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Book thickness of path and cycle powers

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Abstract. Book embeddings with minimum page number are given for powers of path and cycle graphs. We find minimum book embeddings with *star forest pages* using at most two additional pages.

1 Introduction

In the k-th power of a graph, distinct vertices at distance at most k are adjacent. Powers of paths and cycles have been used both theoretically [5, 8, 14] and practically [17, 21]. Here we find improved bounds on their book thickness. A variant of book thickness, where pages are forests of vertex-disjoint stars, is also applied to these powers. For this *star acyclic book thickness*, tight bounds are given under divisibility conditions.

The book thickness of a graph G is the least number of subgraphs in an edge-partition of G such that all the subgraphs (the *pages*) are crossing-free minimized over all drawings that place the vertices on the boundary of a convex polygon [3, 15, 20]. The notion has proven useful for theory and applications, e.g., [1, 2, 4, 6, 7, 9, 13, 19, 23]. Adding constraints to the pages was already considered in [3]. For applications of book thickness (e.g., to vehicle traffic signal control [16]), star forest pages are natural.

To achieve some constraint on page type, the penalty is a multiplicative factor; e.g., with twice the page number, one can achieve acyclic pages [16]. However, for path and cycle powers in some cases, star-forest pages can be achieved with only a small constant *additive* increment in page number over ordinary book thickness.

Section 2 has definitions and gives the book thickness of path and cycle powers; Section 3 obtains analogous results for star acyclic book thickness; and Section 4 is a discussion.

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2 Book thickness of P_n^k and C_n^k

Let P_n and C_n be the path and cycle, respectively, with n vertices. Notation follows that found in [12]. The *k-th power* G^k of a graph G is the supergraph spanned by G with u and w adjacent in G^k if and only if $1 \leq d_G(u,w) \leq k$, where $d_G(u,w)$ is G-distance from u to w [22]. Let $k \mid n$ mean n/k is an integer, $k \nmid n$ not.

A *book embedding* [15] of a graph $G = (V,E)$ consists of a *cyclic order* on the vertices, which are placed along a circle in that order, and an *edge-partition* of E into $r \geq 1$ pairwise-disjoint, non-empty subsets $E_1, \ldots, E_r$ such that, for each i with $1 \leq i \leq r$, the induced drawing of the subgraph $G(E_i)$, determined by edge-subset E_i and the vertex order, is crossing-free. These outerplane *drawings of the subgraphs* are called the *pages* of the book embedding. The term "page" is also used for the corresponding subgraphs. A graph is *outerplanar* if it has an outerplane drawing.

A *star* is a $K_{1,t}$ for $t \geq 0$, which has t edges incident to a central vertex. A *star forest* is a vertex-disjoint union of stars. A *star forest book embedding* is a book embedding where *each page is a star forest*. An *acyclic book embedding* [16] merely requires that the pages be forests. *Book thickness* (or *page* or *stack number*) [3,6,13,15] of G, denoted $bt(G)$ or $bt(G,\omega)$ when fixed cyclic order ω is required, is the least number of pages in any book embedding of G. *Acyclic* (or *star acyclic*) *book thickness*, denoted *abt* (or *sabt*), includes the constraint on pages.

For ν the natural vertex-order for P_n^k, with $2 \leq k \leq n-1$, by arguing as in Theorem 3.6 below, $bt(P_n^k, \nu) \leq k$, but order ω reduces the page-bound by 1.

Theorem 2.1. *Let $k \geq 2$. Then for all n, we have $bt(P_n^k, \omega) \leq k-1$.*

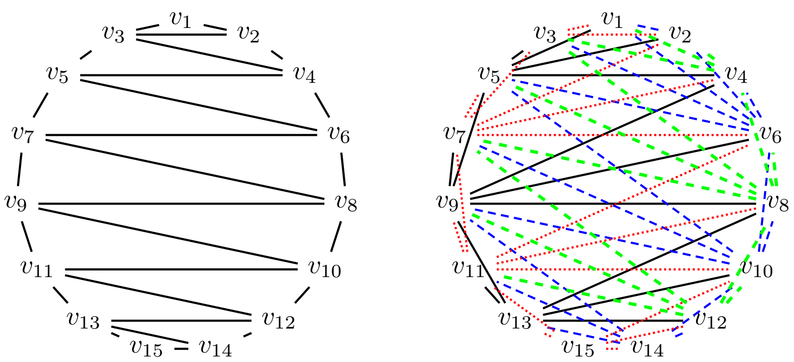
Proof. For n odd, the cyclic vertex ordering ω is *odd-down, even-up*:

$$\omega\colon\ n,\ n-2,\ n-4,\ \ldots,\ 5,\ 3,\ 1,\ 2,\ 4,\ \ldots,\ n-5,\ n-3,\ n-1.$$

(For n even, ω denotes the similar *even-down, odd-up* order.) If $k=2$, order ω gives an outerplane embedding of P_n^2 so $bt(P_n^2) = 1$; see Figure 2.1(a).

For $k \geq 3$ and $1 \leq j \leq k-1$, define the j-th page as the spanning subgraph constituted by the union of star graphs centered at all vertices v_i with $i \equiv j \pmod{k-1}$ where the edges of the star centered at v_i join v_i to v_{i-1}, $v_{i-2}, \ldots, v_{i-k}$. (If $i \leq k$, the stars have fewer than k edges). Note that

v_{i-k+1} is the center of a new star that has no crossings with previous stars and that there are $k-1$ pages. See Figure 2.1(b). □



(a) 1-page embedding of P_{15}^2. (b) 4-page embedding of P_{15}^5.

Figure 2.1: Book embeddings of powers of P_{15}.

For n large enough with respect to k, this upper bound is best possible. Before stating this more formally, we give the following result of Bernhart and Kainen [3] that is useful for our proofs.

Lemma 2.2 (Bernhart and Kainen [3])**.** *For $G = (V, E)$ and $|V| > 3$, we have $bt(G) \geq \left\lceil \frac{|E|-|V|}{|V|-3} \right\rceil$.*

Theorem 2.3. *Let $k \geq 2$ and $n \geq 4 + \binom{k-2}{2}$. Then $bt(P_n^k, \omega) = bt(P_n^k) = k-1$.*

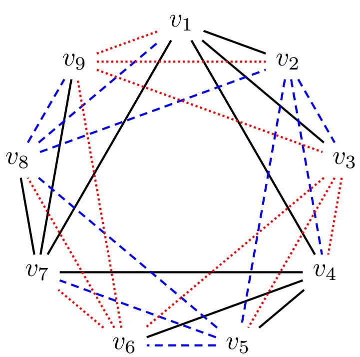
Proof. The number of edges of P_n^k is $kn - \frac{k^2+k}{2}$, where the subtracted term counts the edges missing at the outermost vertices of P_n. By Lemma 2.2 with $|V| = n$ and $|E| = kn - \frac{k^2+k}{2}$, we have $bt(P_n^k)$ is at least

$$\left\lceil \frac{n(k-1) - 3(k-1) + 3(k-1) - \frac{k^2+k}{2}}{n-3} \right\rceil = k - 1 + \left\lceil \frac{3(k-1) - \frac{k^2+k}{2}}{n-3} \right\rceil.$$

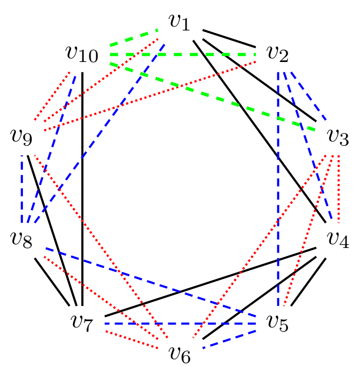
The last term on the right is 0 as the numerator $3(k-1) - \frac{k^2+k}{2} = -\binom{k-2}{2}$ is non-positive while, by hypothesis, the denominator strictly exceeds the absolute value of the numerator. □

Theorem 2.4. *Let $k \geq 2$ and $n \geq 3k$. Then $k \leq bt(C_n^k, \nu) \leq k + r$, where $r := n - k \left\lfloor \frac{n}{k} \right\rfloor$ is the remainder when n is divided by k.*

Proof. The upper bound of $k + r$ is in [16] and needs no hypothesis on n. By Lemma 2.2, we have $bt(C_n^k, \nu) \geq \left\lceil \frac{nk-n}{n-3} \right\rceil = k + \left\lceil \frac{-n+3k}{n-3} \right\rceil = k$, where the last equality holds as the numerator $-n + 3k$ of the ceiling function on the right is not positive and is smaller in absolute value than the denominator because $k > 1$. See Figure 2.2 for $r = 0$ and $r = 1$. □



(a) 3-page embedding of C_9^3.



(b) 4-page embedding of C_{10}^3.

Figure 2.2: Book embeddings of C_n^3, where we add a star to add a vertex. Note that if r is positive, then pages are acyclic.

Corollary 2.5. *If $k \mid n$ with $k \geq 2$ and $\frac{n}{k} \geq 3$, then $bt(C_n^k) = bt(C_n^k, \nu) = k$.*

3 Star arboricity book thickness

Recall that the arboricity $\Gamma(G)$ of a graph is the least number of forests of which it is the union [12, p. 90], and if G is outerplanar, then $\Gamma(G) \leq 2$. Further, every forest is the union of two star forests. Hence, each page of a book embedding can be replaced by 4 star acyclic pages. But this can be reduced to 3 using the next result, due to Hakimi, Mitchem and Schmeichel.

Theorem 3.1 (Hakimi et al. [11])**.** *Every outerplanar graph G is the union of 3 star forests.*

Corollary 3.2. *For every graph G, we have $bt(G) \leq sabt(G) \leq 3bt(G)$.*

Corollaries 2.5 and 3.2 and Theorem 2.4 imply the following.

Corollary 3.3. *If $k \mid n$ with $k \geq 2$ and $n \geq 3k$, then $k \leq sabt(C_n^k) \leq 3k$.*

This will be improved below. First, by [12, p. 90], we have a useful tool.

Lemma 3.4. *If $G = (V, E)$, then $sabt(G) \geq abt(G) \geq \Gamma(G) \geq \left\lceil \frac{|E|}{|V|-1} \right\rceil$.*

With this lemma, one can increase the lower bound in Corollary 3.3.

Theorem 3.5. *If $n \geq 3$ and $1 \leq k \leq \frac{n}{2}$, then $sabt(C_n^k) \geq k + 1$.*

Proof. By Lemma 3.4, $sabt(C_n^k) \geq \left\lceil \frac{nk}{n-1} \right\rceil = k + \left\lceil \frac{k}{n-1} \right\rceil = k + 1$, where the last equality holds since $0 < k < n - 1$. □

Note that $C_n^k = K_n$ for $k \geq \lfloor n/2 \rfloor$ and $sabt(K_n) \leq n - 1$.

Theorem 3.6. *If $k \geq 1$, $s \geq 2$, and $n = s(k + 1)$, then $sabt(C_n^k) = sabt(C_n^k, \nu) = k + 1$.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of C_n and equally spaced in natural order around the circle. The *forward k-star* at v_i is $\{v_i v_{i+1}, \ldots, v_i v_{i+k}\}$ (with addition performed modulo n). The union of the s forward k-stars at $v_1, v_{k+2}, v_{2k+3}, \ldots, v_{n-k}$ is the first of $k + 1$ *edge-disjoint star forest pages* for C_n^k, obtained by rotations of $2\pi/n$ radians. This gives $(k+1)sk = nk = |E(C_n^k)|$ edges. □

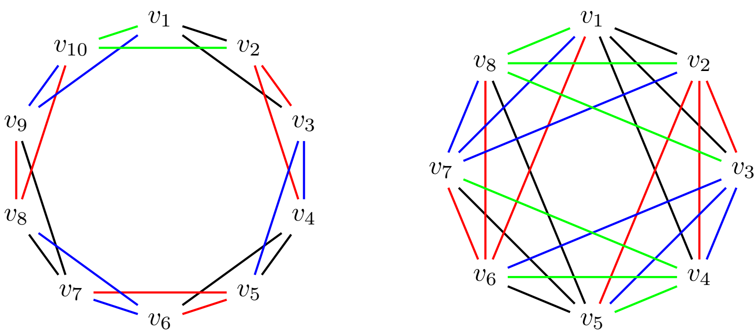
We can now decrease the upper bound in Corollary 3.3 from $3k$ to $2k + 1$ without the divisibility constraint.

Theorem 3.7. *Suppose $k \geq 2$ and $n \geq 2k$.*

(i) *Let $r' := n - (k + 1)\left\lfloor \frac{n}{k+1} \right\rfloor$. Then $sabt(C_n^k, \nu) \leq k + 1 + r'$.*

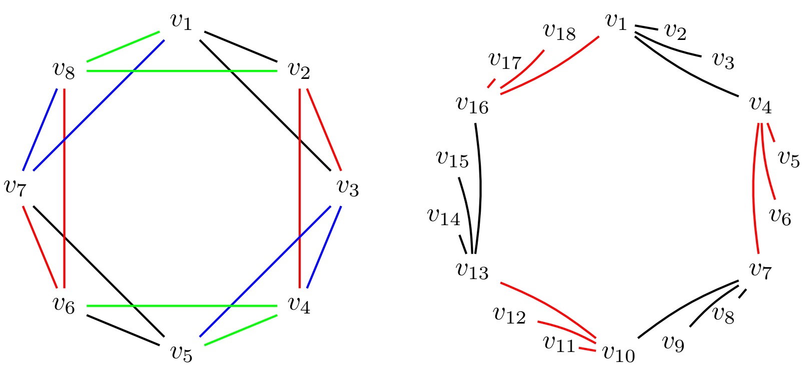
(ii) *Let $k \mid n$ and suppose $\frac{n}{k}$* is even. *Then $sabt(C_n^k, \nu) \leq 2k$.*

Proof. For (i) with $r' = 0$, this is Theorem 3.6. If $r' > 0$, take the forward k-stars at each of the extra r' vertices as single-component star forests. For (ii) the forward k-stars at $v_1, v_{2k+1}, v_{4k+1}, \ldots$ is a star forest, and the forward k-stars at $v_{k+1}, v_{3k+1}, v_{5k+1}, \ldots$ is another edge-disjoint star forest. There are k rotations by $2\pi/n$ radians, so $2k$ star forest pages. See Figures 3.1 and 3.2. □



(a) 4-page embedding of C_{10}^2. (b) 4-page embedding of C_8^3.

Figure 3.1: Star forest book embeddings for case (i) of Theorem 3.7.



(a) 4-page embedding of C_8^2. (b) Two pages of a 6-page embedding of C_{18}^3.

Figure 3.2: Star forest book embedding for case (ii) of Theorem 3.7.

The bound in case (i) of Theorem 3.7 can be improved in some cases. For instance, $sabt(C_{11}^3) \leq 6$. For $n \leq m$, we have $P_n^k \subseteq P_m^k \subset C_m^k$, and this gives a bound for paths.

Theorem 3.8. *If $k \geq 2$ and $n \geq 2 + \binom{k}{2}$, then $k \leq sabt(P_n^k) \leq k+1$.*

Proof. Choose integer $s \geq 2$ large enough that $m := s(k+1)$ satisfies $m \geq n$. Then $sabt(P_n^k) \leq sabt(P_m^k) \leq sabt(C_m^k) \leq k+1$, using Theorem 3.6 for the last inequality.

For the lower bound, by Lemma 3.4 and the proof of Theorem 2.3, we have

$$sabt(P_n^k) \geq \left\lceil \frac{kn - \frac{k(k+1)}{2}}{n-1} \right\rceil = k + \left\lceil \frac{-\binom{k}{2}}{n-1} \right\rceil = k. \qquad \square$$

4 Discussion

By Corollary 2.5 and Theorem 3.6, for all $k \geq 2$,

$$\text{if } \; k(k+1) \text{ divides } n, \; \text{ then } \; sabt(C_n^k) = 1 + bt(C_n^k). \tag{1}$$

By Theorems 2.3 and 3.8, for all $k \geq 3$,

$$\text{if } \; n \geq 2 + \binom{k}{2}, \; \text{ then } \; 1 + bt(P_n^k) \leq sabt(P_n^k) \leq 2 + bt(P_n^k). \tag{2}$$

The condition $k \geq 3$ implies $2 + \binom{k}{2} \geq 4 + \binom{k-2}{2}$ so both theorems apply.

Are there additional graph families, aside from path and cycle powers, such that a small increment in the number of pages in a minimum book embedding can permit a book embedding where each page is a star forest?

Advantages of star forest pages include that component diameter is 2 and, if the edges are oriented so that stars are out-stars, then latency is 1. We believe this could be useful for "graphlet" neural networks (eg., [10, 18]).

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