BULLETIN OF THE ICA
Volume **106** (2026), 43–58

Generating all Eulerian trails avoiding forbidden transitions

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Abstract. Let G be a multigraph without loops and H a graph possibly with loops. We say that G is an H-colored multigraph whenever there exists a function $c\colon E(G) \to V(H)$. A walk (respectively, path, trail) $W = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ in G is an H-walk (respectively, H-path, H-trail) if and only if $\big(c(e_0), c(e_1), \ldots, c(e_{k-2}), c(e_{k-1})\big)$ is a walk in H. W is a closed H-walk (respectively, closed H-trail) if and only if W is an H-walk (respectively, H-trail) such that $v_0 = v_k$ and $c(e_{k-1})c(e_0) \in E(H)$. Notice that W is a properly colored trail whenever H is a complete graph without loops, in particular when H is K_2 we have that W is a properly 2-colored trail. In 1995 Pevzner defined the order transformations, which allow us to generate all properly colored Eulerian trails in a 2-colored multigraph, starting with a fixed one. This result has been fundamental for the study of DNA physical mapping. In this paper we give sufficient conditions on an H-edge coloring of G to generate all Eulerian H-trails of G, starting with a fixed one. As a consequence of the main result we obtain a polynomial-time algorithm to do it.

1 Introduction

For general concepts, terminology, and notation not defined here, we refer the reader to [2] and [3]. In this work we consider multigraphs, multigraphs without parallel edges (called graphs) and graphs without loops (named simple graphs). Let G be a multigraph. Then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The notation $e \backsim uv$ means that u and v are the end-points of the edge e. We say that e *joins* u and v, the edge e is *incident* with u (respectively, with v), u and v are *adjacent*, and e is a *loop* whenever $u = v$. A *walk* is a sequence $W =$

Key words and phrases: Eulerian H-trail, Order transformation, H-coloring
Mathematics Subject Classifications: 05C45, 05C15
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Received: 20 April 2024
Accepted: 23 February 2025

$(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ where $e_i \backsim v_i v_{i+1}$ for every $i \in \{0, \ldots, k-1\}$, and W is *closed* when $v_0 = v_k$. It is called a *path* whenever $v_i \neq v_j$ for all i and j, with $i \neq j$. A *cycle* is a closed walk $(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k, e_k, v_0)$, with $k \geq 2$, such that $v_i \neq v_j$ for all i and j with $i \neq j$. A *trail* is a walk in which no edge is repeated. If $W = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ and $W' = (v_k, e_k, v_{k+1}, e_{k+1}, \ldots, e_{t-1}, v_t)$ are two walks, then the walk $(v_k, e_{k-1}, \ldots, e_1, v_1, e_0, v_0)$, obtained by reversing W, is denoted by W^{-1}; and the walk $(v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k, e_k, v_{k+1}, e_{k+1}, \ldots, e_{t-1}, v_t)$, obtained by concatenating W and W' at v_k, is denoted by either WW' or $W \cup W'$. For vertices v_i and v_j in W with $i < j$, the subwalk $(v_i, e_i, \ldots, e_{j-1}, v_j)$ is denoted by (v_i, W, v_j).

A walk in an edge-colored multigraph G is *properly colored* (or *alternating*) if no two consecutive edges have the same color, including the last and first edges in a closed walk. Several problems have been modeled by edge-colored multigraphs, and the study of the applications of properly colored walks seems to have started in [8], according to [2]. Properly colored walks are of interest in graph theory applications, such as in genetic and molecular biology [5, 9, 10, 12, 13], design of printed circuit and wiring boards [14], channel assignment in wireless networks [1,11], social sciences [4], and graph models for conflict resolutions [15, 16]. They are also of interest in graph theory itself as generalizations of walks in undirected and directed graphs. There is an extensive literature on properly colored walks; for a detailed survey on this topic see for example Chapter 16 of [2].

In [10], Pevzner shows how to generate all properly colored Eulerian trails in an edge-colored multigraph, starting with any one, by means of the following transformations: Let $F = (x_0, e_0, x_1, \ldots, x_{m-1}, e_{m-1}, x_m)$ be a properly colored trail in G and suppose that $\{i, j, k, n\}$ is a subset of $\{0, \ldots, m\}$, with $i < j < k < n$, such that $x_k = x_i$ and $x_n = x_j$. Break up F as follows:

$$F_1 = (x_0, F, x_i), \qquad F_2 = (x_i, F, x_j), \qquad F_3 = (x_j, F, x_k),$$
$$F_4 = (x_k, F, x_n), \qquad F_5 = (x_n, F, x_m).$$

The transformation $F = F_1F_2F_3F_4F_5 \longrightarrow F^* = F_1F_4F_3F_2F_5$ is called an *order exchange* whenever F^* is a properly colored trail. Whenever $x_j = x_i$, break up F as follows:

$$F_1 = (x_0, F, x_i), \qquad F_2 = (x_i, F, x_j), \qquad F_3 = (x_j, F, x_m).$$

The transformation $F = F_1F_2F_3 \longrightarrow F^* = F_1F_2^{-1}F_3$ is called an *order reflection* whenever F^* is an alternating trail.

The main result in [10] is the following.

Theorem 1.1. *Let G be a 2-edge-colored multigraph. Every pair of alternating Eulerian trails X and Y in G can be transformed into each other by means of a sequence of order transformations (exchanges and reflections).*

Let G be a multigraph without loops and H a graph possibly with loops. We say that G is an *H-colored multigraph* whenever there exists a function $c\colon E(G) \to V(H)$. A walk (respectively, path, trail) $W = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k)$ in G is an *H-walk* (respectively, *H-path*, *H-trail*) if and only if $\big(c(e_0), c(e_1), \ldots, c(e_{k-2}), c(e_{k-1})\big)$ is a walk in H. Also, W is a *closed H-walk* (respectively, *closed H-trail*) if and only if W is an H-walk (respectively, H-trail) such that $v_0 = v_k$ and $c(e_{k-1})c(e_0) \in E(H)$. The H-coloring theory is closely related to the automata theory, since each vertex of H represents a state and each edge of H represents an allowed transition, which implies that an H-walk in a multigraph G is a predetermined sequence of possible transitions.

Regarding applications, consider package and messenger services, collecting, and home delivery. Associate an H-colored multigraph G as follows: the branch locations would be the vertices of G, the vertices of H would be the different kinds of products that would need transporting. For $\{a, b\} \subseteq V(G)$, we have $e \backsim ab$ is an edge of G if and only if it is necessary to transport the product $c(e)$ from a to b or from b to a. For $\{i, j\} \subseteq V(H)$, we have $ij \in A(H)$ if and only if the vehicle that leaves the warehouse with product i can then immediately carry the product j and the vehicle that leaves the warehouse with product j can then immediately carry the product i, considering hygiene, security, quality control, etc. For example, if a vehicle transported pets from a to b, it must be cleaned and disinfected before transporting food from b to c. An H-trail $(x_1, e_1, x_2, \ldots, e_{k-1}, x_k)$ in G means that the same vehicle can transport continuously (avoiding any intervention of the vehicle) the product $c(e_i)$ from x_i to x_{i+1} for each $i \in \{1, 2, \ldots, k-1\}$. Thus, an Eulerian H-trail means that a vehicle has transported each product to the desired branch location. In this way, it is possible to minimize time and cost. If an Eulerian H-trail is not optimal for a certain purpose, then we can obtain another route, by means of a sequence of order transformations.

The concepts of H-coloring and H-walks were motivated by the work of Linek and Sands in [7], where they studied paths with restrictions in the color transitions.

Necessary and sufficient conditions for the existence Eulerian H-trails were studied in [6] by considering the following auxiliary graph: Let H be a graph possibly with loops, G an H-colored multigraph without loops, and

u a vertex of G. Let G_u be the simple graph such that $V(G_u) = \{e \in E(G) : e \text{ is incident with } u\}$ and two different vertices a and b are joined by only one edge in G_u if and only if $c(a)c(b) \in E(H)$. The main result in that work was the following:

Theorem 1.2. *Suppose that G is connected and Eulerian and G_u is a complete k_u-partite graph for every $u \in V(G)$ and for some $k_u \in \mathbb{N}$. Then G has a closed Eulerian H-trail if and only if $|C_i^u| \leq \sum_{j \neq i} |C_j^u|$ for every $u \in V(G)$, where $\{C_1^u, \ldots, C_{k_u}^u\}$ is the partition of $V(G_u)$ into independent sets.*

We say that a simple graph G is a 3-*transitive graph* if for every path of length 3, say (u_0, u_1, u_2, u_3), there is an edge e in G such that $e = u_0u_3$.

In this work we show how to generate all Eulerian H-trails, from some initial one, in an H-colored multigraph G, where G_u is a 3-transitive graph for each vertex u of G. We also give an example that shows the condition on the graphs G_u in the main result is tight.

2 Main results

In what follows H is a graph possibly with loops, and G is an H-colored multigraph without loops, with H-coloring $c\colon E(G) \to V(H)$.

Let u be a vertex of G and let E_u denote the set $\{e \in E(G) : e \backsim ux$ for some x in $V(G)\}$.

Lemma 2.1. *Let $X = (x_0, \alpha_0, x_1, \alpha_1, \ldots, \alpha_{q-1}, x_q)$ and $Y = (y_0, \beta_0, y_1, \beta_1, \ldots, \beta_{q-1}, y_q)$ be Eulerian H-trails in G. Also, choose z in $V(G)$ such that G_z is a 3-transitive graph. If $\{a, b, d, e\}$ is a subset of E_z such that*

1. *$\{a, b\} = \{\alpha_i, \alpha_{i+1}\}$ for some i,*
2. *$\{b, d\} = \{\beta_j, \beta_{j+1}\}$ for some j,*
3. *$\{d, e\} = \{\alpha_k, \alpha_{k+1}\}$ for some k,*

then $ae \in E(G_z)$.

Proof. Since X is an Eulerian H-trail (respectively, Y is an Eulerian H-trail), then $\{ab, de\} \subseteq E(G_z)$ (respectively, $bd \in E(G_z)$). So (a, b, d, e) is a path of length 3 contained in G_z. Finally, the 3-transitivity of G_z implies that ae is an edge of G_z. □

Let $F = (x_0, e_0, x_1, \ldots, x_{m-1}, e_{m-1}, x_m)$ be an H-trail in G and suppose that $\{i, j, k, n\}$ is a subset of $\{0, \ldots, m\}$, with $i < j < k < n$, such that $x_k = x_i$ and $x_n = x_j$. Break up F as follows:

$$F_1 = (x_0, F, x_i), \qquad F_2 = (x_i, F, x_j), \qquad F_3 = (x_j, F, x_k),$$
$$F_4 = (x_k, F, x_n), \qquad F_5 = (x_n, F, x_m).$$

The transformation $F = F_1F_2F_3F_4F_5 \longrightarrow F^* = F_1F_4F_3F_2F_5$ is called an *order exchange* whenever F^* becomes an H-trail (see Figure 2.1).

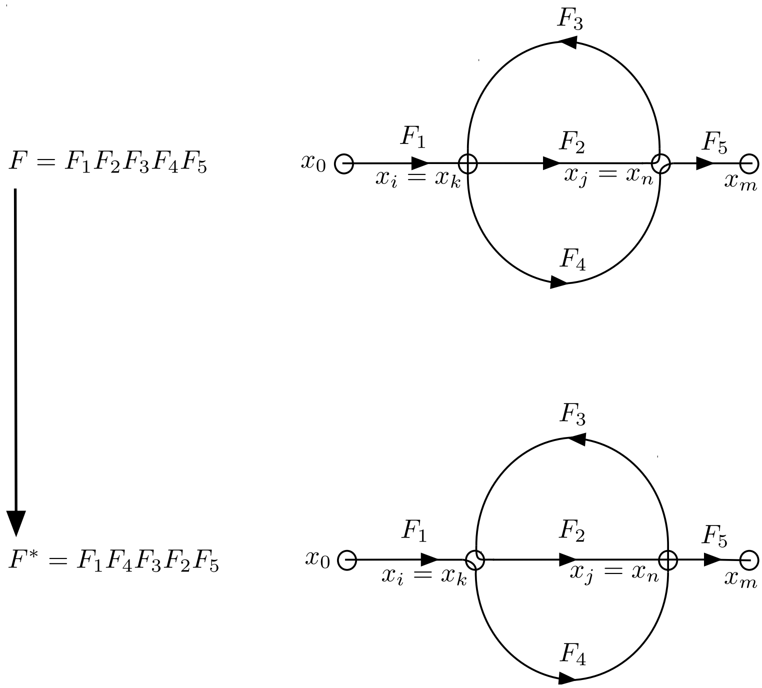

Figure 2.1: The tranformation order exchange.

Whenever $x_j = x_i$, break up F as follows:

$$F_1 = (x_0, F, x_i), \qquad F_2 = (x_i, F, x_j), \qquad F_3 = (x_j, F, x_m).$$

The transformation $F = F_1F_2F_3 \longrightarrow F^* = F_1F_2^{-1}F_3$ is called an *order reflection* whenever F^* is an H-trail, where either F_1 or F_3 can be trivial (see Figure 2.2).

From now on, we use the term order transformation to refer to the order exchange transformations or the order reflection transformations.

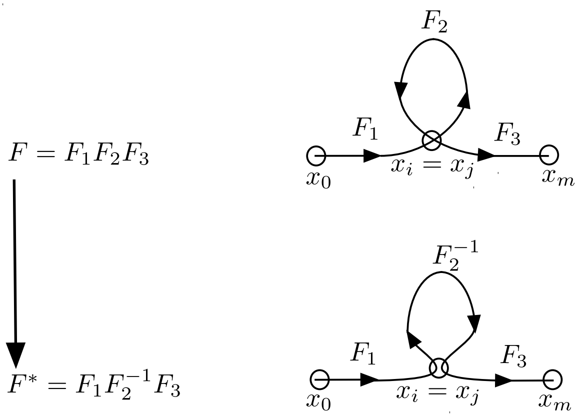

Figure 2.2: The tranformation order reflection.

Let $X = (x_0, e_0, x_1, \ldots, x_{k-1}, e_{k-1}, x_k)$ be an H-trail in G. We denote by Λ_X the set of all the H-trails obtained from X by the application of order transformations.

Remark 2.2. Let X be an H-trail in G.

(i) If X is closed (respectively, open), then every element of Λ_X is a closed H-trail (respectively, open).

(ii) If $X_i \in \Lambda_X$, then $E(X_i) = E(X)$. In particular if X is an Eulerian H-trail, then X_i is an Eulerian H-trail.

Let $X = (x_0, e_0, x_1, \ldots, x_{k-1}, e_{k-1}, x_k)$ be an H-trail. The subtrail (x_0, X, x_j) is denoted by $P_{X,j}$. If X and Y are H-trails of G that begin at x_0, we define the index of X and Y as the maximum element of the set $\{j : P_{X,j} = P_{Y,j}\}$, denoted by $\text{ind}(X, Y)$.

Let c be an H-coloring of G such that for each vertex x of G, we have that G_x is a 3-transitive graph. Let X and Y be two Eulerian H-trails of G beginning in x_0. With the next algorithm we obtain a sequence of Eulerian H-trails, $X_1, X_2, \ldots, X_K$ such that $X_1 = X$ and $X_K = Y$ and for each j in $\{2, \ldots, K\}$ the H-trail X_j is obtained from the H-trail X_{j-1} by means of order transformations.

Algorithm 2.3. *Given an H-colored multigraph G and an Eulerian H-trail Y, this algorithm generates all Eulerian H-trails of G. Suppose that $Y = (y_0, \beta_0, y_1, \ldots, y_{q-1}, \beta_{q-1}, y_q)$, where q is the size of G.*

Step 1. *Define* $X^* = X$ *and* $K = 0$. *Go to Step 2.*

Step 2. *Let* $K = K + 1$. *Define* $X_K = X^*$ *and* $\ell = \text{ind}(X^*, Y)$. *Suppose that* $X^* = (x_0, \alpha_0, x_1, \ldots, x_{q-1}, \alpha_{q-1}, x_q)$.

If $\ell < q$, *go to Step 3, otherwise go to Step 9.*

Step 3. *Find* p *in* $\{\ell + 1, \ldots, q\}$ *such that* $\alpha_p = \beta_\ell$.

If $x_p = y_{\ell+1}$ and $x_{p+1} = y_\ell$, *go to Step 4, otherwise go to Step 5.*

Step 4. *Consider the following* H*-subtrails of* X^**:*

$$T_1 = (x_0, X^*, x_\ell),$$
$$T_2 = (x_\ell, \alpha_\ell, x_{\ell+1}, \ldots, x_p = y_{\ell+1}, \beta_\ell = \alpha_p, x_{p+1} = y_\ell), \text{ and}$$
$$T_3 = (y_\ell = x_{p+1}, X^*, x_q).$$

Define $X^* = T_1 T_2^{-1} T_3$ *and go to Step 2.*

Step 5. *Consider the following* H*-subtrails of* X^**:*

$$P_1 = (x_0, X^*, x_\ell),$$
$$P_2 = (x_\ell, X^*, x_p = y_\ell), \text{ and}$$
$$P_3 = (y_\ell = x_p, \alpha_p = \beta_\ell, y_{\ell+1} = x_{p+1}, \ldots, x_q).$$

Notice that $X^* = P_1 P_2 P_3$.

Let $h \geq \ell + 1$ *be the minimum number satisfying the following conditions:* $y_h \in V(P_2)$ *and* $\beta_h \in E(P_2)$. *We have that* $y_h \in V(P_2) \cap V(P_3)$. *Let* i, *with* $\ell < i \leq p$, *be such that* $y_h = x_i$ *and* $\beta_h \in \{\alpha_{i-1}, \alpha_i\}$. *Let* j, *with* $p < j \leq q$, *be such that* $y_h = x_j$ *and* $\beta_{h-1} \in \{\alpha_{j-1}, \alpha_j\}$.

Now, consider the following H*-trails:*

$$T_1 = (x_0, X^*, x_\ell),$$
$$T_2 = (x_\ell, X^*, x_i),$$
$$T_3 = (x_i, X^*, x_p),$$
$$T_4 = (x_p = y_\ell, \alpha_p = \beta_\ell, x_{p+1} = y_{\ell+1}, \alpha_{p+1}, \ldots, \alpha_{j-1}, x_j), \text{ and}$$
$$T_5 = (x_j, X^*, x_q).$$

Note that $X^* = T_1T_2T_3T_4T_5$*, and* T_3 *or* T_5 *(or both) could be trivial.*

If either $\beta_h = \alpha_{i-1}$ *and* $\beta_{h-1} = \alpha_{j-1}$ *or* $\beta_h = \alpha_i$ *and* $\beta_{h-1} = \alpha_j$*, then go to Step 6.*

If either $\beta_h = \alpha_{i-1}$ *and* $\beta_{h-1} = \alpha_j$ *or* $\beta_h = \alpha_i$ *and* $\beta_{h-1} = \alpha_{j-1}$*, then go to Step 8.*

Step 6. *Let* $K = K + 1$*. Define* $X_K = T_1T_2(T_3T_4)^{-1}T_5$ *and go to Step 7.*

Step 7. *Define* $X^* = T_1(T_2T_4^{-1})^{-1}T_3^{-1}T_5$ *and go to Step 2.*

Step 8. *Define* $X^* = T_1T_4T_3T_2T_5$*, and go to Step 2.*

Step 9. *Output:* $X_1, X_2, \ldots, X_K$.

Lemma 2.4. *By applying Algorithm 2.3 we obtain the following assertions:*

(a) *The trail* X^* *obtained in Step 4 is an Eulerian* H*-trail, an order transformation of the* H*-trail* $T_1T_2T_3$*, satisfying that* $\operatorname{ind}(X^*, Y) > \operatorname{ind}(T_1T_2T_3, Y)$*.*

(b) *The indices* h*,* i*, and* j *defined in Step 5 do exist.*

(c) *The trail* X_K *obtained in Step 6 is an Eulerian* H*-trail, an order transformation of the* H*-trail* $T_1T_2T_3T_4T_5$*, satisfying* $\operatorname{ind}(X_K, Y) = \operatorname{ind}(T_1T_2T_3T_4T_5, Y)$*.*

(d) *The trail* X^* *obtained in Step 7 is an Eulerian* H*-trail, an order transformation of the* H*-trail* X_K*, satisfying* $\operatorname{ind}(X^*, Y) > \operatorname{ind}(X_K, Y)$*.*

(e) *The trail* X^* *obtained in Step 8 is an Eulerian* H*-trail, an order transformation of the* H*-trail* X_K*, satisfying* $\operatorname{ind}(X^*, Y) > \operatorname{ind}(X_K, Y)$*.*

(f) *The time complexity of the Algorithm 2.3 is* $O(q^2)$*, and the storage data is of order* $O(q)$*.*

Proof. Recall $Y = (y_0, \beta_0, y_1, \beta_1, y_2, \ldots, \beta_{q-1}, y_q)$ is an Eulerian H-trail.

(a) The input data at this step is the Eulerian H-trail $W = T_1T_2T_3$, where $T_1 = (x_0, \alpha_0, x_1, \ldots, \alpha_{\ell-1}, x_\ell)$, $T_2 = (x_\ell, \alpha_\ell, x_{\ell+1}, \ldots, x_p = y_{\ell+1}, \beta_\ell = \alpha_p, x_{p+1} = y_\ell)$, $T_3 = (y_\ell = x_{p+1}, \alpha_{p+1}, x_{p+2}, \ldots, x_q)$, and $\operatorname{ind}(T_1T_2T_3, Y) = \ell$. Then $\alpha_i = \beta_i$ for $0 \leq i \leq \ell - 1$ and $x_j = y_j$ for $0 \leq j \leq \ell$.

First, we check to see that $T_1T_2T_3 \longrightarrow X^* = T_1T_2^{-1}T_3$ is an order reflection, and then X^* is an H-trail.

(i) Since Y is an H-trail, then $c(\beta_{\ell-1})c(\beta_\ell) \in A(H)$, and given that $\beta_{\ell-1} = \alpha_{\ell-1}$, we conclude that

$$T_1T_2^{-1} = (x_0, W, x_\ell) \cup (x_\ell = x_{p+1} = y_\ell, \beta_\ell = \alpha_p, y_{\ell+1}) \cup (y_{\ell+1} = x_p, W, x_\ell)$$

is an H-trail.

(ii) Since $x_\ell = x_{p+1}$, we have that $\alpha_\ell, \alpha_{\ell-1}, \alpha_p, \alpha_{p+1}$ are incident with x_ℓ. Given that $\alpha_{\ell-1} = \beta_{\ell-1}$, $\alpha_p = \beta_\ell$, and $T_1T_2T_3$ and Y are H-trails, then Lemma 2.1 implies that $\alpha_\ell\alpha_{p+1} \in E(G_{x_\ell})$, so $c(\alpha_\ell)c(\alpha_{p+1}) \in A(H)$. Hence

$$T_2^{-1}T_3 = (x_{p+1} = y_\ell, \beta_\ell = \alpha_p, y_{\ell+1}) \cup (y_{\ell+1} = x_p, W, x_\ell) \cup (x_\ell = y_\ell = x_{p+1}, W, x_q)$$

is an H-trail.

Therefore, $T_1T_2T_3 \longrightarrow X^*$ is an order reflection, and Remark 2.2 implies that X^* is an Eulerian H-trail. So, since X^* has in common with Y at least the H-trail $(x_0, W, x_\ell) \cup (x_\ell = y_\ell, \beta_\ell, y_{\ell+1})$, then $\text{ind}(X^*, Y) \geq \ell + 1$.

(b) Note that $P_1 = (x_0, \alpha_0, x_1, \ldots, x_\ell)$, $P_2 = (x_\ell, \alpha_\ell, x_{\ell+1}, \ldots, y_\ell = x_p)$, $P_3 = (y_\ell = x_p, \alpha_p = \beta_\ell, y_{\ell+1} = x_{p+1}, \ldots, x_q)$, and h, with $h \geq \ell+1$, is the minimum number satisfying the following conditions: $y_h \in V(P_2)$ and $\beta_h \in E(P_2)$. The existence of the index h follows from the fact that $\alpha_\ell \in E(Y) \cap E(P_2)$, with $\alpha_\ell \sim y_t y_{t+1}$, for some $t \geq \ell + 1$.

Claim 1. $\beta_{h-1} \notin E(P_2)$.

If $h - 1 \geq \ell + 1$ and $\beta_{h-1} \in E(P_2)$, then $y_{h-1} \in V(P_2)$, contradicting the choice of h. If $h - 1 = \ell$, then $\beta_{h-1} = \beta_\ell$, since $\beta_\ell \notin E(P_2)$, and consequently $\beta_{h-1} \notin E(P_2)$. Thus, in any case $\beta_{h-1} \notin E(P_2)$.

Claim 2. $\beta_{h-1} \notin E(P_1)$.

This follows directly from the fact, that $h > \ell$.

Since $X^* = P_1P_2P_3$ is an Eulerian H-trail, this follows from Claims 1 and 2 that $\beta_{h-1} \in E(P_3)$. So, $y_h \in V(P_2) \cap V(P_3)$ implying that

$y_h = x_i$ for some i such that $\ell < i \le p$, and $y_h = x_j$ for some j satisfying $p < j \le q$.

Notice that $\beta_{h-1} \in \{\alpha_{j-1}, \alpha_j\}$.

(c) The input data at this step is H-trail $W = T_1T_2T_3T_4T_5$, where $T_1 = (x_0, \alpha_0, x_1, \alpha_1, \ldots, \alpha_{\ell-1}, x_\ell)$, $T_2 = (x_\ell, \alpha_\ell, x_{\ell+1}, \ldots, \alpha_{i-1}, x_i = y_h)$, $T_3 = (x_i, \alpha_i, x_{i+1}, \ldots, \alpha_{p-1}, x_p)$, $T_4 = (x_p = y_\ell, \alpha_p = \beta_\ell, x_{p+1} = y_{\ell+1}, \alpha_{p+1}, \ldots, \alpha_{j-1}, x_j = y_h)$, $T_5 = (x_j, \alpha_j, x_{j+1}, \alpha_{j+1}, \ldots, x_q)$, with either $\beta_h = \alpha_{i-1}$ and $\beta_{h-1} = \alpha_{j-1}$ or $\beta_h = \alpha_i$ and $\beta_{h-1} = \alpha_j$; satisfying $\operatorname{ind}(W, Y) = \ell$.

First, we check to see that $\{c(\alpha_{i-1})c(\alpha_{j-1}), c(\alpha_i)c(\alpha_j)\} \subseteq E(H)$.

Notice that $\alpha_{i-1}, \alpha_i, \alpha_{j-1}, \alpha_j$ are edges incident with y_h and recall that T_2T_3 and T_4T_5 are H-trails.

Since Y is an H-trail when $\beta_h = \alpha_{i-1}$ and $\beta_{h-1} = \alpha_{j-1}$, we have that $c(\beta_{h-1})c(\beta_h) \in E(H)$, that is, $c(\alpha_{i-1})c(\alpha_{j-1}) \in E(H)$. Therefore, Lemma 2.1 implies $\alpha_i\alpha_j \in E(G_{y_h})$, and hence $c(\alpha_i)c(\alpha_j) \in E(H)$. Analogously, when $\beta_h = \alpha_i$ and $\beta_{h-1} = \alpha_j$, we obtain that $\{c(\alpha_{i-1})c(\alpha_{j-1}), c(\alpha_i)c(\alpha_j)\} \subseteq E(H)$.

Now, we check to see that $T_1T_2T_3T_4T_5 \longrightarrow X_K = T_1T_2(T_3T_4)^{-1}T_5$ is an order reflection.

Notice that $X_K = T_1T_2T_4^{-1}T_3^{-1}T_5$.

(i) $T_2T_4^{-1} = (x_\ell, W, x_i) \cup (x_i = y_h = x_j, W, x_{p+1}) \cup (x_{p+1}, \alpha_p = \beta_\ell, x_p = x_\ell)$ is an H-trail.

This follows from the fact that $c(\alpha_{i-1})c(\alpha_{j-1}) \in E(H)$.

(ii) $T_4^{-1}T_3^{-1}$ is an H-trail.

We have that $T_1T_2T_3T_4T_5$ is an H-trail. Hence also T_3T_4 and $T_4^{-1}T_3^{-1}$ are H-trails.

(iii) $T_3^{-1}T_5 = (x_p, W, x_i) \cup (x_i = x_j, W, x_q)$ is an H-trail.

This follows from the fact that $c(\alpha_i)c(\alpha_j) \in E(H)$.

Thus, $T_1T_2T_3T_4T_5 \longrightarrow X_K$ is an order reflection, and Remark 2.2 implies that X_K is an Eulerian H-trail. Now, X_K has in common with Y the H-trail $T_1 = (x_0, \alpha_0, x_1, \ldots, \alpha_{\ell-1}, x_\ell)$, as $\alpha_\ell \neq \beta_\ell$, and so $\operatorname{ind}(X_K, Y) = \ell$.

(d) The input data at this step is Eulerian H-trail $X_K = T_1T_2T_4^{-1}T_3^{-1}T_5$, where $T_1, \dots, T_5$ were described above, with $\text{ind}(X_K, Y) = \ell$.

Recall that $T_1T_2T_3T_4T_5$ is an H-trail and either $\beta_h = \alpha_{i-1}$ and $\beta_{h-1} = \alpha_{j-1}$ or $\beta_h = \alpha_i$ and $\beta_{h-1} = \alpha_j$.

First, we check to see that $X_K \longrightarrow X^* = T_1(T_2T_4^{-1})^{-1}T_3^{-1}T_5$ is an order reflection. Notice that $X^* = T_1T_4T_2^{-1}T_3^{-1}T_5$.

(i) $T_1T_4 = (x_0, X^*, x_\ell) \cup (x_\ell = y_\ell = x_p, \alpha_p = \beta_\ell, x_{p+1}) \cup (x_{p+1}, X^*, x_j)$ is an H-trail.

This follows from the fact that $\alpha_{\ell-1} = \beta_{\ell-1}$, $\alpha_p = \beta_\ell$, and Y is an H-trail.

(ii) $T_4T_2^{-1}$ is an H-trail.

Since X_K is an H-trail, we get that $T_2T_4^{-1}$ is an H-trail, which implies that $T_4T_2^{-1}$ is an H-trail.

(iii) $T_2^{-1}T_3^{-1} = (y_h = x_i, \alpha_{i-1}, \dots, x_{\ell+1}, \alpha_\ell, x_\ell = x_p, \alpha_{p-1}, \dots, x_{i+1}, \alpha_i, x_i = y_h)$ is an H-trail.

Recall that $\alpha_{\ell-1}, \alpha_\ell, \alpha_{p-1}, \alpha_p$ are edges incident with y_ℓ. Since $\beta_{\ell-1} = \alpha_{\ell-1}$, $\beta_\ell = \alpha_p$, and X_K and Y are H-trails, then Lemma 2.1 implies that $\alpha_\ell\alpha_{p-1} \in E(G_{y_\ell})$. Therefore, we have that $c(\alpha_\ell)c(\alpha_{p-1}) \in E(H)$ and $T_2^{-1}T_3^{-1}$ is an H-trail.

(iv) $T_3^{-1}T_5$ is an H-trail.

Notice that $T_3^{-1}T_5$ is a subtrail of X_K, and hence it is an H-trail.

We conclude that $X_K \longrightarrow X^* = T_1(T_2T_4^{-1})^{-1}T_3^{-1}T_5$ is an order reflection.

Remark 2.2 implies that X^* is an Eulerian H-trail. Now, X^* has in common with Y at least the H-trail $(x_0, \alpha_0, x_1, \dots, \alpha_{\ell-1}, x_\ell = y_\ell = x_p, \alpha_p = \beta_\ell, x_{p+1})$, with $\text{ind}(X^*, Y) \geq \ell + 1$ and $\text{ind}(X^*, Y) > \text{ind}(X_K, Y)$.

(e) The input data at this step are the H-trails $X_K = T_1T_2T_3T_4T_5$, where $T_1 = (x_0, \alpha_0, x_1, \alpha_1, \dots, \alpha_{\ell-1}, x_\ell)$, $T_2 = (x_\ell, \alpha_\ell, x_{\ell+1}, \dots, \alpha_{i-1}, x_i = y_h)$, $T_3 = (x_i, \alpha_i, x_{i+1}, \dots, \alpha_{p-1}, x_p)$, $T_4 = (x_p = y_\ell, \alpha_p = \beta_\ell, x_{p+1} = y_{\ell+1}, \alpha_{p+1}, \dots, \alpha_{j-1}, x_j = y_h)$, $T_5 = (x_j, \alpha_j, x_{j+1}, \alpha_{j+1}, \dots, x_q)$, and $\text{ind}(X_K, Y) = \ell$.

Recall that either $\beta_h = \alpha_i$ and $\beta_{h-1} = \alpha_{j-1}$ or $\beta_h = \alpha_{i-1}$ and $\beta_{h-1} = \alpha_j$. We consider two cases.

Case 1. $\beta_h = \alpha_i$ and $\beta_{h-1} = \alpha_{j-1}$.

In this case we prove that $X_K \longrightarrow X^* = T_1T_4T_3T_2T_5$ is an order reflection.

(i) $T_1T_4 = (x_0, X^*, x_\ell) \cup (x_\ell = y_\ell = x_p, \alpha_p = \beta_\ell, x_{p+1}, \alpha_{p+1}, \ldots, \alpha_{j-1}, y_h = x_j)$ is an H-trail.

Since $\beta_{\ell-1} = \alpha_{\ell-1}$, $\beta_\ell = \alpha_p$, and Y is an H-trail, then T_1T_4 is an H-trail.

(ii) $T_4T_3 = (x_\ell = y_\ell = x_p,\ \alpha_p = \beta_\ell,\ x_{p+1},\ \alpha_{p+1},\ \ldots,\ \alpha_{j-1},\ x_j = y_h = x_i, \alpha_i, \ldots, x_{p-1}, \alpha_{p-1}, x_p = x_\ell)$ is an H-trail.

Since $\beta_{h-1} = \alpha_{j-1}$, $\beta_h = \alpha_i$, and Y is an H-trail, then T_4T_3 is an H-trail.

(iii) $T_3T_2 = (x_j = y_h = x_i, \alpha_i, \ldots, x_{p-1}, \alpha_{p-1}, x_p = x_\ell, \alpha_\ell, x_{\ell+1}, \ldots, \alpha_{i-1}, x_i = y_h)$ is an H-trail.

Recall that $\alpha_{\ell-1}, \alpha_\ell, \alpha_{p-1}, \alpha_p$ are edges incident with y_ℓ. Since $\beta_{\ell-1} = \alpha_{\ell-1}$, $\beta_\ell = \alpha_p$, and X_K and Y are H-trails, it follows from Lemma 2.1 that $\alpha_{p-1}\alpha_\ell \in E(G_{y_\ell})$, which implies that $c(\alpha_{p-1})c(\alpha_\ell) \in E(H)$.

(iv) $T_2T_5 = (x_\ell, \alpha_\ell, x_{\ell+1}, \ldots, \alpha_{i-1}, x_i = y_h = x_j, \alpha_j, x_{j+1}, \alpha_{j+1}, \ldots, x_q)$ is an H-trail.

Recall that $\alpha_i, \alpha_{i-1}, \alpha_j, \alpha_{j-1}$ are edges incident with y_h. Since $\beta_{h-1} = \alpha_{j-1}$, $\beta_h = \alpha_i$, and X_K and Y are H-trails, then Lemma 2.1 implies that $\alpha_{i-1}\alpha_j \in E(G_{y_h})$. Therefore, we have that $c(\alpha_{i-1})c(\alpha_j) \in E(H)$.

Therefore, $X_K \longrightarrow X^* = T_1T_4T_3T_2T_5$ is an order reflection. Remark 2.2 implies that X^* is an Eulerian H-trail, having in common with Y at least the H-trail $(x_0, \alpha_0, x_1, \ldots, \alpha_{\ell-1}, x_\ell = y_\ell = x_p, \alpha_p = \beta_\ell, x_{p+1})$, with $\text{ind}(X^*, Y) \geq \ell + 1$ and $\text{ind}(X^*, Y) > \text{ind}(X_K, Y)$.

Case 2. $\beta_h = \alpha_{i-1}$ and $\beta_{h-1} = \alpha_j$.

Proceeding in a completely similar way as in Case 1, we get that $X^* = T_1T_4T_3T_2T_5$ is an H-trail.

(f) Consider the execution of Algorithm 2.3:

(I) Step 1 is executed at most once.

(II) Step 2 is executed at most once for each edge in the graph; hence, it runs at most q times. After each execution of Step 2, one of the following sequences of steps is executed:
- Step 9;
- Steps 3 and 4;
- Steps 3, 5, 6, and 7;
- Step 3, 5, and 8.

(III) Step 9 is executed at most once.

In Step 2, in order to find the index, $\text{ind}(X^*, Y)$, each edge α_i of X^* is compared with the corresponding edge β_i of Y. Therefore, the time complexity of Step 2 is $O(q)$.

In Step 3, the edge β_ℓ is compared with each edge of X^*, becoming at most q comparisons. Therefore, the time complexity of Step 3 is $O(q)$.

In Step 5, finding the subindex h involves at most q comparisons, since it is a particular case of the problem of looking for common elements between two sets, whose time complexity is $O(q)$.

The execution of Steps 4, 6, 7, 8, and 9 is of constant complexity, i.e., $O(1)$.

Thus, considering the nested structure and the fact that Step 2 is called q times, the overall time complexity for Algorithm 2.3 is $O(q^2)$, and the storage data is of order $O(q)$. □

Notice that Algorithm 2.3 works as follows: Given an H-colored graph G that contains at least two Eulerian H-trails, namely X and Y both of them starting in a fixed vertex x_0, the algorithm constructs a succession of Eulerian H-trails $X = X_1, X_2, \ldots, X_K = Y$ where, for each $i \in \{1, \ldots, K-1\}$, the trail X_{i+1} is obtained of X_i by means of order transformations and $\text{ind}(X_{i+1}, Y) > \text{ind}(X_i, Y)$.

Theorem 2.5. *Suppose that c is an H-coloring of G such that for each vertex x of G, we have that G_x is a 3-transitive graph. Let X and Y be two Eulerian H-trails in G beginning in the same vertex. Then the sequence $X_1, X_2, \ldots, X_K$ obtained by applying Algorithm 2.3 satisfies the following:*

1. *For each $j \in \{2, \ldots, K\}$, we have that X_j is an Eulerian H-trail obtained from X_{j-1} through order transformations.*
2. *$X_K = Y$ and thus $Y \in \Lambda_X$.*

Remark 2.6. In Theorem 2.5 we cannot remove the condition “for each vertex x of G, we have that G_x is a 3-transitive graph,” as it is shown in the following example.

Consider the multigraph G in Figure 2.3. Note that G_{x_1} is not a 3-transitive graph.

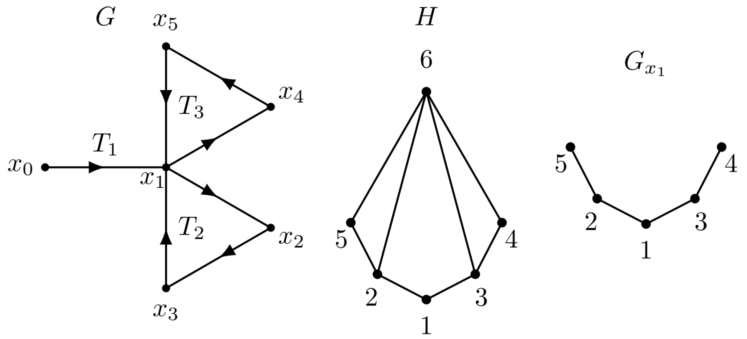

Figure 2.3: H-colored multigraph G.

Let $X = (x_0, x_1, x_2, x_3, x_1, x_4, x_5, x_1)$ and $Y = (x_0, x_1, x_3, x_2, x_1, x_5, x_4, x_1)$ be two Eulerian H-trails in G, where $T_1 = (x_0, x_1)$, $T_2 = (x_1, x_2, x_3, x_1)$, and $T_3 = (x_1, x_4, x_5, x_1)$. Notice that $Y = T_1T_2^{-1}T_3^{-1}$. We intend that $Y \notin \Lambda_X$. From the definition of order reflection, we have that the following transformations are the only possible order reflections:

1) $X \longrightarrow T_1T_2^{-1}T_3$ is not an order reflection, since $T_2^{-1}T_3$ is not an H-trail.

2) $X \longrightarrow T_1(T_2T_3)^{-1} = T_1T_3^{-1}T_2^{-1}$ is not an order reflection, since $T_1T_3^{-1}$ is not an H-trail.

3) $X \longrightarrow T_1T_2T_3^{-1}$ is not an order reflection, since $T_2T_3^{-1}$ is not an H-trail.

On the other hand, from the definition of order exchange we have that it is not possible to implement an order exchange.

Acknowledgments

Funding: Research was suported by UNAM-DGAPA-PAPIIT IN104717, CONACyT CB-2013\219840, and CONACyT scholarship for postgraduate studies 289847. The authors wish to thank an anonymous referee for many comments that improved the rewriting of this paper.

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