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The metamorphoses of maximum packings of $2K_n$ with triples to maximum packings of $2K_n$ with 4-cycles for $n \equiv 2$, 3, 6, 7, or 10 (mod 12)

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Dedicated to the memory of Curt Lindner, our dear friend and advisor.

Abstract. Gionfriddo and Lindner detailed the idea of the metamorphosis of 2-fold triple systems with no repeated triples into 2-fold 4-cycle systems of all orders where each system exists (2003). Couch expanded on this idea for $n \equiv 5$, 8, or 11 (mod 12), proving that when $n \equiv 11$ (mod 12), a maximum packing of $2K_n$ with triples has a metamorphosis into a maximum packing of $2K_n$ with 4-cycles, with the leave of a double edge being preserved throughout the metamorphosis, and for $n \equiv 5$ or 8 (mod 12), a maximum packing of $2K_n$ with triples has a metamorphosis into a a 2-fold 4-cycle system of order n, except for when $n = 5$ or $n = 8$, when no such metamorphosis is possible (2016). In this paper, we prove that all remaining orders, i.e. $n \equiv 2$, 3, 6, 7, or 10 (mod 12), can be similarly addressed. For $n \equiv 3$, 6, 7, or 10 (mod 12), a 2-fold triple system (moreover, a hinge system) of order n has a metamorphosis to a maximum packing of $2K_n$ with 4-cycles with the leave a double edge, except for $n = 3$, $n = 6$, and $n = 7$, where no such metamorphosis is possible. When $n \equiv 2$ (mod 12), a maximum packing of $2K_n$ with triples (and as before, with hinges) has a metamorphosis into a maximum packing of $2K_n$ with 4-cycles, with the leave of a double edge being preserved throughout the metamorphosis.

1 Introduction

A λ-fold k-cycle system of order n is a pair (X, C), where C is a collection of edge-disjoint k-cycles that partitions the edge set of λK_n with vertex set

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X (λK_n denotes the graph on n vertices in which each pair of vertices is joined by exactly λ edges). It is well-known that the spectrum for 2-fold 3-cycle (or *triple*) systems is the set of all $n \equiv 0$ or $1 \pmod 3$, and the spectrum for 2-fold 4-cycle systems—i.e., the values of n for which such a system exists—is the set of all $n \equiv 0$ or $1 \pmod 4$ (see [6]). A *hinge* is the multigraph comprised of 2 edge-disjoint 3-cycles with exactly 2 vertices in common (see Figure 1.1 below); the two edges joining the common vertices are naturally called the *double edge* of the hinge. A *maximum packing* of a graph G with a subgraph C is an ordered triple $\big(V(G), T, L\big)$, where T is a collection of copies of C whose edges partition $E(G) \setminus L$. Note that if L is empty, G is λK_n, and C is a 3-cycle; this would correspond to a λ-fold triple system. Thus, a maximum packing of a graph G with C is simply a collection of edge-disjoint copies of C that cover as many edges of G as possible. The uncovered edges L are called the *leave* of the packing.

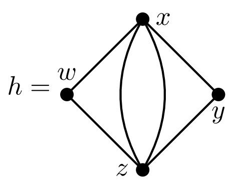


Figure 1.1: The hinge $h = \langle x, z, y, w\rangle$.

The following notation is used throughout, as in [4]. Figure 1.1 above is a hinge, h, and is denoted by $h = \langle x, z, y, w\rangle$, $\langle x, z, w, y\rangle$, $\langle z, x, y, w\rangle$, or $\langle z, x, w, y\rangle$. A double edge between vertices x and y is denoted by $\langle x, y\rangle$. Triples are denoted by any cyclic shift of (x, y, z), 4-cycles by any cyclic shift of (x, y, z, w), and single edges by (x, y) or (y, x).

Let G be a graph and suppose $H^\star$ is a set of hinges. For a hinge

$$h = \langle x, z, y, w\rangle,$$

let

$$\Delta(h) = \{(x, z, y), (x, z, w)\}$$

and define

$$\Delta(H^\star) = \bigcup_{h \in H^\star} \Delta(h).$$

Similarly, let

$$\square(h) = \{(x, y, z, w)\}$$

and

$$D(h) = \{\langle x, z\rangle\}$$

and define

$$\square(H^\star) = \bigcup_{h \in H^\star} \square(h)$$

and

$$D(H^\star) = \bigcup_{h \in H^\star} D(h).$$

Suppose $\big(V(G), \Delta(H^\star), L^\Delta\big)$ is a maximum packing of G with triples (with leave L^Δ) and $\big(V(G), \square(H^\star) \cup D^\star, L^\square\big)$ is a maximum packing of G with 4-cycles (with leave $L^\square$), where either the edges in $D(H^\star)$ or $D(H^\star) \cup L^\Delta$ can be partitioned into the set $D^\star$, each element of which induces a 4-cycle. Further suppose $L^\Delta \subseteq L^\square$ or $L^\square \subseteq L^\Delta$. Then we call $\big(V(G), H^\star, L^\Delta, D^\star, L^\square\big)$ a *metamorphosis* of a maximum packing of G with hinges to a maximum packing of G with 4-cycles. Note that any packing with hinges corresponds to a packing with triples.

In [5], Gionfriddo and Lindner proved the following:

Theorem 1.1. *$2K_n$ has a metamorphosis $(X, H^\star, \emptyset, D^\star, \emptyset)$ from a 2-fold hinge system to a 2-fold 4-cycle system for all $n \equiv 0, 1, 4$, or $9 \pmod{12}$ except $n = 4$.*

When $n \equiv 2 \pmod 3$, there exists a maximum packing of $2K_n$ with triples with only a double edge in the leave L^Δ (see [6]); otherwise, there exists a maximum packing with $L^\Delta = \emptyset$ (i.e., a 2-fold triple system of order n). When $n \equiv 0$ or $1 \pmod 4$, the goal is to form a metamorphosis in which these 2 edges are used in 4-cycles; otherwise, they form $L^\square$. Similarly, when $n \equiv 2$ or $3 \pmod 4$, there exists a maximum packing of $2K_n$ with 4-cycles with only a double edge in the leave $L^\square$; otherwise, there exists a maximum packing with $L^\square = \emptyset$ (i.e., a 2-fold 4-cycle system of order n). When $n \equiv 11 \pmod{12}$, a maximum packing of $2K_n$ with hinges has a metamorphosis into a maximum packing of $2K_n$ with 4-cycles, with the leave of a double edge being preserved throughout the metamorphosis, and for $n \equiv 5$ or $8 \pmod{12}$, a maximum packing of $2K_n$ with hinges has a metamorphosis into a a 2-fold 4-cycle system of order n, except for when $n = 5$ or 8, when no such metamorphosis is possible [4]. We aim to show that all remaining orders, i.e. $n \equiv 2, 3, 6, 7,$ or $10 \pmod{12}$, can be similarly addressed. These constructions conclude the trilogy and proof of the overarching result:

Theorem 1.2. *$2K_n$ has a metamorphosis of a maximum packing with hinges to a maximum packing with 4-cycles for all $n \geq 0$ except $n = 3$, 4, 5, 6, 7, or 8.*

The following notions and theorems are used in the constructions throughout this work. Let $G \setminus H$ denote the graph $\big(V(G), E(G) \setminus E'\big)$, where $V' \subseteq V(G)$ and (V', E') is isomorphic to H. Call (V', E') the *hole* and say the hole is *on* V' and the *size* of the hole is $|V'|$. Let Q be a set of integers and let $H(Q)$ be a partition of Q into pairwise disjoint sets, also called *holes*. A *quasigroup with holes* $H(Q)$ is a quasigroup $(Q, \circ)$ in which, for each $h \in H$, $(h, \circ)$ is a subquasigroup of $(Q, \circ)$. For the purposes of this work, each $(h, \circ)$ can be thought of as a set of "forbidden" products in $(Q, \circ)$.

It is nice that in the situation where $L^{\triangle} \neq \emptyset$, either $L^{\triangle}$ is "used up" in the metamorphosis or "preserved" as $L^{\square}$. We are always able to decompose $2K_{n,n,\ldots,n}$ into 4-cycles as long as n is even, due to a theorem of Dominique Sotteau found below. Furthermore, the quasigroups we need always exist (except possibly for small cases).

Theorem 1.3 (Sotteau [8])**.** *Necessary and sufficient conditions for the complete bipartite graph $K_{m,n}$ to be partitioned into $(2k)$-cycles are*

(1) *m and n are even,*

(2) *$k \leq m$ and $k \leq n$, and*

(3) *$2k \mid mn$.*

Theorem 1.4 (Lindner and Rodger [7] and Dinitz [1])**.** *There exists a commutative quasigroup $(Q, \circ)$ with holes of size 2 if $|Q| \equiv 0 \pmod 2$ and $|Q| > 4$.*

2 Case: $n \equiv 2 \pmod{12}$

For the case where $n = 2$, let $L = \langle 1, 2 \rangle$. Then $(\{1, 2\}, \emptyset, L, \emptyset, L)$ is a metamorphosis of a maximum packing of G with hinges to a maximum packing of G with 4-cycles. We begin with some necessary ingredients for a general construction for $n \equiv 2 \pmod{12}$: a direct construction of a metamorphosis of a maximum packing of $2K_{14}$ with hinges to a maximum packing of $2K_{14}$ with 4-cycles, a 2-fold maximum packing with hinges for a system of order 5, and a lemma concerning a special decomposition of $2K_{4n}$.

Lemma 2.1. *There exists a metamorphosis of a maximum packing of* $2K_{14}$ *with hinges to a maximum packing of* $2K_{14}$ *with* 4*-cycles.*

Proof. Let the vertex set $V = \{\infty_1, \infty_2\} \cup \{0, 1, 2, \ldots, 11\}$. A 2-fold maximum packing with triples on V can be paired in the natural way to produce the following set of hinges:

$$H^\star = \left\{ \begin{array}{c} \langle 0, \infty_1, 11, 1\rangle,\ \langle 2, \infty_1, 1, 3\rangle,\ \langle 4, \infty_1, 3, 5\rangle,\ \langle 6, \infty_1, 5, 7\rangle, \\ \langle 8, \infty_1, 7, 9\rangle,\ \langle 10, \infty_1, 9, 11\rangle,\ \langle 0, \infty_2, 5, 7\rangle,\ \langle 2, \infty_2, 7, 9\rangle, \\ \langle 4, \infty_2, 9, 11\rangle,\ \langle 6, \infty_2, 11, 1\rangle,\ \langle 8, \infty_2, 1, 3\rangle,\ \langle 10, \infty_2, 3, 5\rangle, \\ \langle 0, 3, 11, 10\rangle,\ \langle 1, 4, 0, 11\rangle,\ \langle 2, 5, 1, 0\rangle,\ \langle 3, 6, 2, 1\rangle,\ \langle 4, 7, 3, 2\rangle, \\ \langle 5, 8, 4, 3\rangle,\ \langle 6, 9, 5, 4\rangle,\ \langle 7, 10, 6, 5\rangle,\ \langle 8, 11, 7, 6\rangle,\ \langle 9, 0, 8, 7\rangle, \\ \langle 10, 1, 9, 8\rangle,\ \langle 11, 2, 10, 9\rangle,\ \langle 0, 6, 2, 8\rangle,\ \langle 1, 7, 3, 9\rangle, \\ \langle 2, 8, 4, 10\rangle,\ \langle 3, 9, 5, 11\rangle,\ \langle 4, 10, 6, 0\rangle,\ \langle 5, 11, 7, 1\rangle \end{array} \right\}$$

The missing double edge in the system is the leave $L^\Delta = L = \langle \infty_1, \infty_2\rangle$.

Removing the double edge from each of the hinges yields $\square(H^\star)$, and we can then rearrange the removed double edges, $D(H^\star)$, into the following set of 4-cycles:

$$D^\star = \left\{ \begin{array}{c} (0, \infty_1, 2, \infty_2),\ (2, \infty_1, 4, \infty_2),\ (4, \infty_1, 6, \infty_2), \\ (6, \infty_1, 8, \infty_2),\ (8, \infty_1, 10, \infty_2),\ (10, \infty_1, 0, \infty_2), \\ (0, 3, 6, 9),\ (1, 4, 7, 10),\ (2, 5, 8, 11),\ (0, 3, 9, 6),\ (1, 4, 10, 7), \\ (2, 5, 11, 8),\ (0, 6, 3, 9),\ (1, 7, 4, 10),\ (2, 8, 5, 11) \end{array} \right\}$$

Now, we have a maximum packing of $2K_{14}$ with 4-cycles $(V, \square(H^\star) \cup D^\star, L)$ and $(V, H^\star, L, D^\star, L)$ is a metamorphosis of a maximum packing of $2K_{14}$ with hinges to a maximum packing of $2K_{14}$ with 4-cycles. $\square$

Example 2.2 (2-fold maximum packing of hinges on a system of order 5)**.** Let

$$V = \{\infty_1, \infty_2\} \cup \{1, 2, 3\},$$
$$H^\star = \{\langle 1, 2, \infty_1, \infty_2\rangle, \langle 1, 3, \infty_1, \infty_2\rangle, \langle 2, 3, \infty_1, \infty_2\rangle\},$$
$$L^\Delta = L = \langle \infty_1, \infty_2\rangle.$$

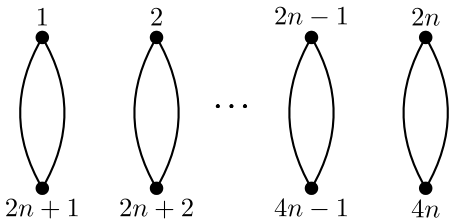
Then $(V, \Delta(H^\star), L)$ is a maximum packing with triples. While there is no metamorphosis of this maximum packing with triples, this system, as is, proves to be satisfactory for application in this section's general construction.

Lemma 2.3. *$2K_{4n}$ can be decomposed into two double 1-factors and $4n^2-3n$ total 4-cycles.*

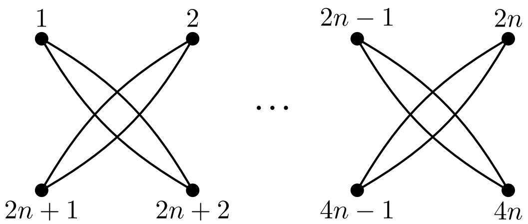
Proof. We prove this lemma by construction. Let

$$V(2K_{4n}) = \{1, 2, \ldots, 4n\}.$$

Our first double 1-factor has edges of the form $\langle i,\, i+2n\rangle$, for $1 \le i \le 2n$.

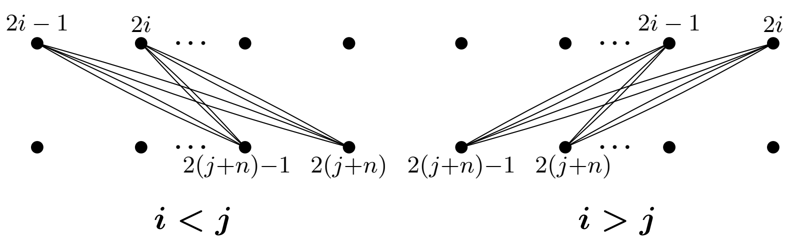


Our second double 1-factor has edges of the form $\langle 2i-1,\, 2(i+n)\rangle$ and $\langle 2i,\, 2(i+n)-1\rangle$, for $1 \le i \le n$.



For $n \equiv 0 \pmod 2$:

(1) Form two copies of 4-cycles $\big(2i-1,\, 2(j+n)-1,\, 2i,\, 2(j+n)\big)$, for $1 \le i \le n$, $1 \le j \le n$, and $i \ne j$.

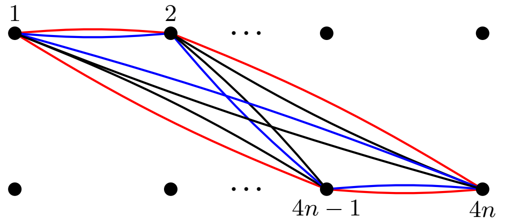


(2) The remaining edges form $2K_{2n} = 2K_{4k}$ on $\{1, 2, \ldots, 2n\}$ and $2K_{4k}$ on $\{2n+1,\, 2n+2,\, \ldots,\, 4n\}$, which can be decomposed into 4-cycles.

For $n \equiv 1 \pmod 2$:

(1) Form two copies of 4-cycles $\big(2i-1,\, 2(j+n)-1,\, 2i,\, 2(j+n)\big)$, for $1 \le i \le n-1$, $1 \le j \le n$, and $i \ne j$, except when $i = 1$ and $j = n$.

(2) Form the following 4-cycles: $(1,\ 2,\ 4n,\ 4n-1)$, $(1,\ 2,\ 4n-1,\ 4n)$, and $(1,\ 4n,\ 2,\ 4n-1)$.



(3) The remaining edges form $2K_{2n} = 2K_{4k+2}$ on $\{1, 2, \ldots, 2n\}$ minus $\langle 1, 2\rangle$ and $2K_{4k+2}$ on $\{2n+1, 2n+2, \ldots, , 4n-1, 4n\}$ minus $\langle 4n-1, 4n\rangle$, which can be decomposed into 4-cycles [6]. □

2.1 Construction

We now begin the construction for the $n \equiv 2 \pmod{12}$ case. Note that n of this form can be written as $12k+2$ or equivalently $3(4k)+2$. Now define $\infty = \{\infty_1, \infty_2\}$ and $Q = \{1, 2, 3, \ldots, 4k\}$. Let $V = \infty \cup \big(Q \times \{1, 2, 3\}\big)$ and $(Q, \circ)$ be an idempotent antisymmetric quasigroup of order $4k$ (see [3]). This is to say for $i, j \in Q$, we have $i \circ i = i$ and $i \circ j \neq j \circ i$ for $i \neq j$.

(1) For $1 \leq i \leq 4k$, use Example 2.2 where $\{1, 2, 3\}$ are renamed

$$\{(i, 1), (i, 2), (i, 3)\},$$

respectively, to place a copy on $\infty \cup \{(i, 1),\ (i, 2),\ (i, 3)\}$. Place these hinges in $H^\star$.

(2) Now for all $x, y \in Q$, with $x < y$, place the hinges

$$\langle (x, 1),\ (y, 1),\ (x \circ y, 2),\ (y \circ x, 2)\rangle,$$
$$\langle (x, 2),\ (y, 2),\ (x \circ y, 3),\ (y \circ x, 3)\rangle,$$
$$\langle (x, 3),\ (y, 3),\ (x \circ y, 1),\ (y \circ x, 1)\rangle$$

in $H^\star$. Let $L^\Delta = L = \langle \infty_1, \infty_2\rangle$ and note that $\big(V, \Delta(H^\star), L\big)$ is a maximum packing of $2K_{12k+2}$ with triples.

(3) Note that the remaining double edges from the previous step, that is, edges in $D(H^\star)$, are precisely a $2K_{4k}$ on each of levels 1, 2, and 3. Decompose this graph into the two 1-factors F_1 and F_2 and the $4k^2 - 3n$ total 4-cycles as shown in Lemma 2.3. Place the resulting

4-cycles in $D^\star$. Finally, for each $\{a,b\} \in F_1 \cup F_2$, place the 4-cycles

$$\big((a,1),\,(b,1),\,(b,2),\,(a,2)\big),$$
$$\big((a,1),\,(b,1),\,(b,3),\,(a,3)\big),$$
$$\big((a,2),\,(b,2),\,(b,3),\,(a,3)\big)$$

in $D^\star$ (which also uses up the remaining double edges from the first step).

Thus $(V, H^\star, L, D^\star, L)$ is a metamorphosis of a maximum packing of $2K_{12k+2}$ with hinges to a maximum packing of $2K_{12k+2}$ with 4-cycles, as desired, yielding the following result.

Theorem 2.4. *There exists a metamorphosis of a maximum packing of $2K_{12k+2}$ with hinges to a maximum packing of $2K_{12k+2}$ with 4-cycles for all $k \geq 0$.*

3 Case: $n \equiv 3 \pmod{12}$

For the case where $n = 3$, we cannot produce even one hinge. Before we introduce the construction that works for all $n \equiv 3 \pmod{12}$ at least 15, we establish the following terminology, example, and associated lemma.

Let $H = \{h_1, h_2, h_3, \ldots, h_t\}$ be a collection of pairwise disjoint subsets of the set V called *holes*. We define $2h_i = \langle x, y\rangle$ whenever $h_i = \{x, y\}$ and $2H = \{2h_1, 2h_2, 2h_3, \ldots, 2h_t\}$. Let $2K_n$ have vertex set V and let C be a collection of 4-cycles that partitions $2K_n \setminus 2H$ based on V. We then call (V, C) a 2-fold 4-cycle system with holes $2H$.

Example 3.1 (2-fold 4-cycle system of order 5 with two holes of size 2)**.** Let

$$V = \{1,2,3,4,5\},$$
$$2H = \{\langle 2,3\rangle, \langle 4,5\rangle\},$$
$$C = \{(1,2,4,3),\,(1,3,5,2),\,(1,4,3,5),\,(1,5,2,4)\}.$$

Lemma 3.2. *There exists a 2-fold 4-cycle system of order $4n+1$ with $2n$ holes of size 2 for all $4n+1 \geq 5$.*

Proof. Let $X = \{1,2,3,\ldots,n\}$, $V = \{\infty\} \cup \big(X \times \{1,2,3,4\}\big)$ and define a collection C of 4-cycles as follows:

(1) For each $x \in X$, define a copy of Example 3.1 on

$$\{\infty\} \cup \big(\{x\} \times \{1, 2, 3, 4\}\big),$$

where $2H = \{\langle(x, 1), (x, 2)\rangle, \langle(x, 3), (x, 4)\rangle\}$, and place these 4-cycles in C.

(2) For each $x \neq y$, partition $2K_{4,4}$ with parts $\{x\} \times \{1, 2, 3, 4\}$ and $\{y\} \times \{1, 2, 3, 4\}$ into 4-cycles and place these 4-cycles in C.

Then (V, C) is a 2-fold 4-cycle system of order $4n + 1$ with $2n$ holes of size 2. □

3.1 Construction

Let $12k + 3 \geq 15$ and note then that $k \geq 1$. Let $Q = \{1, 2, 3, \dots, 4k + 1\}$. Then let $(Q, \circ_1)$ be an antisymmetric idempotent quasigroup of order $4k+1$ and $(Q, \circ_2)$ be a commutative idempotent quasigroup of order $4k + 1$ (see [7]). Finally, let α be the permutation $(1\ 2\ 3\ \dots\ 4k\ 4k + 1)$ and $V = Q \times \{1, 2, 3\}$ and proceed as follows:

(1) For each $x \neq y \in Q$,

$$\langle(x, 1), (y, 1), (x \circ_1 y, 2), (y \circ_1 x, 2)\rangle \in H^\star.$$

Note that $\langle(x, 1), (x, 2)\rangle$ is not covered for $x \in Q$.

(2) For each $x \neq y \in Q$,

$$\big\langle(x, 2), (y, 2), (x \circ_2 y, 3), \big((x \circ_2 y)\alpha, 3\big)\big\rangle \in H^\star.$$

Note that $\big((x, 2), (x, 3)\big)$ and $\big((x, 2), (x\alpha, 3)\big)$ are not covered for all $x \in Q$.

(3) For each $x \neq y \in Q$,

$$\big\langle(x, 3), (y, 3), (x \circ_2 y, 1), \big((x \circ_2 y)\alpha^{-1}, 1\big)\big\rangle \in H^\star.$$

Note that $\big((x, 3), (x, 1)\big)$ and $\big((x, 3), (x\alpha^{-1}, 1)\big)$ are not covered for all $x \in Q$.

(4) For each $x \in Q$,

$$\langle(x, 1), (x, 2), (x, 3), (x\alpha, 3)\rangle \in H.$$

Then $(V, \Delta(H^\star), \emptyset)$ is a 2-fold triple system of order $12k + 3$. We now perform a metamorphosis into a maximum packing with 4-cycles as follows:

(5) Use Lemma 3.2 to construct a 2-fold 4-cycle system of order $4k+1$ on $Q \times \{i\}$ with holes $\langle (2j,i), (2j+1,i) \rangle$ for $1 \le i \le 2$, $1 \le j \le 2k$. Place two copies of the 4-cycles $\big((2j,1), (2j+1,1), (2j+1,2), (2j,2)\big)$ for $1 \le j \le 2k$ into $D^\star$. This uses the double edges from (1), (2), and (4), except for $L^\square = L = \langle (1,1), (1,2) \rangle$.

(6) Since Q is of order $4k+1$, we know also that we can rearrange our remaining double edges from (3) into 4-cycles. Place these 4-cycles into $D^\star$.

Thus $(V, H^\star, \emptyset, D^\star, L)$ is a metamorphosis of a hinge system of order $12k+3$ to a maximum packing of $2K_{12k+3}$ with 4-cycles, as desired, yielding the following result.

Theorem 3.3. *There exists a metamorphosis of a hinge system of order $12k+3$ to a maximum packing of $2K_{12k+3}$ with 4-cycles for all $k \ge 1$.*

4 Case: $n \equiv 6 \pmod{12}$

It is first shown that there exists no metamorphosis of order 6, and then we proceed to the iterative construction for a metamorphosis for $n \equiv 6 \pmod{12}$ whenever $n \ge 18$.

Lemma 4.1. *There does not exist a metamorphosis of a hinge system of order 6 to a maximum packing of $2K_6$ with 4-cycles.*

Proof. Let $(V, H^\star)$ be a hinge system of order 6 and let $D(H^\star)$ be the 5 double edges in the hinges. It is important to note that, considered as a 2-fold triple system $\big(V, \Delta(H^\star)\big)$, each triple contains exactly one edge from the double edges in $D(H^\star)$. Now suppose that $D(H^\star)$ contains a 4-cycle $(1,2,3,4)$. Since each of $(1,2)$, $(2,3)$, $(3,4)$, and $(4,1)$ is half of a double edge in $D(H^\star)$, we have that $D(H^\star)$ contains the 4-cycle of double edges $(\langle 1,2 \rangle, \langle 2,3 \rangle, \langle 3,4 \rangle, \langle 4,1 \rangle)$. Since each of the ten triples in $\Delta(H^\star)$ contain exactly one edge from $D(H^\star)$, the two triples in $\Delta(H^\star)$ containing the edge $(1,2)$ must look like $(1,2,5)$ and $(1,2,6)$. Similarly, $\Delta(H^\star)$ must contain triples that look like $(3,4,5)$, $(3,4,6)$, $(2,3,5)$, $(2,3,6)$, $(1,4,5)$, and $(1,4,6)$. This forces the remaining edges in $2K_6$ to look like $\langle 1,4 \rangle$, $\langle 2,3 \rangle$, and $\langle 5,6 \rangle$. These edges cannot be paired into two triples, much less a hinge. It follows that $D(H^\star)$ cannot contain even one 4-cycle. □

4.1 Construction

Let $12k + 6 \geq 18$ and note then that $k \geq 1$. Let $Q = \{1, 2, 3, \ldots, 4k + 2\}$. Then let $(Q, \circ_1)$ be an antisymmetric idempotent quasigroup of order $4k+2$ and $(Q, \circ_2)$ be an antisymmetric quasigroup of order $4k+2$ with holes $H = \{h_i : 1 \leq i \leq 2k+1\}$ of size 2 defined in the usual manner found in [7] (these can be formed by simply taking the direct product of an antisymmetric quasigroup of order $2k + 1$ with the quasigroup of order 2 and renaming the symbols appropriately). Finally, let $V = Q \times \{1, 2, 3\}$ and proceed as follows:

(1) For each $x \neq y \in Q$,

$$\langle (x, 1),\ (y, 1),\ (x \circ_1 y, 2),\ (y \circ_1 x, 2) \rangle \in H^\star.$$

Note that $\langle (x, 1),\ (x, 2) \rangle$ is not covered for $x \in Q$.

(2) For each x and y belonging to the same $h_i \in H$,

$$\langle (x, 2),\ (y, 2),\ (x, 3),\ (y, 3) \rangle \in H^\star,$$

and for each x and y belonging to different holes of H,

$$\langle (x, 2),\ (y, 2),\ (x \circ_2 y, 3),\ (y \circ_2 x, 3) \rangle \in H^\star.$$

Note that the edges in $\big((x, 2),\ (y, 3),\ (y, 2),\ (x, 3)\big)$ are not covered for any $\{x, y\} \in H$.

(3) We now repeat (2) for edges between $Q \times \{3\}$ and $Q \times \{1\}$. For each x and y belonging to the same $h_i \in H$,

$$\langle (x, 3),\ (y, 3),\ (x, 1),\ (y, 1) \rangle \in H^\star,$$

and for each x and y belonging to different holes of H,

$$\langle (x, 3),\ (y, 3),\ (x \circ_2 y, 1),\ (y \circ_2 x, 1) \rangle \in H^\star.$$

Note that the edges in $\big((x, 1),\ (y, 3),\ (y, 1),\ (x, 3)\big)$ are not covered for any $\{x, y\} \in H$.

(4) For each x and y belonging to the same $h_i \in H$, we have both

$$\langle (x, 1),\ (x, 2),\ (x, 3),\ (y, 3) \rangle \in H^\star,$$
$$\langle (y, 1),\ (y, 2),\ (x, 3),\ (y, 3) \rangle \in H^\star.$$

This covers the edges remaining from the previous steps.

Then $(V, \Delta(H^\star), \emptyset)$ is a 2-fold triple system of order $12k + 6$. We now perform a metamorphosis into a maximum packing with 4-cycles as follows:

(5) For each x and y belonging to the same $h_i \in H$, place two copies of the 4-cycle $\big((x,1),(y,1),(y,2),(x,2)\big)$ into $D^\star$.

(6) After accounting for the 4-cycles in step (5) above, 2 copies of the complete $(2n+1)$-partite graph with all partite sets having size 2 on $Q \times \{1\}$ and $Q \times \{2\}$ remain. For each pair of partite sets $\{x,y\}$ and $\{z,w\}$ in these $2K_{2,2,\ldots,2}$, place 2 copies of (x,w,y,z) into $D^\star$.

(7) Since Q is of order $4k+2$, we can rearrange our remaining double edges from the hinges created in (3) into 4-cycles with leave $L^\square = L = \langle(1,3),\ (2,3)\rangle$ (see [1]). Place these 4-cycles into $D^\star$.

Thus $(V, H^\star, \emptyset, D^\star, L)$ is a metamorphosis of a hinge system of order $12k+6$ to a maximum packing of $2K_{12k+6}$ with 4-cycles, as desired, yielding the following result.

Theorem 4.2. *There exists a metamorphosis of a hinge system of order $12k+6$ to a maximum packing of $2K_{12k+6}$ with 4-cycles for all $k \geq 1$.*

5 Case: $n \equiv 7 \pmod{12}$

It is first shown that there exists no metamorphosis of order 7, and then we proceed to the iterative construction for a metamorphosis for $n \equiv 7 \pmod{12}$ whenever $n \geq 19$.

Lemma 5.1. *There does not exist a metamorphosis of a hinge system of order 7 to a maximum packing of $2K_7$ with 4-cycles.*

Proof. Any 2-fold triple system of order 7 with no repeated triples consists of a pair of disjoint Steiner triple systems [2]. So let (V, T_1) and (V, T_2) be a pair of disjoint triple systems of order 7 and $(V, H^\star)$ any hinge system constructed from T_1 and T_2. Let D be the collection of 7 double edges from the hinges. Now suppose D contains a 4-cycle $(1,2,3,4)$. Then D must also contain the 4-cycle of double edges $(\langle 1,2\rangle,\ \langle 2,3\rangle,\ \langle 3,4\rangle,\ \langle 4,1\rangle)$. Since the Steiner triple systems of order 7 is a projective plane and since there is only one such plane (up to isomorphism), both T_1 and T_2 must contain the Pasch configuration (see [9]) as shown in Figure 5.1.

Let $\{x,a,y\} \in T_1$ and $\{z,b,w\} \in T_2$. Since (V,T_1) and (V,T_2) have order 7, we must have $\{x,a,y\} = \{z,b,w\} = \{5,6,7\}$, which is a contradiction, since T_1 and T_2 are disjoint. $\square$

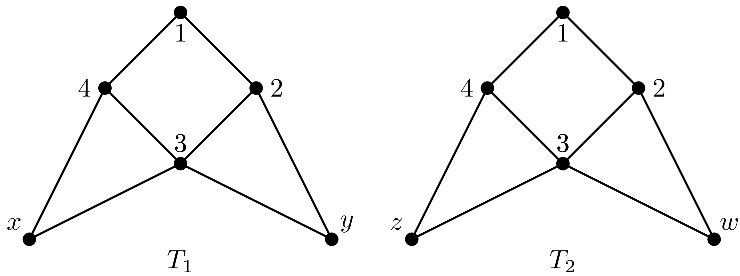


Figure 5.1: Iterations of the Pasch configuration.

5.1 Construction

Let $12k+7 \geq 19$ and note then that $k \geq 1$. Let $Q = \{1, 2, 3, \ldots, 4k+2\}$. Then let $(Q, \circ)$ be an antisymmetric idempotent quasigroup of order $4k+2$. Finally, let $V = \{\infty\} \cup \big(Q \times \{1, 2, 3\}\big)$ and proceed as follows:

(1) For each $x \in Q$, we have both

$$\langle \infty,\, (x,1),\, (x,2),\, (x,3) \rangle \in H^\star,$$
$$\langle (x,2),\, (x,3),\, (x,1),\, \infty) \rangle \in H^\star.$$

(2) For each $x \neq y \in Q$,

$$\langle (x,1),\, (y,1),\, (x \circ y, 2),\, (y \circ x, 2) \rangle \in H^\star,$$
$$\langle (x,2),\, (y,2),\, (x \circ y, 3),\, (y \circ x, 3) \rangle \in H^\star,$$
$$\langle (x,3),\, (y,3),\, (x \circ y, 1),\, (y \circ x, 1) \rangle \in H^\star.$$

Then $(V, \Delta(H^\star), \emptyset)$ is a 2-fold triple system of order $12k+7$. We now perform a metamorphosis into a maximum packing with 4-cycles as follows:

(3) For $1 \leq j \leq 2k+1$, place two copies of the 4-cycle $\big((2j-1, 2),\, (2j, 2),\, (2j, 3),\, (2j-1, 3)\big)$ into $D^\star$.

(4) After accounting for the 4-cycles in step (3) above, 2 copies of the complete $(2n+1)$-partite graph with all partite sets having size 2 on $Q \times \{2\}$ and $Q \times \{3\}$ remain. For each pair of partite sets $\{x, y\}$ and $\{z, w\}$ in these $2K_{2,2,\ldots,2}$, place 2 copies of (x, w, y, z) into $D^\star$.

(5) The remaining double edges are of the form either $\langle \infty,\, (x,1) \rangle$ or $\langle (x,1),\, (y,1) \rangle$ where $x \neq y \in Q$. We can consider these double edges as $2K_{4k+3}$, which can be rearranged into 4-cycles and placed in $D^\star$, with leave $L^\square = L = \langle \infty, (x,1) \rangle$ (see [1]).

Thus $(V, H^\star, \emptyset, D^\star, L)$ is a metamorphosis of a hinge system of order $12k+7$ to a maximum packing of $2K_{12k+7}$ with 4-cycles, as desired, yielding the following result.

Theorem 5.2. *There exists a metamorphosis of a hinge system of order $12k+7$ to a maximum packing of $2K_{12k+7}$ with 4-cycles for all $k \geq 1$.*

6 Case: $n \equiv 10 \pmod{12}$

We begin with some necessary ingredients: some 2-fold 4-cycle systems with holes, a related lemma, and a direct construction of a metamorphosis for $n = 10$.

Example 6.1 (2-fold 4-cycle system of order 7 with three holes of size 2)**.** Let

$$\begin{aligned} V &= \{\infty\} \cup \{1,2,3,4,5,6\},\\ 2H &= \{\langle 1,2\rangle,\ \langle 3,4\rangle,\ \langle 5,6\rangle\},\\ C &= \{(\infty,1,4,2),\ (\infty,2,3,1),\ (\infty,3,6,4),\ (\infty,4,5,3),\\ &\qquad (\infty,5,4,6),\ (\infty,6,3,5),\ (1,3,2,4),\ (1,5,2,6),\ (1,5,2,6)\}. \end{aligned}$$

Example 6.2 (2-fold 4-cycle system of order 7 with one hole of size 3 and two holes of size 2)**.** Let

$$\begin{aligned} V &= \{\infty\} \cup \{1,2,3,4,5,6\},\\ 2H &= \{\langle \infty,1,2\rangle,\ \langle 3,4\rangle,\ \langle 5,6\rangle\},\\ C &= \{(\infty,3,6,4),\ (\infty,4,5,3),\ (\infty,5,3,6),\ (\infty,6,4,5),\\ &\qquad (1,3,2,4),\ (1,3,2,4),\ (1,5,2,6),\ (1,5,2,6)\}. \end{aligned}$$

where $\langle \infty,1,2\rangle$ (in $2H$) represents two copies of the 3-cycle $(\infty,1,2)$.

Lemma 6.3. *There exists a 2-fold 4-cycle system of order $4n+3$ with $2n$ holes of size 2 for all $4n+3 \geq 7$.*

Proof. Let $X = \{1,2,3,\ldots,n\}$ and $V = \{\infty_1,\infty_2,\infty_3\} \cup \big(X \times \{1,2,3,4\}\big)$ and define a collection C of 4-cycles as follows:

(1) Define a copy of Example 6.1 on $\{\infty_1,\infty_2,\infty_3\} \cup \big(\{1\} \times \{1,2,3,4\}\big)$ and place these 4-cycles in C.

(2) For each $i \geq 2$, define a copy of Example 6.2 on $\{\infty_1, \infty_2, \infty_3\} \cup \big(\{i\} \times \{1,2,3,4\}\big)$, with the proviso that the hole of size 3 is on $\{\infty_1, \infty_2, \infty_3\}$ and that the holes of size 2 are on $\langle (i,1), (i,2) \rangle$ and $\langle (i,3), (i,4) \rangle$, and place these 4-cycles in C.

(3) For each $x \neq y$, partition $2K_{4,4}$ with parts $\{x\} \times \{1,2,3,4\}$ and $\{y\} \times \{1,2,3,4\}$ into 4-cycles and place these 4-cycles in C.

Then (V, C) is a 2-fold 4-cycle system of order $4n+3$ with $2n+1$ holes of size 2. □

Lemma 6.4. *There exists a metamorphosis of a hinge system of order* 10 *to a maximum packing of* $2K_{10}$ *with* 4*-cycles.*

Proof. Let the vertex set $V = \{0, 1, \ldots, 9\}$. Then, let

$$H^\star = \begin{Bmatrix} \langle 0,2,1,3\rangle, \langle 0,4,5,6\rangle, \langle 0,8,7,9\rangle, \langle 2,4,1,7\rangle, \langle 2,8,3,6\rangle, \\ \langle 4,8,6,7\rangle, \langle 1,5,7,8\rangle, \langle 3,5,7,8\rangle, \langle 3,9,4,6\rangle, \langle 1,9,6,8\rangle, \\ \langle 5,6,0,2\rangle, \langle 6,7,1,3\rangle, \langle 7,9,0,2\rangle, \langle 5,9,2,4\rangle, \langle 1,3,0,4\rangle \end{Bmatrix}$$

and with $L^\square = L = \langle 1,3 \rangle$ we let

$$\{(0,2,8,4), (0,8,2,4), (0,8,4,2), (1,5,3,9), \\ (1,5,3,9), (5,6,7,9), (5,6,7,9)\} \subseteq D^\star.$$

Thus $(V, H^\star, \emptyset, D^\star, L)$ is a metamorphosis of a hinge system of order 10 to a maximum packing of $2K_{10}$ with 4-cycles. □

6.1 Construction

We now proceed directly to the iterative construction for a metamorphosis for $n \equiv 10 \pmod{12}$ whenever $n \geq 22$. Let $12k+10 \geq 22$ and note then that $k \geq 1$. Let $Q = \{1, 2, 3, \ldots, 4k+3\}$. Then let $(Q, \circ)$ be an antisymmetric idempotent quasigroup of order $4k+3$. Finally, let $V = \{\infty\} \cup \big(Q \times \{1,2,3\}\big)$ and proceed as follows:

(1) For each $x \in Q$, we have both

$$\langle \infty, (x,1), (x,2), (x,3) \rangle \in H^\star,$$
$$\langle (x,2), (x,3), (x,1), \infty) \rangle \in H^\star.$$

(2) For each $x \neq y \in Q$,

$$\langle (x,1),\, (y,1),\, (x\circ y,2),\, (y\circ x,2)\rangle \in H^\star,$$
$$\langle (x,2),\, (y,2),\, (x\circ y,3),\, (y\circ x,3)\rangle \in H^\star,$$
$$\langle (x,3),\, (y,3),\, (x\circ y,1),\, (y\circ x,1)\rangle \in H^\star.$$

Then $(V, \Delta(H^\star), \emptyset)$ is a 2-fold triple system of order $12k+10$. We now perform a metamorphosis into a maximum packing with 4-cycles as follows:

(3) For $1 \leq j \leq 2k+1$, place two copies of the 4-cycle $\big((2j,2),\, (2j+1,2),\, (2j+1,3),\, (2j,3)\big)$ into $D^\star$. Note that we have used all "vertical" double edges between $Q \times \{2\}$ and $Q \times \{3\}$ except for $L^\square = L = \langle (1,2),\, (1,3)\rangle$.

(4) For $i \in \{2,3\}$, the remaining double edges on $Q \times \{i\}$ form

$$2K_{4n+3} \setminus \big\{\big\langle (2,i),(3,i)\big\rangle,\, \big\langle (4,i),(5,i)\big\rangle,\, \ldots,\, \big\langle (4n+2,i),(4n+3,i)\big\rangle\big\}.$$

Using Lemma 6.3, we can guarantee a rearrangement of these edges into 4-cycles and place these 4-cycles into $D^\star$.

(5) The remaining double edges are of the form either $\langle \infty,\, (x,1)\rangle$ or $\langle (x,1),\, (y,1)\rangle$ where $x \neq y \in Q$. We can consider these double edges as $2K_{4k+4}$, which can be rearranged into 4-cycles and placed in $D^\star$.

Thus $(V, H^\star, \emptyset, D^\star, L)$ is a metamorphosis of a hinge system of order $12k+10$ to a maximum packing of $2K_{12k+10}$ with 4-cycles, as desired, yielding the following result.

Theorem 6.5. *There exists a metamorphosis of a hinge system of order $12k+10$ to a maximum packing of $2K_{12k+10}$ with 4-cycles for all $k \geq 0$.*

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