j=2}^{k} l_j = l$, we have $\mathrm{SD}(G) < \mathrm{SD}\big(T(h,d)\big)$. It can be easily observed that $\mathrm{SD}(G_j) = \mathrm{SD}\big(UC(l_j,d)\big)$ for every $j \in \{2,\dots,k\}$. In other words,

$$\mathrm{SD}(G_1) + \sum_{j=2}^{k} \mathrm{SD}\big(UC(l_j,d)\big) < \mathrm{SD}\big(T(h,d)\big).$$

It is direct that $\sum_{j=2}^{k} \mathrm{SD}\big(UC(l_j,d)\big) = \mathrm{SD}\big(UC(l,d)\big)$. Thus, we have

$$\mathrm{SD}(G_1) + \mathrm{SD}\big(UC(l,d)\big) < \mathrm{SD}\big(T(h,d)\big). \tag{1}$$

Case 2: $G_2, \dots, G_k$ all have two types of degrees, d and 1, and contain more than one cycle. That is, there are more than one tree components in G (by Lemma 2.11). In this particular case, we compare the star degree of $G_2 \cup \cdots \cup G_k$ with $\mathrm{SD}\big(UC(l,d)\big)$.

Sub-case 2a: None of the tree components is a K_2. If none of the tree components G_i, for $i > 1$, is isomorphic to the graph K_2, then all the pendant vertices in $G_2 \cup \cdots \cup G_k$ contribute to its star degree. In other words, the reduction in star degree due to non-tree components is balanced by the increase in star degree due to tree components. Thus, $\mathrm{SD}(G_2 \cup \cdots \cup G_k) = \mathrm{SD}\big(UC(l,d)\big)$. From (1), we have

$$\mathrm{SD}(G_1) + \mathrm{SD}(G_2 \cup \cdots \cup G_k) < \mathrm{SD}\big(T(h,d)\big),$$

which implies that $\mathrm{SD}(G) < \mathrm{SD}\big(T(h,d)\big)$.

Sub-case 2b: At least one of the tree components is a K_2. If at least one of the tree components G_i, for $i \geq 1$, is isomorphic to the graph K_2, then at least two pendant vertices in $G_2 \cup \cdots \cup G_k$ will not contribute to its star degree. In this case, $\mathrm{SD}(G_2 \cup \cdots \cup G_k) < \mathrm{SD}\big(UC(l,d)\big)$, which implies that

$$\mathrm{SD}(G_1) + \mathrm{SD}(G_2 \cup \cdots \cup G_k) < \mathrm{SD}(G_1) + \mathrm{SD}\big(UC(l,d)\big).$$

Applying (1) we obtain, $\mathrm{SD}(G) < \mathrm{SD}\big(T(h,d)\big)$.

From the above cases, we conclude that $\mathrm{SD}(G) < \mathrm{SD}\big(T(h,d)\big)$ holds true for any disconnected graph G having the same degree sequence as that of $T(h,d)$. □

Theorem 4.10. *$T(2,d)$ is determined by its signless Laplacian permanental polynomial.*

Proof. Let G be a graph having the same signless Laplacian permanental polynomial as that of $T(2,d)$. Then, the constant term of $\psi\big(Q(G);x\big)$ is the same as that of $\psi\big(Q\big(T(2,d)\big);x\big)$, which is non-zero by Lemma 2.2. Thus, the constant term of $\psi\big(Q(G);x\big)$ is not equal to zero.

Now, by Theorems 2.6 and 3.2, we conclude that G is triangle free and has the same degree sequence as that of $T(2,d)$. Since $\psi\big(Q(G);x\big)$ =

$\psi\big(Q\big(T(2,d)\big);x\big)$, by Theorem 2.3, $\mathrm{SD}(G) = \mathrm{SD}\big(T(2,d)\big)$. By Theorem 4.9, G can not be disconnected. Finally, by Corollary 4.7, G must be isomorphic to $T(2,d)$. □

Since $T(2,d)$ is a tree, it is also bipartite. Thus, by Theorem 2.12, the following result is direct.

Corollary 4.11. $T(2,d)$ *is determined by its Laplacian permanental polynomial.*

5 Conclusion

In this article, we applied the star degrees of possible graphs on the same degree sequence to prove that $T(2,d)$ is determined by its signless Laplacian permanental polynomial. It is observed by Example 4.8 that two non-isomorphic graphs can have the same star degree. Therefore, the star degree does not help distinguish these graphs through their signless Laplacian permanental polynomial. We need to find some other factor to distinguish graphs based on their signless Laplacian permanental polynomial. This leads to the following open problems:

Open Problem 5.1. Is the graph $T(h,d)$ determined by its signless Laplacian permanental polynomial for $h > 2$?

Consider a graph $T(n_1,\ldots,n_h,d)$ where n_i denotes the number of vertices of degree d with depth i, for each $i \in \{1,2,\ldots,h\}$, and the remaining vertices are pendants.

Open Problem 5.2. Is the graph $T(n_1,\ldots,n_h,d)$ determined by its signless Laplacian permanental polynomial?

If there is no restriction on the vertex degrees in a tree, then the same problem can be extended to the following:

Open Problem 5.3. Are all trees determined by their (signless) Laplacian permanental polynomials?

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