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On S-magic labeling of graph products

Maurice Almeida

Abstract. Let $G = (V, E)$ be a graph and let S be a set of positive integers with $|S| = |V|$. The graph G is said to be S-magic if there exists a bijection $l\colon V \to S$ such that the weight of any vertex u, which is defined as the sum of labels on vertices adjacent to u, is a constant k for all $u \in V$. The constant k is called an S-magic constant. The set of all S-magic constants of G for different labeling sets is denoted by $M(G)$. In this paper, we study S-magic labelings of various graph products like lexicographic products of graphs with C_4, direct products of graphs with C_4, Cartesian products of graphs with C_4, corona products of graphs, and joins of graphs. We find various classes of the above graph products that do not admit an S-magic labeling. We also give S-magic labeling conditions for several classes of the above graph products that do admit S-magic labelings, and we determine $M(G)$ for these classes of graphs.

1 Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph $G = (V, E)$, where $V(G)$ denotes the vertex set of the graph G and $E(G)$ denotes the edge set of the graph G. The *order* of the graph G, is defined as $|V(G)|$, while the *size* of the graph G is defined by $|E(G)|$. For graph theoretic notations we refer to Chartrand and Lesniak [11]. The *open neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the *degree* $d(x)$ of x is $|N(x)|$, the size of the open neighborhood of x.

Let $G = (V, E)$ be a graph of order n and let f be a bijection from $V(G)$ onto $\{1, 2, 3, \ldots, n\}$. For such a bijection we define the *weight* of a vertex v as $w(v) = \sum_{u \in N(v)} f(u)$, i.e., the sum of labels of vertices adjacent to u. The bijection is called a *distance magic labeling* if $w(v) = k$ for all vertices $v \in V$ and for some constant k (see [13]).

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Corresponding author: Maurice Almeida <p20230078@goa.bits-pilani.ac.in>

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A large class of graphs cannot be magic labeled with respect to the labeling set $\{1, 2, \ldots, n\}$. For example, the graph $K_{1,3}$ is not magic with respect to the labeling set $\{1, 2, 3, 4\}$. However, if we extend the set of labels to arbitrary sets such as $\{1, 2, 3, 6\}$ or $\{1, 2, 4, 7\}$, the graph can be magic labeled with magic constants 6 or 7, respectively. In fact a set S of positive integers induces a magic labeling on $K_{1,3}$ if and only if it is of the form $\{a, b, c, a + b + c : a, b, c \in \mathbb{N}\}$ and $\{6, 7, 8, \ldots\}$ are all the magic constants that can be obtained from $K_{1,3}$. Motivated by this idea, Godinho et al. [1] introduced the concept of S-magic labeling:

Definition 1.1. Let $G = (V, E)$ be a graph and let S be a set of positive integers with $|S| = |V|$. The graph G is said to be S-*magic* if there exists a bijection $l\colon V \to S$ such that $\sum_{u \in N(v)} l(u) = k$, a constant, for all $v \in V$. The set of all S-magic constants of G for different label sets is denoted by $M(G)$.

For any non-negative integer k, an illustration of S-magic labeling of cycle C_4 and path P_3 is shown in Figure 1.1.

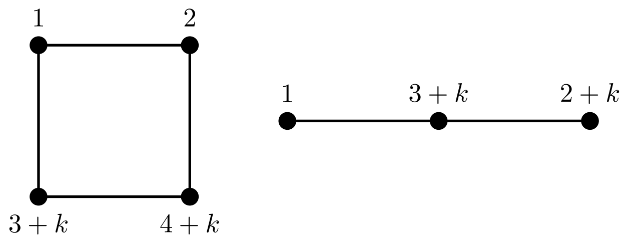

Figure 1.1: S-magic labeling of the graphs C_4 and P_3.

Godinho et al. [1] proved the following useful result to identify graphs that do not admit an S-magic labeling.

Theorem 1.2. [1] *If u and v are two vertices in the graph G such that $|N(u) \Delta N(v)| = 2$, where Δ is the symmetric difference of sets, then G is not S-magic.*

Godinho et al. [1] also proved that the Petersen graph is not S-magic, the cycle C_n is S-magic if and only if $n = 4$, the complete graph K_n for $n \geq 2$ is not S-magic, and they also stated the conditions under which a complete r-partite graph $K_{m_1, m_2, \ldots, m_r}$ admits an S-magic labeling. In this paper, we shall study S-magic labelings of five particular graph operations:

corona product of graphs, join of graphs, lexicographic product of graphs, direct product of graphs, and Cartesian product of graphs. We list their definitions below.

Definition 1.3. Let G and H be two graphs. The *corona* $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H and joining each vertex of the i^{th} copy of H to the i^{th} vertex of G.

Definition 1.4. Let G and H be two graphs. The *join* $G + H$ of G and H has vertex set $V(G) \cup V(H)$, and its edge set includes edges in G and H and all those edges which join a vertex in G with a vertex in H.

Definition 1.5. Let G and H be two graphs. The *lexicographic product* of G and H is defined as a graph having vertex set $V(G) \times V(H)$ in which two vertices (g, h) and (g', h') are adjacent if either g is adjacent to g' in G or $g = g'$ and h is adjacent to h' in H. The lexicographic product of graphs is also called the composition of graphs and is denoted by $G[H]$.

Definition 1.6. Let G and H be two graphs. The *direct product* of G and H is defined as a graph having vertex set $V(G) \times V(H)$ in which two vertices (g, h) and (g', h') are adjacent if g is adjacent to g' in G and h is adjacent to h' in H. The direct product of graphs is also known as Kronecker product of graphs and is denoted by $G \times H$.

Definition 1.7. Let G and H be two graphs. The *Cartesian product* $G \Box H$ of G and H is defined as a graph having vertex set $V(G) \times V(H)$ in which two vertices (g, h) and (g', h') are adjacent if either $g = g'$ in G and h is adjacent to h' in H or g is adjacent to g' in G and $h = h'$ in H.

In our investigation, we require the concept of the magic rectangle and magic rectangle set. These concepts are discussed in detail in [9] and [10]. Here we state their definitions.

Definition 1.8. A *magic rectangle* $R_{a,b}$ is an $a \times b$ array with entries

$$\{1, 2, \ldots, ab\},$$

each appearing once such that the sum of entries in each row is equal to $\frac{b(ab+1)}{2}$ and the sum of entries in each column is equal to $\frac{a(ab+1)}{2}$.

Definition 1.9. A *magic rectangle set*, $MRS(a, b; c)$ is a collection of c arrays of size $a \times b$ whose entries are elements of $\{1, 2, \ldots, abc\}$, each appearing once, with each row sum in every rectangle equal to the constant $\rho = \frac{b(abc+1)}{2}$ and each column sum in every rectangle equal to the constant $\sigma = \frac{a(abc+1)}{2}$.

We now present some useful results on magic rectangles and magic rectangle sets, which we require for our work. The following theorem gives the necessary and sufficient conditions for the existence of a magic rectangle $R_{a,b}$.

Theorem 1.10. [10] *For $a, b > 1$, there is a magic rectangle $R_{a,b}$ if and only if $a \equiv b \pmod 2$ and $(a, b) \neq (2, 2)$.*

The following theorems, state the conditions necessary for the existence of magic rectangle sets $MRS(a, b; c)$.

Theorem 1.11. [9] *If $a \equiv b \equiv 0 \pmod 2$ and $b \geq 4$, then a magic rectangle set $MRS(a, b; c)$ exists for every c.*

Theorem 1.12. [9] *If a or b is odd and abc is even, then no magic rectangle set $MRS(a, b; c)$ exists.*

Theorem 1.13. [9] *Let a, b, c be positive odd integers such that $1 < a \leq b$, then the magic rectangle set $MRS(a, b; c)$ exists.*

In this paper, we shall study the S-magic labeling of the lexicographic product, direct product, and the Cartesian product of the complete graph K_n, the complete r-partite graph $K_{(n,r)}$, and the windmill graph C_3^t with the cycle C_4. We shall also present some results related to S-magic labelings of the join and corona products of graphs.

2 Main results

2.1 Lexicographic products of graphs

Let $V(G) = \{x_1, x_2, \ldots, x_p\}$. Let $C_n = v_0v_1v_2\ldots v_{n-1}v_0$ be the cycle on n vertices. Let $H = G[C_n]$. Let x_i^j, where $1 \leq i \leq p$ and $0 \leq j \leq n-1$, be the vertices of H that correspond with vertices x_i in G.

Theorem 2.1. *Let $r \geq 1$ and $n \geq 4$. If G is an r-regular graph on p vertices and C_n, the cycle of length n, then $G[C_n]$ admits S-magic labeling if and only if $n = 4$.*

Proof. If $n \neq 4$, then there exist vertices x_i^2 and x_i^4 in the i^{th} copy of C_n, such that $N(x_i^2) \triangle N(x_i^4) = \{x_i^1, x_i^5\}$. Therefore by Theorem 1.2, $G[C_n]$ has no S-magic labeling.

Suppose $n = 4$. Then $G[C_4]$ is $4r + 2$ regular. Label the vertices x_i^j, where $1 \leq i \leq p$ and $0 \leq j \leq 3$, of $G[C_4]$ in the following way: For $k \in \mathbb{Z}_{\geq 0}$, define $f\colon V(G[C_4]) \to \{1, 2, 3, \dots, 2p-1, 2p, 2p+1+k, 2p+2+k, \dots, 4p-1+k, 4p+k\}$ such that

$$f\left(x_i^j\right) = \begin{cases} i & \text{for } 1 \leq i \leq p \text{ and } j = 0, \\ 2p + 1 - i & \text{for } 1 \leq i \leq p \text{ and } j = 1, \\ 4p + 1 - i + k & \text{for } 1 \leq i \leq p \text{ and } j = 2, \\ 2p + i + k & \text{for } 1 \leq i \leq p \text{ and } j = 3. \end{cases}$$

The sum of labels of vertices in the i^{th} copy of C_4 is $8p + 2 + 2k$, which is independent of i. Additionally, for $1 \leq i \leq p$, we have $f(x_i^0) + f(x_i^2) = 4p + 1 + k = f(x_i^1) + f(x_i^3)$. Therefore for every $x \in H$,

$$w(x) = r(8p + 2 + 2k) + (4p + 1 + k) = (2r + 1)(4p + 1 + k).$$

This gives an S-magic labeling. □

Observation 2.2. If $k = 0$, the above labeling is a distance magic labeling of $G[C_4]$ with magic constant $(2r + 1)(4p + 1)$.

Let $V(K_n) = \{x_1, x_2, \dots, x_n\}$. Let $C_4 = v_0 v_1 v_2 v_3 v_0$ be the cycle on four vertices. Let x_i^j, with $0 \leq j \leq 3$ and $1 \leq i \leq n$, be the vertices of $K_n[C_4]$ that correspond with vertices x_i in K_n.

Theorem 2.3. *For $n \geq 1$, a bijection l is an S-magic labeling for the graph $K_n[C_4]$ if and only if $l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = a$ for all $1 \leq i \leq n$ and for some constant a. Also $M(K_n[C_4]) = \big\{(2n - 1)(4n + 1 + k) : k \in \mathbb{Z}_{\geq 0}\big\}$.*

Proof. Suppose l is an S-magic labeling for the graph $K_n[C_4]$ with S-magic constant μ. Then since

$$\mu = w(x_i^1) = \sum_{\substack{1 \leq p \leq n,\\ p \neq i}} \sum_{j=0}^{3} l(x_p^j) + \big(l(x_i^0) + l(x_i^2)\big),$$

$$\mu = w(x_i^2) = \sum_{\substack{1 \leq p \leq n,\\ p \neq i}} \sum_{j=0}^{3} l(x_p^j) + \big(l(x_i^1) + l(x_i^3)\big),$$

we get $l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = a_i$, for all $1 \leq i \leq n$ and for some constants a_i. Now

$$\mu = w(x_i^j) = \sum_{\substack{1 \leq p \leq n, \\ p \neq i}} 2a_p + a_i, \qquad \mu = w(x_h^j) = \sum_{\substack{1 \leq p \leq n, \\ p \neq i}} 2a_p + a_h$$

implies $2a_h + a_i = 2a_i + a_h$; hence, we have $a_h = a_i = a$, for all $1 \leq h, i \leq n$ and for some constant a.

Conversely, suppose l is a labeling of $K_n[C_4]$ with

$$l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = a,$$

for all $1 \leq i \leq n$ and for some constant a. Then we have

$$w(x_i^j) = 2a(n-1) + a = a(2n-1),$$

for all $1 \leq i \leq n$ and $0 \leq j \leq 3$. Hence $K_n[C_4]$ is S-magic.

To find $M(K_n[C_4])$, label vertices of $K_n[C_4]$ in the following way: For $k \in \mathbb{Z}_{\geq 0}$, define $f\colon V(K_n[C_4]) \to \{1, 2, \ldots, 2n, 2n+1+k, 2n+2+k, \ldots, 4n-1+k, 4n+k\}$ such that

$$f(x_i^j) = \begin{cases} i & \text{for } 1 \leq i \leq n \text{ and } j = 0, \\ 2n+1-i & \text{for } 1 \leq i \leq n \text{ and } j = 1, \\ 4n+1-i+k & \text{for } 1 \leq i \leq n \text{ and } j = 2, \\ 2n+i+k & \text{for } 1 \leq i \leq n \text{ and } j = 3. \end{cases}$$

Under this labeling we get $l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = (4n+1+k)$, and hence $K_n[C_4]$ is S-magic with S-magic constant $(2n-1)(4n+1+k)$. If $k = 0$, we get a distance magic labeling of $K_n[C_4]$, with magic constant $(2n-1)(4n+1)$, which is the smallest S-magic constant. Also, from the first part of the proof, it follows that $(2n-1)$ should divide any S-magic constant. Therefore

$$M(K_n[C_4]) = \{(2n-1)(4n+1+k) : k \in \mathbb{Z}_{\geq 0}\}. \qquad \square$$

Let $V(K_{(n,r)}) = \{x_{1k}, x_{2k}, \ldots, x_{nk} : 1 \leq k \leq r\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \leq j \leq 3$, let x_{1k}^j, x_{1k}^j,. . .,x_{nk}^j denote vertices of $K_{(n,r)}[C_4]$ that correspond with $x_{1k}, x_{2k}, \ldots, x_{nk}$ in $K_{(n,r)}$, respectively, for $1 \leq k \leq r$.

Theorem 2.4. *For $n \geq 2$ and $r \geq 2$, a bijection l is an S-magic labeling of $K_{(n,r)}[C_4]$ if and only if $l(x^0_{i,h}) + l(x^2_{i,h}) = l(x^1_{i,h}) + l(x^3_{i,h}) = a$, for all $1 \leq i \leq n$ and $1 \leq h \leq r$ and for some constant a. Also $M(K_{(n,r)}[C_4]) = \{(2n(r-1)+1)(4rn+1+k) : k \in \mathbb{Z}_{\geq 0}\}$.*

Proof. Suppose $K_{(n,r)}[C_4]$ has an S-magic labeling l with S-magic constant μ. Then since

$$\mu = w(x^0_{i,p}) = \sum_{\substack{1\leq h\leq r,\\ h\neq p}} \sum_{k=1}^{n} \sum_{j=0}^{3} l(x^j_{k,h}) + \big(l(x^1_{i,p}) + l(x^3_{i,p})\big),$$

$$\mu = w(x^1_{i,p}) = \sum_{\substack{1\leq h\leq r,\\ h\neq p}} \sum_{k=1}^{n} \sum_{j=0}^{3} l(x^j_{k,h}) + \big(l(x^0_{i,p}) + l(x^2_{i,p})\big),$$

we have $l(x^1_{i,p}) + l(x^3_{i,p}) = l(x^0_{i,p}) + l(x^2_{i,p}) = a_{i,p}$, for all $1 \leq i \leq n$ and $1 \leq p \leq r$. Also,

$$w(x^0_{i,p}) = \sum_{\substack{1\leq h\leq r,\\ h\neq p}} \sum_{k=1}^{n} 2a_{k,h} + a_{i,p} = w(x^0_{j,p}) = \sum_{\substack{1\leq h\leq r,\\ h\neq p}} \sum_{k=1}^{n} 2a_{k,h} + a_{j,p}$$

implies $a_{i,p} = a_{j,p} = a_p$, for all $1 \leq i, j \leq n$. Now as

$$w(x^0_{i,p}) = \sum_{\substack{1\leq h\leq r,\\ h\neq p}} 2na_h + a_p = w(x^0_{i,m}) = \sum_{\substack{1\leq h\leq r,\\ h\neq m}} 2na_h + a_m,$$

we have $2na_m + a_p = 2na_p + a_m$; hence, $a_m = a_p = a$, for all $1 \leq m, p \leq n$.

Conversely, suppose l is a labeling with $l(x^0_{i,k}) + l(x^2_{i,k}) = l(x^1_{i,k}) + l(x^3_{i,k}) = a$, for all $1 \leq i \leq n$ and $1 \leq k \leq r$ and for some constant a. Then for any vertex $x^j_{i,p}$ in $K_{(n,r)}[C_4]$ we have

$$w(x^j_{i,p}) = \sum_{\substack{1\leq h\leq r,\\ h\neq p}} \sum_{k=1}^{n} \sum_{m=0}^{3} l(x^m_{k,h}) + \big(l(x^{j-1}_{i,p}) + l(x^{j+1}_{i,p})\big) = (2n(r-1)+1)a,$$

where $j-1$ and $j+1$ are taken modulo 4. Hence $K_{(n,r)}[C_4]$ is S-magic with S-magic constant $(2n(r-1)+1)a$.

To find $M(K_{(n,r)}[C_4])$, label vertices of $K_{(n,r)}[C_4]$ in the following way: For $k \in \mathbb{Z}_{\geq 0}$, define $f\colon V(K_{(n,r)}[C_4]) \to \{1, 2, 3, \ldots, 2rn-1, 2rn, 2rn+1+k, 2rn+2+k, \ldots, 4rn-1+k, 4rn+k\}$ such that

$$f(x_{i,j}^h) = \begin{cases} i + (j-1)n & \text{for } 1 \le i \le n,\ 1 \le j \le r,\ h = 0, \\ 2rn + 1 - i - (j-1)n & \text{for } 1 \le i \le n,\ 1 \le j \le r,\ h = 1, \\ 4rn + 1 - i - (j-1)n + k & \text{for } 1 \le i \le n,\ 1 \le j \le r,\ h = 2, \\ 2rn + i + (j-1)n + k & \text{for } 1 \le i \le n,\ 1 \le j \le r,\ h = 3. \end{cases}$$

Under this labeling we get $l(x_{i,h}^0) + l(x_{i,h}^2) = l(x_{i,h}^1) + l(x_{i,h}^3) = (4rn+1+k)$ for all $1 \le i \le n$ and $1 \le h \le r$, and hence $K_{(n,r)}[C_4]$ is S-magic with S-magic constant $(2n(r-1)+1)(4rn+1+k)$. If $k = 0$, we get a distance magic labeling of $K_{(n,r)}[C_4]$ with magic constant $(2n(r-1)+1)(4rn+1)$, which is the smallest S-magic constant. Also, from the first part of the proof, it follows that $(2n(r-1)+1)$ should divide any S-magic constant. Therefore

$$M(K_{(n,r)}[C_4]) = \big\{(2n(r-1)+1)(4rn+1+k) : k \in \mathbb{Z}_{\ge 0}\big\} \qquad \square$$

We denote by C_3^t, the graph obtained by taking t copies of the cycle C_3 and joining them by selecting one vertex from every cycle and identifying the selected vertices. Let $V(C_3^t) = \{x, y_i, z_i : 0 \le i \le t-1\}$ where x is the central vertex. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \le i \le t-1$ and $0 \le j \le 3$, let y_i^j and z_i^j be the vertices of $C_3^t[C_4]$ that correspond with vertices y_i and z_i in C_3^t and let x^j, for $0 \le j \le 3$, be the vertices of $C_3^t[C_4]$ that correspond with vertex x in C_3^t.

Theorem 2.5. *For $t > 1$, a bijection l is an S-magic labeling of $C_3^t[C_4]$ if and only if the following conditions hold:*

1. *$l(x^0) + l(x^2) = l(x^1) + l(x^3) = b$, for some constant b.*
2. *$l(y_i^0) + l(y_i^2) = l(y_i^1) + l(y_i^3) = l(z_i^0) + l(z_i^2) = l(z_i^1) + l(z_i^3) = a$, for some constant a.*
3. *$b = (4t-3)a$.*

Also $M(C_3^t[C_4]) = \big\{(8t-3)(8t+1+k) : k \in \mathbb{Z}_{\ge 0}\big\}$.

Proof. Suppose $C_3^t[C_4]$ has an S-magic labeling l with S-magic constant μ. Then

$$\mu = w(x^1) = \sum_{i=0}^{t-1}\sum_{j=0}^{3}\big(l(y_i^j) + l(z_i^j)\big) + l(x^0) + l(x^2),$$

$$\mu = w(x^2) = \sum_{i=0}^{t-1}\sum_{j=0}^{3}\big(l(y_i^j) + l(z_i^j)\big) + l(x^1) + l(x^3)$$

implies $l(x^0)+l(x^2)=l(x^1)+l(x^3)=b$, for some constant b.

Now for $0\le k\le t-1$, since

$$\mu = w(y_k^1)=\sum_{j=0}^{3}\big(l(x^j)+l(z_k^j)\big)+l(y_k^0)+l(y_k^2),$$
$$\mu = w(y_k^2)=\sum_{j=0}^{3}\big(l(x^j)+l(z_k^j)\big)+l(y_k^1)+l(y_k^3),$$

we get $l(y_k^0)+l(y_k^2)=l(y_k^1)+l(y_k^3)=a_y^k$, for some constant a_y^k. Similarly, we get $l(z_k^0)+l(z_k^2)=l(z_k^1)+l(z_k^3)=a_z^k$, for some constant a_z^k.

Since $a_y^i+2a_z^i+2b=w(y_i^j)=a_z^i+2a_y^i+2b=w(z_i^j)$, we obtain $a_y^i=a_z^i=a^i$. Now for any $0\le i,l\le t-1$ and $0\le j,h\le 3$, we have $3a^i+2b=w(z_i^j)=w(z_l^h)=3a^l+2b$; hence, $a^i=a^l=a$. Also since $3a+2b=b+4ta$, we have $b=(4t-3)a$.

Conversely, suppose l is a labeling such that conditions (1), (2), and (3) hold. Then for $0\le j\le 3$ and $0\le i\le t-1$, we have $w(x^j)=4ta+b=4ta+(4t-3)a=(8t-3)a$ and $w(y_i^j)=w(z_i^j)=2b+3a=2(4t-3)a+3a=(8t-3)a$. Therefore $C_3^t[C_4]$ is S-magic with S-magic constant $(8t-3)a$ for some constant a.

To determine $M(C_3^t[C_4])$, for $k\in\mathbb{Z}_{\ge 0}$, define a labeling f for $V(C_3^t[C_4])$ to the set

$$\Big\{1,2,\ldots,4t-1,4t,4t+1+k,4t+2+k,\ldots,8t-1+k,8t+k, \left\lfloor\tfrac{(4t-3)(8t+1+k)}{2}\right\rfloor-2,\left\lfloor\tfrac{(4t-3)(8t+1+k)}{2}\right\rfloor-1, \left\lceil\tfrac{(4t-3)(8t+1+k)}{2}\right\rceil+1,\left\lceil\tfrac{(4t-3)(8t+1+k)}{2}\right\rceil+2\Big\}$$

such that

$$f(y_i^j)=\begin{cases} i+1 & \text{for } j=0,\\ 2t+i+1 & \text{for } j=1,\\ 8t-i+k & \text{for } j=2,\\ 6t-i+k & \text{for } j=3,\end{cases}\qquad f(x^j)=\begin{cases} \left\lfloor\frac{(4t-3)(8t+1+k)}{2}\right\rfloor-2 & \text{for } j=0,\\ \left\lfloor\frac{(4t-3)(8t+1+k)}{2}\right\rfloor-1 & \text{for } j=1,\\ \left\lceil\frac{(4t-3)(8t+1+k)}{2}\right\rceil+2 & \text{for } j=2,\\ \left\lceil\frac{(4t-3)(8t+1+k)}{2}\right\rceil+1 & \text{for } j=3.\end{cases}$$

$$f(z_i^j)=\begin{cases} f(y_i^j)+t & \text{for } j=0,1,\\ f(y_i^j)-t & \text{for } j=2,3,\end{cases}$$

Under this labeling $f(x^0)+f(x^2) = f(x^1)+f(x^3) = (4t-3)(8t+1+k)$ and $f(y_i^0)+f(y_i^2) = f(y_i^1)+f(y_i^3) = f(z_i^0)+f(z_i^2) = f(z_i^1)+f(z_i^3) = (8t+1+k)$ and hence $C_3^t[C_4]$ is S-magic with S-magic constant $(8t-3)(8t+1+k)$. If $k=0$ we get the smallest S-magic constant. Also, from the first part of the proof, it follows that $(8t-3)$ divides any S-magic constant. Therefore

$$M(C_3^t[C_4]) = \big\{(8t-3)(8t+1+k) : k \in \mathbb{Z}_{\geq 0}\big\}. \qquad \square$$

2.2 Direct products of graphs

M. Anholcer et al. [8] proved that, for any r-regular graph G on n-vertices, $G \times C_4$ is distance magic.

Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. Let x_i^j, with $0 \leq j \leq 3$ and $1 \leq i \leq n$, be the vertices of $K_n \times C_4$ that correspond with vertices x_i in K_n.

Theorem 2.6. *For $n \geq 1$, a bijection l is an S-magic labeling of $K_n \times C_4$ if and only if $l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = a$, for all $1 \leq i \leq n$ and for some constant a. Also $M(K_n \times C_4) = \big\{(n-1)(4n+1+k) : k \in \mathbb{Z}_{\geq 0}\big\}$.*

Proof. Suppose $K_n \times C_4$ has an S-magic labeling l with S-magic constant μ. Since

$$w(x_i^0) = \sum_{\substack{1\leq h\leq n,\\ h\neq i}} \big(l(x_h^1) + l(x_h^3)\big) = w(x_p^0) = \sum_{\substack{1\leq h\leq n,\\ h\neq p}} \big(l(x_h^1) + l(x_h^3)\big),$$

we have $l(x_p^1) + l(x_p^3) = l(x_i^1) + l(x_i^3) = a$, for all $1 \leq p, i \leq n$ and for some constant a. Analogously, we have, $l(x_p^0) + l(x_p^2) = l(x_i^0) + l(x_i^2) = b$, for all $1 \leq p, i \leq n$ and for some constant b. Now since $\mu = w(x_i^0) = a(n-1)$ and $\mu = w(x_i^1) = b(n-1)$, we have $a = b$.

Conversely, suppose l is a labeling such that $l(x_i^0)+l(x_i^2) = l(x_i^1)+l(x_i^3) = a$, for all $1 \leq i \leq n$ and for some constant a. Then for any vertex x_i^j in $K_n \times C_4$, we have

$$w(x_i^j) = \sum_{\substack{1\leq h\leq n,\\ h\neq i}} \big(l(x_h^{j-1}) + l(x_h^{j+1})\big) = (n-1)a,$$

where $j-1$ and $j+1$ are taken modulo 4. Therefore $K_n \times C_4$ is S-magic with S-magic constant $(n-1)a$, for some constant a.

To find $M(K_n \times C_4)$, label vertices of $K_n \times C_4$ in the following way: For $k \in \mathbb{Z}_{\geq 0}$, define $f\colon V(K_n \times C_4) \to \{1, 2, 3, \ldots, 2n-1, 2n, 2n+1+k, 2n+2+k, \ldots, 4n-1+k, 4n+k\}$ such that

$$f(x_i^j) = \begin{cases} i & \text{for } 1 \leq i \leq n \text{ and } j = 0, \\ 2n+1-i & \text{for } 1 \leq i \leq n \text{ and } j = 1, \\ 4n+1-i+k & \text{for } 1 \leq i \leq n \text{ and } j = 2, \\ 2n+i+k & \text{for } 1 \leq i \leq n \text{ and } j = 3. \end{cases}$$

Under this labeling we get $l(x_i^0) + l(x_i^2) = l(x_i^1) + l(x_i^3) = (4n+1+k)$ and hence $K_n \times C_4$ is S-magic with S-magic constant $(n-1)(4n+1+k)$. If $k = 0$, we get a distance magic labeling of $K_n[C_4]$ with magic constant $(n-1)(4n+1)$, which is the smallest S-magic constant. Also, from the first part of the proof, it follows that $(n-1)$ should divide any S-magic constant. Therefore

$$M(K_n \times C_4) = \big\{(n-1)(4n+1+k) : k \in \mathbb{Z}_{\geq 0}\big\} \qquad \square$$

Let $V(K_{(n,r)}) = \{x_{1k}, x_{2k}, \ldots, x_{nk} : 1 \leq k \leq r\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \leq j \leq 3$, let x_{1k}^j, x_{1k}^j,…,x_{nk}^j denote vertices of $K_{(n,r)} \times C_4$ that correspond with $x_{1k}, x_{2k}, \ldots, x_{nk}$ in $K_{(n,r)}$, respectively, for $1 \leq k \leq r$.

Theorem 2.7. *For $n \geq 2$ and $r \geq 2$, a bijection l is an S-magic labeling of $K_{(n,r)} \times C_4$ if and only if $\sum_{k=1}^{n}\big(l(x_{k,p}^1) + l(x_{k,p}^3)\big) = \sum_{k=1}^{n}\big(l(x_{k,p}^0) + l(x_{k,p}^2)\big) = a$, for all $1 \leq p \leq n$ and for some constant a.*

Proof. Suppose the graph $K_{(n,r)} \times C_4$ admits an S-magic labeling l with S-magic constant μ. Since

$$\mu = w(x_{i,p}^1) = \sum_{\substack{1 \leq h \leq r,\\ h \neq p}} \sum_{k=1}^{n} \big(l(x_{k,h}^0) + l(x_{k,h}^2)\big),$$

$$\mu = w(x_{i,s}^1) = \sum_{\substack{1 \leq h \leq r,\\ h \neq s}} \sum_{k=1}^{n} \big(l(x_{k,h}^0) + l(x_{k,h}^2)\big),$$

we have

$$\sum_{k=1}^{n}\big(l(x_{k,p}^0) + l(x_{k,p}^2)\big) = \sum_{k=1}^{n}\big(l(x_{k,s}^0) + l(x_{k,s}^2)\big) = a,$$

for all $1 \leq p, s \leq r$ and for some constant a. Analogously we have

$$\sum_{k=1}^{n}\big(l(x_{k,p}^1)+l(x_{k,p}^3)\big)=\sum_{k=1}^{n}\big(l(x_{k,s}^1)+l(x_{k,s}^3)\big)=b,$$

for all $1 \leq p, s \leq r$ and for some constant b. Also now, since $\mu = w(x_{i,p}^1) = (r-1)a$ and $\mu = w(x_{i,p}^2) = (r-1)b$, we get $a = b$.

Conversely, suppose l is a labeling such that

$$\sum_{k=1}^{n}\big(l(x_{k,p}^1)+l(x_{k,p}^3)\big)=\sum_{k=1}^{n}\big(l(x_{k,p}^0)+l(x_{k,p}^2)\big)=a,$$

for any $1 \leq p \leq r$ and for some constant a. Then, for any vertex $x_{i,p}^j$ in $K_{(n,r)} \times C_4$, we have

$$w(x_{i,p}^j)=\sum_{1\leq h\leq r, h\neq p}\sum_{k=1}^{n}\big(l(x_{k,h}^{j-1})+l(x_{k,h}^{j+1})\big)=(r-1)a,$$

where, $j-1$ and $j+1$ are taken modulo 4. Hence $K_{(n,r)} \times C_4$ is S-magic with S-magic constant $(r-1)a$, for some constant a. □

Unfortunately, we are not able to find $M(K_{(n,r)} \times C_4)$. Therefore, we pose the following problem.

Problem 2.8. For $n \geq 2$ and $r \geq 2$, determine $M(K_{(n,r)} \times C_4)$.

Let $V(C_3^t) = \{x, y_i, z_i : 0 \leq i \leq t-1\}$ where x is the central vertex. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \leq i \leq t-1$ and $0 \leq j \leq 3$, let y_i^j,z_i^j be the vertices of $C_3^t \times C_4$ that correspond with vertices y_i and z_i in C_3^t and x^j for $0 \leq j \leq 3$ be the vertices of $C_3^t \times C_4$ that correspond with vertex x in C_3^t

Theorem 2.9. *For $t > 1$, a bijection l is an S-magic labeling of $C_3^t \times C_4$ if and only if the following conditions hold:*

1. $l(x^0) + l(x^2) = l(x^1) + l(x^3) = s$, *for some constant s.*
2. $l(y_i^0) + l(y_i^2) = l(y_i^1) + l(y_i^3) = l(z_i^0) + l(z_i^2) = l(z_i^1) + l(z_i^3) = a$, *for some constant a.*
3. $s = (2t-1)a$.

Also $M((C_3^t \times C_4) = \big\{(2t)(8t+1+k) : k \in \mathbb{Z}_{\geq 0}\big\}$.

Proof. Suppose $C_3^t \times C_4$ has an S-magic labeling l.

Let $l(x^0)+l(x^2) = s_1$, $l(x^1)+l(x^3) = s_2$, $l(y_i^0)+l(y_i^2) = a_i^1$, $l(y_i^1)+l(y_i^3) = a_i^2$, $l(z_i^0) + l(z_i^2) = b_i^1$,$l(z_i^1) + l(z_i^3) = b_i^2$ for $0 \leq i \leq t-1$.

Since $a_i^1 + s_1 = w(z_i^1) = w(y_i^1) = b_i^1 + s_1$ and $a_i^2 + s_2 = w(z_i^2) = w(y_i^2) = b_i^2 + s_2$, we have $a_i^1 = b_i^1$ and $a_i^2 = b_i^2$. Also, for $i \neq j$, as $w(z_i^1) = w(z_j^1)$, we have $a_i^1 = a_j^1$. Similarly, for $i \neq j$, as $w(y_i^1) = w(y_j^1)$, we have $b_i^1 = b_j^1$. Therefore, we can assume $a_i^1 = b_i^1 = a_1$ for all $0 \leq i \leq t-1$ and for some constant a_1. Similarly, we can show $a_i^2 = b_i^2 = a_2$ for all $0 \leq i \leq t-1$ and for some constant a_2. Furthermore $w(x^0) = 2ta_2 = w(x^1) = 2ta_1$, we have $a_1 = a_2 = a$ and hence $s_1 = s_2 = s$.

Therefore we have $l(x^0) + l(x^2) = l(x^1) + l(x^3) = s$ and $l(y_i^0) + l(y_i^2) = l(y_i^1) + l(y_i^3) = l(z_i^0) + l(z_i^2) = l(z_i^1) + l(z_i^3) = a$.

Also as, $w(x^j) = 2ta = w(z_i^j) = a + s$, we have $s = (2t-1)a$.

Conversely, suppose l is a labeling such that conditions (1),(2), and (3) hold.

Then for $0 \leq j \leq 3$ and $0 \leq i \leq t-1$ we have $w(x^j) = 2ta$ and $w(y_i^j) = w(z_i^j) = a + s = a + (2t-1)a = 2ta$. Therefore $C_3^t \times C_4$ is S-magic with S-magic constant $2ta$ for some constant a.

To determine $M(C_3^t \times C_4)$ define a labeling f for $V(C_3^t \times C_4)$ to the set

$$\Big\{1, 2, \ldots, 4t-1, 4t, 4t+1+k, 4t+2+k, \ldots, 8t-1+k, 8t+k, \left\lfloor \tfrac{(2t-1)(8t+1+k)}{2} \right\rfloor - 2, \left\lfloor \tfrac{(2t-1)(8t+1+k)}{2} \right\rfloor - 1, \left\lceil \tfrac{(2t-1)(8t+1+k)}{2} \right\rceil + 1, \left\lceil \tfrac{(2t-1)(8t+1+k)}{2} \right\rceil + 2\Big\}$$

such that

$$f(y_i^j) = \begin{cases} i+1 & \text{for } j = 0, \\ 2t+1+i & \text{for } j = 1, \\ 8t-i+k & \text{for } j = 2, \\ 6t-i+k & \text{for } j = 3, \end{cases}$$

$$f(z_i^j) = \begin{cases} f(y_i^j) + t & \text{for } j = 0, 1, \\ f(y_i^j) - t & \text{for } j = 2, 3, \end{cases}$$

$$f(x^j) = \begin{cases} \left\lfloor \frac{(2t-1)(8t+1+k)}{2} \right\rfloor - 2 & \text{for } j = 0, \\ \left\lfloor \frac{(2t-1)(8t+1+k)}{2} \right\rfloor - 1 & \text{for } j = 1, \\ \left\lceil \frac{(2t-1)(8t+1+k)}{2} \right\rceil + 2 & \text{for } j = 2, \\ \left\lceil \frac{(2t-1)(8t+1+k)}{2} \right\rceil + 1 & \text{for } j = 3. \end{cases}$$

Under this labeling $f(x^0)+f(x^2) = f(x^1)+f(x^3) = (2t-1)(8t+1+k)$ and $f(y_i^0)+f(y_i^2) = f(y_i^1)+f(y_i^3) = f(z_i^0)+f(z_i^2) = f(z_i^1)+f(z_i^3) = (8t+1+k)$, and hence $C_3^t[C_4]$ is S-magic with S-magic constant $(2t)(8t+1+k)$. If $k=0$, we get the smallest S-magic constant. Also, from the first part of the proof, it follows that $2t$ divides any S-magic constant. Therefore

$$M(C_3^t \times C_4) = \{(2t)(8t+1+k) : k \in \mathbb{Z}_{\geq 0}\}. \qquad \square$$

2.3 Cartesian products of graphs

Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. Let x_i^j, with $0 \leq j \leq 3$ and $1 \leq i \leq n$, be the vertices of $K_n \square C_4$ that correspond with vertices x_i in K_n.

Theorem 2.10. *For $n \geq 1$, the graph $K_n \square C_4$ is not S-magic.*

Proof. Suppose $K_n \square C_4$ is S-magic with labeling l and S-magic constant μ. Let $l(x_i^0)+l(x_i^2) = a_{02}^i$ and $l(x_i^1)+l(x_i^3) = a_{13}^i$, for any $1 \leq i \leq n$. Since

$$\mu = w(x_i^0) = \sum_{\substack{1\leq k\leq n,\\ k\neq i}} l(x_k^0) + a_{13}^i,$$

we have

$$0 = w(x_i^0) - w(x_h^0) = l(x_h^0) - l(x_i^0) + a_{13}^i - a_{13}^h.$$

Analogously, we have

$$0 = w(x_i^2) - w(x_h^2) = l(x_h^2) - l(x_i^2) + a_{13}^i - a_{13}^h.$$

Therefore we get

$$l(x_h^0) - l(x_h^2) = l(x_i^0) - l(x_i^2),$$

for any $1 \leq i, h \leq n$; hence, $l(x_i^0) = k + l(x_i^2)$ for some constant k and for any $1 \leq i \leq n$. On the other hand, we have

$$\mu = w(x_i^2) = \sum_{\substack{1\leq p\leq n,\\ p\neq i}} l(x_p^2) + a_{13}^i,$$

$$\mu = w(x_i^0) = \sum_{\substack{1\leq p\leq n,\\ p\neq i}} l(x_p^0) + a_{13}^i = \sum_{\substack{1\leq p\leq n,\\ p\neq i}} (k + l(x_p^2)) + a_{13}^i,$$

which implies $k = 0$; hence, $l(x_i^0) = l(x_i^2)$, a contradiction. $\square$

Note that $K_{(2,2)} \cong C_4$, and we shall now prove that $C_4 \square C_4$ is not S- magic. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices and let

$$V(C_4 \square C_4) = \{v_{i,j} : 0 \leq i \leq 3 \text{ and } 0 \leq j \leq 3\}.$$

Lemma 2.11. *The Cartesian product $C_4 \square C_4$ is not S-magic.*

Proof. Suppose $C_4 \square C_4$ is S-magic with S-magic labeling l and S-magic constant μ. Then $w(v_{0,1}) = l(v_{0,0}) + l(v_{1,1}) + l(v_{3,1}) + l(v_{0,2})$ and $w(v_{3,2}) = l(v_{2,2}) + l(v_{3,1}) + l(v_{3,3}) + l(v_{0,2})$, which implies that

$$l(v_{0,0}) + l(v_{1,1}) = l(v_{2,2}) + l(v_{3,3}). \tag{1}$$

Similarly, since $w(v_{2,1}) = l(v_{2,2}) + l(v_{2,0}) + l(v_{1,1}) + l(v_{3,1})$ and $w(v_{3,0}) = l(v_{0,0}) + l(v_{2,0}) + l(v_{3,1}) + l(v_{3,3})$, we have

$$l(v_{1,1}) + l(v_{2,2}) = l(v_{0,0}) + l(v_{3,3}). \tag{2}$$

Subtracting (2) from (1) we get

$$\begin{aligned} l(v_{0,0}) - l(v_{2,2}) &= l(v_{2,2}) - l(v_{0,0}) \\ 2l(v_{0,0}) &= 2l(v_{2,2}) \\ l(v_{0,0}) &= l(v_{2,2}). \end{aligned}$$

This is a contradiction; hence, $C_4 \square C_4$ is not S-magic. □

Let $V(K_{(2,r)}) = \{v_{1k}, v_{2k} : 1 \leq k \leq r\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \leq j \leq 3$, let $v_{1,k}^j$ and $v_{2,k}^j$ denote vertices of H that correspond with vertices $v_{1,k}$ and $v_{2,k}$ in $K_{(2,r)}$, respectively.

Lemma 2.12. *For $r \geq 3$, the graph $K_{(2,r)} \square C_4$ is not S-magic.*

Proof. Suppose $K_{(2,r)} \square C_4$ is S-magic with S-magic labeling l and S-magic constant μ. Since

$$\begin{aligned} \mu = w(v_{1,j}^1) &= \sum_{\substack{1 \leq p \leq r, \\ p \neq j}} \left(l(v_{1,p}^1) + l(v_{2,p}^1) \right) + l(v_{1,j}^0) + l(v_{1,j}^2), \\ \mu = w(v_{2,j}^1) &= \sum_{\substack{1 \leq p \leq r, \\ p \neq j}} \left(l(v_{1,p}^1) + l(v_{2,p}^1) \right) + l(v_{2,j}^0) + l(v_{2,j}^2), \end{aligned}$$

we have

$$l(v_{1,j}^0) + l(v_{1,j}^2) = l(v_{2,j}^0) + l(v_{2,j}^2). \tag{3}$$

Also, as $w(v_{1,j}^0) = w(v_{1,j}^2)$, we get, for $1 \leq j \leq r$,

$$\sum_{\substack{1\leq p\leq r,\\ p\neq j}} \big(l(v_{1,p}^0) + l(v_{2,p}^0)\big) = \sum_{\substack{1\leq p\leq r,\\ p\neq j}} \big(l(v_{1,p}^2) + l(v_{2,p}^2)\big). \tag{4}$$

Hence,

$$(r-1)\sum_{j=1}^{r}\big(l(v_{1,j}^0) + l(v_{2,j}^0)\big) = (r-1)\sum_{j=1}^{r}\big(l(v_{1,j}^2) + l(v_{2,j}^2)\big)$$

implying

$$\sum_{j=1}^{r}\big(l(v_{1,j}^0) + l(v_{2,j}^0)\big) = \sum_{j=1}^{r}\big(l(v_{1,j}^2) + l(v_{2,j}^2)\big). \tag{5}$$

Subtracting equation (4) from equation (5), we get, $l(v_{1,j}^0) + l(v_{2,j}^0) = l(v_{1,j}^2) + l(v_{2,j}^2)$, which together with equality (3) implies that $l(v_{1,j}^2) = l(v_{2,j}^0)$, contradicting the fact that l is a bijection. □

Theorem 2.13. *For $r \geq 2$, the graph $K_{(2,r)} \square C_4$ is not S-magic.*

Proof. The proof follows from Lemmas 2.11 and 2.12. □

Let $V(K_{(n,r)}) = \{x_{1k}, x_{2k}, \ldots, x_{nk} : 1 \leq k \leq r\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \leq j \leq 3$, let x_{1k}^j, x_{2k}^j,…,x_{nk}^j denote vertices of $K_{(n,r)} \square C_4$ that correspond with vertices $x_{1k}, x_{2k}, \ldots, x_{nk}$ in $K_{(n,r)}$, respectively, for $1 \leq k \leq r$.

Theorem 2.14. *For $r \geq 2$ and $n \geq 3$, a bijection l is an S-magic labeling of $K_{(n,r)} \square C_4$ if and only if $l(x_{i,j}^1) + l(x_{i,j}^3) = l(x_{i,j}^0) + l(x_{i,j}^2) = a$, for any $1 \leq i \leq n$ and $1 \leq j \leq r$ and for some constant a. Also for n even, $M(K_{(n,r)} \square C_4) = \left\{\frac{(n(r-1)+2)(4nr+1+k)}{2} : k \in \mathbb{Z}_{\geq 0}\right\}$.*

Proof. Suppose $K_{(n,r)} \square C_4$ is S-magic with labeling l and S-magic constant μ. Since $w(x_{i,j}^0) = w(x_{k,j}^0)$, we have

$$\sum_{\substack{1\leq p\leq r,\\ p\neq j}} \sum_{h=1}^{n} l(x_{h,p}^0) + \big(l(x_{i,j}^1) + l(x_{i,j}^3)\big) = \sum_{\substack{1\leq p\leq r,\\ p\neq j}} \sum_{h=1}^{n} l(x_{h,p}^0) + \big(l(x_{k,j}^1) + l(x_{k,j}^3)\big);$$

hence, $l(x^1_{i,j}) + l(x^3_{i,j}) = l(x^1_{k,j}) + l(x^3_{k,j})$, for any $1 \le i \le n$ and $1 \le j \le r$. Similarly, $l(x^0_{i,j})+l(x^2_{i,j}) = l(x^0_{k,j})+l(x^2_{k,j})$, for any $1 \le i \le n$ and $1 \le j \le r$. So let $l(x^1_{i,j}) + l(x^3_{i,j}) = a^j_{1,3}$ and $l(x^0_{i,j}) + l(x^2_{i,j}) = a^j_{0,2}$, for any $1 \le i \le n$ and $1 \le j \le r$. Then

$$\mu = w(x^0_{i,j}) = \sum_{\substack{1\le p\le r,\\ p\ne j}} \sum_{h=1}^{n} l(x^0_{h,p}) + a^j_{1,3}.$$

For $1 \le j, h \le r$ and $1 \le i \le n$, we have

$$2\mu = w(x^0_{i,h}) + w(x^2_{i,h}) = n \sum_{\substack{1\le p\le r,\\ p\ne h}} a^p_{0,2} + 2a^h_{1,3}, \tag{6}$$

$$2\mu = w(x^0_{i,j}) + w(x^2_{i,j}) = n \sum_{\substack{1\le p\le r,\\ p\ne j}} a^p_{0,2} + 2a^j_{1,3}. \tag{7}$$

Subtracting (6) from (7) we get $n(a^j_{0,2} - a^h_{0,2}) = 2(a^j_{1,3} - a^h_{1,3})$, for any $1 \le j, h \le r$. Analogously, $2(a^j_{0,2} - a^h_{0,2}) = n(a^j_{1,3} - a^h_{1,3})$, for any $1 \le j, h \le r$. Therefore, for any $1 \le j, h \le r$, we have $(n-2)(a^j_{0,2} - a^h_{0,2}) = -(n-2)(a^j_{1,3} - a^h_{1,3})$. Since $n \ne 2$, we have $a^j_{0,2} + a^j_{1,3} = a$, for any $1 \le j \le r$ and for some constant a. If $a^j_{0,2} = \frac{a}{2} - c^j$ for any $1 \le j \le r$ and some constants c^j, then $a^j_{1,3} = \frac{a}{2} + c^j$. Then we get $\sum_{i=1}^{n}\big(l(x^0_{i,j}) + l(x^2_{i,j})\big) = n\big(\frac{a}{2} - c^j\big)$ and $\sum_{i=1}^{n}\big(l(x^1_{i,j}) + l(x^3_{i,j})\big) = n\big(\frac{a}{2} + c^j\big)$, for any $1 \le j \le r$.

Now for any $1 \le j \le r$ and $1 \le i \le n$, we have

$$2\mu = w(x^0_{i,j}) + w(x^2_{i,j}) = n \sum_{\substack{1\le p\le r,\\ p\ne j}} \big(\tfrac{a}{2} + c^p\big) + 2\big(\tfrac{a}{2} - c^j\big), \tag{8}$$

$$2\mu = w(x^0_{i,h}) + w(x^2_{i,h}) = n \sum_{\substack{1\le p\le r,\\ p\ne h}} \big(\tfrac{a}{2} + c^p\big) + 2\big(\tfrac{a}{2} - c^h\big). \tag{9}$$

Subtracting (8) from (9), we obtain $(n+2)c^h = (n+2)c^j$, for any $1 \le j, h \le r$. Hence $c^j = c$ for any $1 \le j \le r$. So now if $a^j_{0,2} = \frac{a}{2} - c$ for any $1 \le j \le r$ and some constants c, then $a^j_{1,3} = \frac{a}{2} + c$. Then we get $\sum_{i=1}^{n}\big(l(x^0_{i,j}) + l(x^2_{i,j})\big) = n\big(\frac{a}{2} - c\big)$ and $\sum_{i=1}^{n}\big(l(x^1_{i,j}) + l(x^3_{i,j})\big) = n\big(\frac{a}{2} + c\big)$, for any $1 \le j \le r$.

Now for any $1 \le j \le r$ and $1 \le i \le n$, we have

$$2\mu = w(x^0_{i,j}) + w(x^2_{i,j}) = n(r-1)\big(\tfrac{a}{2} + c\big) + 2\big(\tfrac{a}{2} - c\big), \tag{10}$$

$$2\mu = w(x^1_{i,j}) + w(x^3_{i,j}) = n(r-1)\big(\tfrac{a}{2} - c\big) + 2\big(\tfrac{a}{2} + c\big). \tag{11}$$

Subtracting (10) from (11), we get $(2-n(r-1))c=0$, and since $n\geq 3$ and $r\geq 2$, we get $c=0$. Hence $a^j_{0,2}=\frac{a}{2}$ and $a^j_{1,3}=\frac{a}{2}$, that is, $l(x^1_{i,j})+l(x^3_{i,j})=l(x^0_{i,j})+l(x^2_{i,j})$ for any $1\leq i\leq n$ and $1\leq j\leq r$.

Conversely, suppose $l(x^1_{i,j})+l(x^3_{i,j})=l(x^0_{i,j})+l(x^2_{i,j})=a$, for any $1\leq i\leq n$ and $1\leq j\leq r$ and for some constant a. Then since

$$w(x^m_{i,j})=\sum_{\substack{1\leq p\leq r,\\ p\neq j}}\sum_{h=1}^{n} l(x^m_{h,p})+a$$

for any $0\leq m\leq 3$ and $1\leq i\leq n$ and $1\leq j\leq r$, it is easy to check that $\sum_{i=1}^{n} l(x^m_{i,j})=\frac{na}{2}$, for any $0\leq m\leq 3$ and $1\leq j\leq r$. Now for any vertex $x^m_{i,j}$ in $K(n,r)\square C_4$ we have

$$\begin{aligned} w(x^m_{i,j}) &= \sum_{\substack{1\leq p\leq r,\\ p\neq j}}\sum_{h=1}^{n} l(x^m_{h,p})+\big(l(x^{m-1}_{i,j})+l(x^{m+1}_{i,j})\big)\\ &= \tfrac{na(r-1)}{2}+a=\tfrac{(n(r-1)+2)}{2}a, \end{aligned}$$

where both $m-1$ and $m+1$ are taken modulo 4. Therefore $K_{(n,r)}\square C_4$ is S-magic with S-magic constant $\frac{(n(r-1)+2)a}{2}$ for some constant a.

If n is even, then there exists a magic rectangle set $MRS(2,n;2r)$ by Theorem 1.11. Denote by $a^l_{1,j}$ and $a^l_{2,j}$ the entries in the first and second row, respectively, of the j^{th} column of the l^{th} rectangle, where $1\leq j\leq n$ and $1\leq l\leq 2r$. Such a rectangle has constant column sum equal to $(4nr+1)$. For $k\in\mathbb{Z}_{\geq 0}$ and $1\leq i\leq n$ and $1\leq j\leq r$, label the vertices

$$l(x^0_{i,j})=\begin{cases} a^j_{1,i}+k & \text{if } j \text{ odd,}\\ a^j_{1,i} & \text{if } j \text{ even,}\end{cases}\qquad l(x^2_{i,j})=\begin{cases} a^j_{2,i} & \text{if } j \text{ odd,}\\ a^j_{2,i}+k & \text{if } j \text{ even,}\end{cases}$$

$$l(x^1_{i,j})=\begin{cases} a^{j+r}_{1,i}+k & \text{if } j \text{ odd,}\\ a^{j+r}_{1,i} & \text{if } j \text{ even,}\end{cases}\qquad l(x^3_{i,j})=\begin{cases} a^{j+r}_{2,i} & \text{if } j \text{ odd,}\\ a^{j+r}_{2,i}+k & \text{if } j \text{ even.}\end{cases}$$

Under this labeling $l(x^1_{i,j})+l(x^3_{i,j})=l(x^0_{i,j})+l(x^2_{i,j})=(4nr+1+k)$. Hence the labeling is an S-magic labeling for $K_{(n,r)}\square C_4$ with S-magic constant $\frac{(n(r-1)+2)(4nr+1+k)}{2}$. If $k=0$, we get a distance magic labeling for $K_{(n,r)}\square C_4$, which is the smallest S-magic labeling. Also, from the first part of the proof, it follows that $\frac{(n(r-1)+2)}{2}$ should divide any S-magic constant of $K_{(n,r)}\square C_4$. Hence, for n even, $M(K_{(n,r)}\square C_4)=\big\{\frac{(n(r-1)+2)(4nr+1+k)}{2} : k\in\mathbb{Z}_{\geq 0}\big\}$. $\square$

For n odd, by Theorem 1.12, such a magic rectangle set does not exist, and finding $M(K_{(n,r)}\square C_4)$ is not easy. Hence, we state the following problem.

Problem 2.15. For $r \geq 2$ and $n \geq 3$ odd, determine $M(K_{(n,r)}\square C_4)$.

Let $V(C_3^t) = \{x, y_i, z_i : 0 \leq i \leq t-1\}$ where x is the central vertex. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. For $0 \leq i \leq t-1$ and $0 \leq j \leq 3$, let y_i^j and z_i^j be the vertices of $C_3^t\square C_4$ that correspond, respectively, with vertices y_i and z_i in C_3^t and let x^j, for $0 \leq j \leq 3$, be the vertices of $C_3^t\square C_4$ that correspond with vertex x in C_3^t.

Theorem 2.16. *For $t > 1$, the graph $C_3^t\square C_4$ is not S-magic.*

Proof. Suppose $C_3^t\square C_4$ is S-magic with S-magic labeling l and S-magic constant μ. Then, we have

$$\mu = w(x^1) = l(x^0) + l(x^2) + \textstyle\sum_{i=1}^{t}\big[l(y_i^1) + l(z_i^1)\big],$$
$$\mu = w(x^3) = l(x^0) + l(x^2) + \textstyle\sum_{i=1}^{t}\big[l(y_i^3) + l(z_i^3)\big].$$

Hence we have,

$$\sum_{i=1}^{t}\big[l(y_i^1) + l(z_i^1)\big] = \sum_{i=1}^{t}\big[l(y_i^3) + l(z_i^3)\big]. \tag{12}$$

Now since $w(y_i^1) = l(y_i^0) + l(y_i^2) + l(x^1) + l(z_i^1) = w(y_i^3) = l(y_i^0) + l(y_i^2) + l(x^3) + l(z_i^3)$, we have, for $1 \leq i \leq t$,

$$l(x^1) + l(z_i^1) = l(x^3) + l(z_i^3). \tag{13}$$

Also since $w(z_i^1) = l(z_i^0) + l(z_i^2) + l(x^1) + l(y_i^1) = w(z_i^3) = l(z_i^0) + l(z_i^2) + l(x^3) + l(y_i^3)$, we have, for $1 \leq i \leq t$,

$$l(x^1) + l(y_i^1) = l(x^3) + l(y_i^3). \tag{14}$$

Adding all t terms of (13) and (14), we get

$$2t \times l(x^1) + \sum_{i=1}^{t}\big[l(z_i^1) + l(y_i^1)\big] = 2t \times l(x^3) + \sum_{i=1}^{t}\big[l(z_i^3) + l(y_i^3)\big].$$

Using (12), we get $2t \times l(x^1) = 2t \times l(x^3)$, and this leads to $l(x^1) = l(x^3)$, which is a contradiction. □

2.4 Joins and corona products of graphs

Theorem 2.17. *Let G be a graph having two vertices u and v such that $|N(u) \cap N(v)| = d(u) - 1 = d(v) - 1$. Then for any graph H, we have that $G + H$ is not S-magic.*

Proof. Since $|N(u) \cap N(v)| = d(u) - 1 = d(v) - 1$, there exist vertices x, y in G such that $x \in N(u)$ but $x \notin N(v)$ and $y \in N(v)$ but $y \notin N(u)$. Now in $G + H$, the vertices u and v will have the same neighbors in G, but in addition, all vertices of H will be adjacent to u and v. Therefore $|N(u) \triangle N(v)| = 2$, and by Theorem 1.2, $G + H$ is not S-magic. □

Observation 2.18. Based on the above theorem, we make the following observations:

1. For any graph G and $n \neq 4$, the join $G + C_n$ is not S-magic.
2. For any graph G and $n \neq 1, 3$, the join $G + P_n$ is not S-magic.
3. For any graph G and $n \neq 1$, the join $G + K_n$ is not S-magic.
4. For any graph G and $t \geq 1$, the join $G + C_3^t$ is not S-magic.

For any two graphs G and H, we denote by $N_G(x)$ the vertices adjacent to x in G in the join of $G + H$.

Theorem 2.19. *Let G be a graph with $V(G) = \{x_1, x_2, \ldots, x_n\}$. Let $C_4 = v_0v_1v_2v_3v_0$ be the cycle on four vertices. A bijection l is an S-magic labeling of $G + C_4$ if and only if the following conditions hold:*

(1) $\sum_{x \in V(G)} l(x) - \sum_{y \in N_G(p)} l(y) = a$*, for all $p \in V(G)$ and for some constant a.*

(2) $l(v_0) + l(v_2) = l(v_1) + l(v_3) = b$*, for some constant b.*

(3) $a = b$.

Proof. For any $x_i, x_j \in V(G)$, with $1 \leq i, j \leq n$, since

$$w(x_i) = \sum_{y \in N_G(x_i)} l(y) + \sum_{k=0}^{3} l(v_k) = w(x_j) = \sum_{y \in N_G(x_j)} l(y) + \sum_{k=0}^{3} l(v_k),$$

we have, for any $1 \leq i, j \leq n$ and for some constant k,

$$\sum_{y \in N_G(x_i)} l(y) = \sum_{y \in N_G(x_j)} l(y) = k.$$

Therefore we get, for $1 \leq i \leq n$,

$$\sum_{x\in V(G)} l(x) \; - \sum_{y\in N_G(x_i)} l(y) \; = \sum_{x\in V(G)} l(x) - k = a,$$

for some constant a. Also since

$$w(v_0) = \sum_{x\in V(G)} l(x) + l(v_1) + l(v_3), \quad w(v_1) = \sum_{x\in V(G)} l(x) + l(v_0) + l(v_2),$$

we have $l(v_1) + l(v_3) = l(v_0) + l(v_2) = b$, for some constant b. Now since $w(x_i) = w(v_0)$, we have, for $1 \leq i \leq n$,

$$\sum_{x\in V(G)} l(x) \; - \sum_{y\in N_G(x_i)} l(y) = l(v_0) + l(v_2) = b.$$

Hence,

$$\sum_{x\in V(G)} l(x) - k = a = b.$$

Conversely, suppose conditions (1), (2), and (3) hold. Then we have, for any $1 \leq i \leq n$,

$$w(x_i) = \sum_{y\in N_G(x_i)} l(y) + \sum_{k=0}^{3} l(v_k) \; = \sum_{x\in V(G)} l(x) - a + 2a \; = \sum_{x\in V(G)} l(x) + a,$$

$$w(v_m) = \sum_{x\in V(G)} l(x) + l(v_{m-1}) + l(v_{m+1}) \; = \sum_{x\in V(G)} l(x) + a,$$

where $m-1$ and $m+1$ are taken modulo 4. Hence $G + C_4$ is S-magic with S-magic constant $\sum_{x\in V(G)} l(x) + a$, for some constant a. □

Corollary 2.20. *For $n \geq 2$ and $r \geq 2$, a bijection l is an S-magic labeling of the join $K_{(n,r)} + C_4$ if and only the following conditions hold:*

(1) *The sum of labels in every partite part equals the same constant a.*

(2) *$l(v_1) + l(v_3) = l(v_0) + l(v_2) = b$, for some constant b.*

(3) *$a = b$.*

Theorem 2.21. *Let G be any graph. Let H be a graph having two vertices u and v such that $|N(u) \cap N(v)| = d(u) - 1 = d(v) - 1$. Then $G \circ H$ is not S-magic.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. Let $H_1, H_2, \ldots, H_n$ be n copies of H such that v_i is adjacent to each vertex of H_i, for $1 \le i \le n$. Let u_j^i, for $1 \le j \le m$, denote vertices in H_i, for $1 \le i \le n$. Since $|N(u) \cap N(v)| = d(u) - 1 = d(v) - 1$, there exist vertices x, y in H such that $x \in N(u)$ but $x \notin N(v)$ and $y \in N(v)$ but $y \notin N(u)$. Consider the i^{th} copy of H, i.e., H_i. Let u_{j1}^i, u_{j2}^i, u_{j3}^i, u_{j4}^i be the vertices in H_i corresponding to vertices u, v, x, y in H, respectively. Then $u_{j3}^i \in N(u_{j1}^i)$ but $u_{j3}^i \notin N(u_{j2}^i)$ and $u_{j4}^i \in N(u_{j2}^i)$ but $u_{j4}^i \notin N(u_{j1}^i)$, while the remaining neighbors of u_{j1}^i and u_{j2}^i are the same. Therefore $|N(u_{j1}^i) \triangle N(u_{j2}^i)| = 2$, and by Theorem 1.2, $G \circ H$ is not S-magic. □

Observation 2.22. Based on the above theorem, we make the following observations:

1. For any graph G and $n \neq 4$, the corona product $G \circ C_n$ is not S-magic.
2. For any graph G and $n \neq 1, 3$, the corona product $G \circ P_n$ is not S-magic.
3. For any graph G and $n \neq 1$, the corona product $G \circ K_n$ is not S-magic.
4. For any graph G and $t \geq 1$, the corona product $G \circ C_3^t$ is not S-magic.

Theorem 2.23. *Let O_m denote the empty graph on m vertices. Then, for any graph G of order at least 2, the corona product $G \circ O_m$ is not S-magic.*

Proof. Let $H = O_m$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. Let $H_1, H_2, \ldots, H_n$ be n copies of H such that v_i is adjacent to each vertex of H_i, for $1 \le i \le n$. Let u_j^i, for $1 \le j \le m$, denote vertices in H_i, for $1 \le i \le n$. Then $N(u_j^i) = \{v_i\}$ and $N(u_j^p) = \{v_p\}$. Therefore, for $i \neq p$, $|N(u_j^i) \triangle N(u_j^p)| = 2$, and by Theorem 1.2, $G \circ O_m$ is not S-magic. □

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Maurice Almeida
Department of Mathematics,
Rosary College of Commerce and Arts,
Navelim-Salcete, Goa-India
p20230078@goa.bits-pilani.ac.in