BULLETIN OF THE ICA
Volume **104** (2025), 37–50

An algorithmic approach in constructing infinitely many even size graphs with local antimagic chromatic number 3

Dedicated to Prof. S. Arumugam on the occasion of his 80th birthday

Gee-Choon Lau

Abstract. An edge labeling of a connected graph $G = (V, E)$ is said to be local antimagic if it is a bijection $f : E \to \{1, \dots, |E|\}$ such that for any pair of adjacent vertices x and y, $f^+(x) \neq f^+(y)$, where the induced vertex label $f^+(x) = \sum f(e)$, with e ranging over all the edges incident to x. The local antimagic chromatic number of G, denoted by $\chi_{la}(G)$, is the minimum number of distinct induced vertex labels over all local antimagic labelings of G. In this paper, we first introduce an algorithmic approach to construct a family of infinitely many even size non-regular tripartite graphs with $t \geq 1$ component(s) in which every component is of odd order $p \geq 9$ and of size $q = n(p + 1)$ for $n \geq 2$. We show that every graph in this family has local antimagic chromatic number 3. We then allow the m-th component to have order $p_m \geq 9$ and size $n_m(p_m + 1)$ for $n_m \geq 2$, $1 \leq m \leq t$. We prove that every such graph with all components having same order and size also has local antimagic chromatic number 3. Lastly, we construct another family of infinitely many graphs such that different components may have different order and size all of which having local antimagic chromatic number 3. Consequently, many other families of (possibly disconnected) graphs with local antimagic chromatic number 3 are also constructed.

1 Introduction

Let $G = (V, E)$ be a connected graph of order p and size q. A bijection $f : E \to \{1, 2, \cdots, q\}$ is called a *local antimagic labeling* if $f^+(u) \neq f^+(v)$ whenever $uv \in E$, where $f^+(u) = \sum_{e \in E(u)} f(e)$ and $E(u)$ is the set of

Key words and phrases: Local antimagic chromatic number, disconnected, non-regular, tripartite.
Mathematics Subject Classifications: 05C78; 05C69.

Received: 18 March 2024
Accepted: 18 June 2024

edges incident to u. The mapping f^+, which is also denoted by f_G^+, is called a *vertex labeling of* G *induced by* f, and the labels assigned to vertices are called *induced colors* under f. The *color number* of a local antimagic labeling f is the number of distinct induced colors under f, denoted by $c(f)$. Moreover, f is called a *local antimagic* $c(f)$*-coloring* and G is local antimagic $c(f)$-colorable. The *local antimagic chromatic number* $\chi_{la}(G)$ is defined to be the minimum number of colors taken over all colorings of G induced by local antimatic labelings of G (see [1, 3]). Affirmative solutions on some problems raised in [1] can be found in [6]. Let $G + H$ and mG denote the disjoint union of graphs G and H, and m copies of G, respectively. In [9], the author proved that every connected graph of order at least 3 has a local antimagic labeling. Thus, local antimagic chromatic number is well defined for all graphs without a K_2 component. For integers $a < b$, let $[a, b] = \{a, a + 1, \ldots, b\}$. For an edge e of G, let $G - e$ be the graph with edge e deleted. Very little is known about the local antimagic chromatic number of disconnected graphs (see [2, 4, 8] for some results on 2-regular graphs and forests).

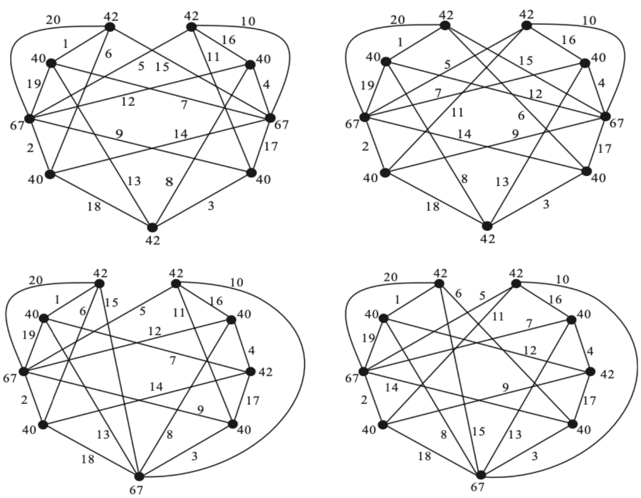

Figure 1.1: Four tripartite graphs with local antimagic chromatic number 3.

Consider the four non-isomorphic graphs in Figure 1.1. They are tripartite with exactly two degree 6 vertices and all others of degree 4. The associated edge-labeling shows that they have local antimagic chromatic number 3.

For $n \geq 2$, we can now define a family of non-regular disconnected graphs with $t \geq 1$ component(s) in which every component is a bi-regular graph of odd order $p \geq 9$ and size $q = n(p+1)$, having exactly n degree $2n+2$ vertices and $p-n$ degree $2n$ vertices, denoted $\mathcal{L}_t(p,n)$. Thus, the graphs in Figure 1.1 are in $\mathcal{L}_1(9,2)$. The m-th component of each graph in $\mathcal{L}_t(p,n)$, denoted $L_t(p,n|m,a_m^1,a_m^2,\ldots,a_m^n)$, has vertex set

$$V = \{u_{m,i} : 1 \leq m \leq t, 1 \leq i \leq p\}$$

and edge set

$$E_m = E_m^{\phi} \cup F_m \ (1 \leq \phi \leq n)$$

such that

$$\begin{aligned} E_m^1 &= \{u_{m,i}u_{m,i+1} : 1 \leq m \leq t, 1 \leq i \leq p-1\}, \\ E_m^{\phi} &= \big\{u_{m,j_i^{\phi}}u_{m,j_{i+1}^{\phi}} : 1 \leq i \leq p-1, \quad |j_{i+1}^{\phi} - j_i^{\phi}| \text{ is odd} \geq 3, \\ &\qquad\qquad 1 = j_1^{\phi} \neq j_2^{\phi} \neq \cdots \neq j_{p-1}^{\phi} \neq j_p^{\phi} = p\big\}, \end{aligned}$$

where j_i^{ϕ} and j_{i+1}^{ϕ} are of distinct parity and $E_m^{\phi} \cap E_m^{\phi'} = \varnothing$ for $2 \leq \phi \neq \phi' \leq n$, and such that

$$F_m^{\phi} = \big\{u_{m,1}u_{m,2a_m^{\phi}+1},\ u_{m,p}u_{m,2a_m^{\phi}+1}\big\}$$

for $1 \leq a_m^1 \neq \cdots \neq a_m^n \leq \frac{p-3}{2}$. Thus, the $p-1$ edges in E_m^{ϕ}, $2 \leq \phi \leq n$, also induce a path of order p with pending vertices $u_{m,1}$ and $u_{m,p}$ alternating in odd and even subscripts. Note that each graph in $\mathcal{L}_t(p,n)$ has size $tn(p+1)$. Thus, one can check that if $p = 9$, then $n = 2$ so that $|\mathcal{L}_1(9,2)| = 4$ up to isomorphism with either $a_1^1 = 1$ and $a_1^2 = 3$ or $a_1^1 = 1$ and $a_1^2 = 2$ as in Figure 1.1.

Example 1.1. A graph in $\mathcal{L}_1(13,2)$ is shown in Figure 1.2. The edges in $E_{1,2}$ induce an order 13 path

$$u_{1,1}\, u_{1,4}\, u_{1,7}\, u_{1,2}\, u_{1,5}\, u_{12}\, u_{1,9}\, u_{1,6}\, u_{1,11}\, u_{1,8}\, u_{1,3}\, u_{1,10}\, u_{1,13}$$

and $F_1^1 = \{u_{1,1}\, u_{1,5},\ u_{1,13}\, u_{1,5}\}$ and $F_1^2 = \{u_{1,1}\, u_{1,9},\ u_{1,13}\, u_{1,9}\}$.

Observe that each graph G in $\mathcal{L}_t(p,n)$ is tripartite so that $\chi_{la}(G) \geq 3$. In this paper, we prove that every such graph has local antimagic chromatic number 3. Consequently, we also define many infinite families of (possibly disconnected) related graphs and prove that all of them are tripartite graphs with local antimagic chromatic number 3.

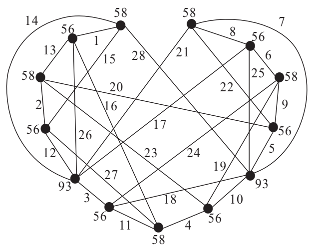

Figure 1.2: A graph in $\mathcal{L}_1(13, 2)$ with local antimagic chromatic number 3.

2 Main results

We first give an algorithmic approach to construct the m-th component of each graph in $\mathcal{L}_t(p, n)$ using $n = 2$ paths, say $P_{m,\phi}$ of order $p + 2$ with vertex set $V \cup \{x^{\phi}_{m,1}, x^{\phi}_{m,2}\}$ for $1 \le \phi \le n$. The edge set of $P_{m,1}$ is

$$\{x^1_{m,1}\, u_{m,1},\ x^1_{m,2}\, u_{m,p}\} \cup \{u_{m,i}\, u_{m,i+1} : 1 \le i \le p - 1\}.$$

For $2 \le \phi \le n$, the edge set of $P_{m,\phi}$ is

$$\begin{aligned}&\{x^{\phi}_{m,1}\, u_{m,1},\ x^{\phi}_{m,2}\, u_{m,p}\}\\ &\qquad \cup \Big\{u_{m,j^{\phi}_i}\, u_{m,j^{\phi}_{i+1}} : 1 \le i \le p-1, \quad |j^{\phi}_{i+1} - j^{\phi}_i| \text{ is odd and} \ge 3,\\ &\qquad\qquad\qquad\qquad\qquad 1 = j^{\phi}_1 \ne j^{\phi}_2 \ne \cdots \ne j^{\phi}_{p-1} \ne j^{\phi}_p = p\Big\}\end{aligned}$$

respectively such that every edge appears once in $P_{m,\phi}$ ($2 \le \phi \le n$). By merging vertices $x^{\phi}_{m,1}, x^{\phi}_{m,2}$ to $u_{m,2a^{\phi}_m+1}$ for $1 \le a^1_m \ne \cdots \ne a^n_m \le \frac{p-3}{2}$, we now get the component as required. Note that the maximum value of n is a function of p.

Theorem 2.1. *For $t \ge 1$, $n \ge 2$, and odd $p \ge 9$, every graph in $\mathcal{L}_t(p, n)$ has local antimagic chromatic number 3.*

Proof. Recall that $\chi_{la}(G) \ge 3$ for each $G \in \mathcal{L}_t(p, n)$. Take $n = 2$. Consider the m-th component $L_t(p, n|m, 1, \frac{p-3}{2})$. Let the consecutive edges of $P_{m,\phi}$, $1 \le \phi \le n$, be $e_{m,(\phi-1)(p+1)+i}$, $1 \le i \le p + 1$. Define a bijection $f \colon E\Big(L_t\big(p, n|1, \frac{p-3}{2}\big)\Big) \to \big[1, tn(p + 1)\big]$ such that for $1 \le i \le n(p + 1)/2$, $1 \le m \le t$,

$$\begin{aligned} f(e_{m,2i}) &= (m-1)n(p+1)/2 + i,\\ f(e_{m,2i-1}) &= (2t-m+1)n(p+1)/2 + 1 - i.\end{aligned}$$

Observe that

$$\begin{aligned} f^+(x_{m,1}) + f^+(x_{m,2}) &= (2t-m+1)n\tfrac{p+1}{2} + (m-1)n\tfrac{p+1}{2} + \tfrac{p+1}{2} \\ &= (2tn+1)(p+1)/2, \\ f^+(y_{m,1}) + f^+(y_{m,2}) &= (2t-m+1)n\tfrac{p+1}{2} - \tfrac{p+1}{2} + (m-1)n\tfrac{p+1}{2} + p + 1 \\ &= (2tn+1)(p+1)/2, \end{aligned}$$

and in $P_{m,\phi}$, $1 \le \phi \le n$,

$$\begin{aligned} f^+(u_{m,2i-1}) &= (m-1)n\tfrac{p+1}{2} + i + (2t-m+1)n\tfrac{p+1}{2} + 1 - i \\ &= tn(p+1) + 1 \qquad\qquad \text{for } 1 \le i \le (p+1)/2, \\ f^+(u_{m,2i}) &= (m-1)n\tfrac{p+1}{2} + i + (2t-m+1)n\tfrac{p+1}{2} + 1 - (i+1) \\ &= tn(p+1) \qquad\qquad \text{for } 1 \le i \le (p-1)/2. \end{aligned}$$

Combining the above, we can now conclude that in $L_t\big(p, n|m, 1, \tfrac{p-3}{2}\big)$

$$\begin{aligned} f^+(u_{m,2i-1}) &= tn^2(p+1) + n \text{ for } i \ne 2,\ (p-1)/2, \\ f^+(u_{m,2i}) &= tn^2(p+1), \\ f^+(u_{m,3}) &= f^+(u_{m,p-2}) \\ &= (2tn+1)(p+1)/2 + tn^2(p+1) + n \\ &= (2tn^2 + 2tn + 1)(p+1)/2 + n. \end{aligned}$$

Thus, $L_t\big(p, n|m, 1, \tfrac{p-3}{2}\big)$ admits a local antimagic 3-coloring. Furthermore, since $L_t\big(p, n|m, 1, \tfrac{p-3}{2}\big)$ is tripartite, $\chi_{la}\Big(L_t\big(p, n|m, 1, \tfrac{p-3}{2}\big)\Big) = 3$.

In general, for $n \ge 3$, each component of every graph in $\mathcal{L}_t(p, n)$ can be obtained by defining a suitable $P_{m,\phi}$ $(3 \le \phi \le n)$. Labeling the edges according to the function defined above, we immediately get a required local antimagic 3-coloring for every graph in $\mathcal{L}_t(p, n)$. This completes the proof. □

Example 2.2. For $p = 11$ and $t = 2$, using the path $P_{1,2}$ with edge set

$$\big\{x^2_{m,1}\, u_{m,1},\ u_{m,i}\, u_{m,i+3},\ u_{m,p}\, x^2_{m,2} : 1 \le i \le p-1\big\}$$

(modulo $p-1$ when $i = p-2$ or $p-1$), we can get

$$\mathcal{L}_2(11, 2) = \big\{L_2(11, 2|1, a, b) + L_2(11, 2|2, a', b')\big\}$$

with $(a, b), (a', b') \in \{(1, 2),\ (1, 3),\ (1, 4),\ (2, 3),\ (2, 4),\ (3, 4)\}$. The graph $L_2(11, 2|1, 1, 4) + L_2(11, 2|2, 1, 3)$ is given in Figure 2.1.

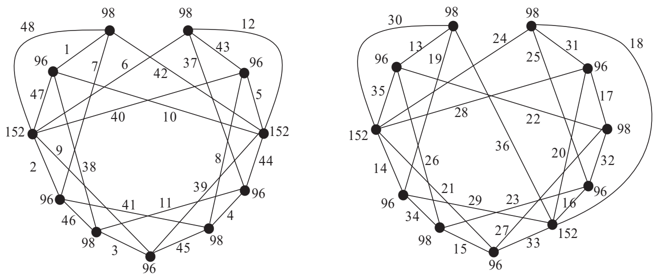

Figure 2.1: Graph $L_2(11, 2|1, 1, 4) + L_2(11, 2|2, 1, 3)$ in $\mathcal{L}_2(11, 2)$.

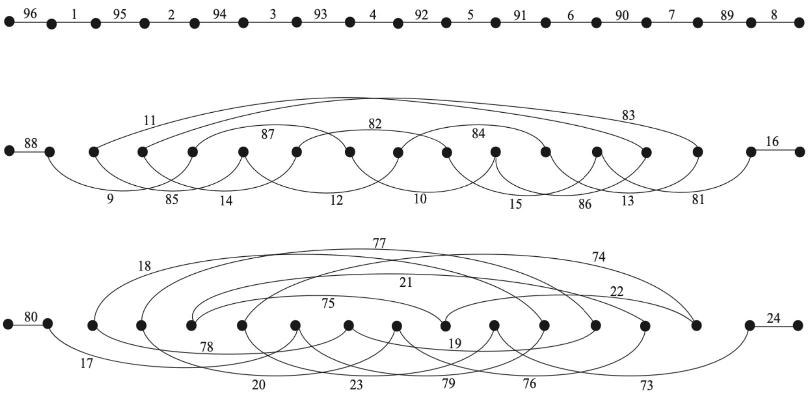

Figure 2.2: Paths $P_{1,1}$, $P_{1,2}$, $P_{1,3}$ that give the first component.

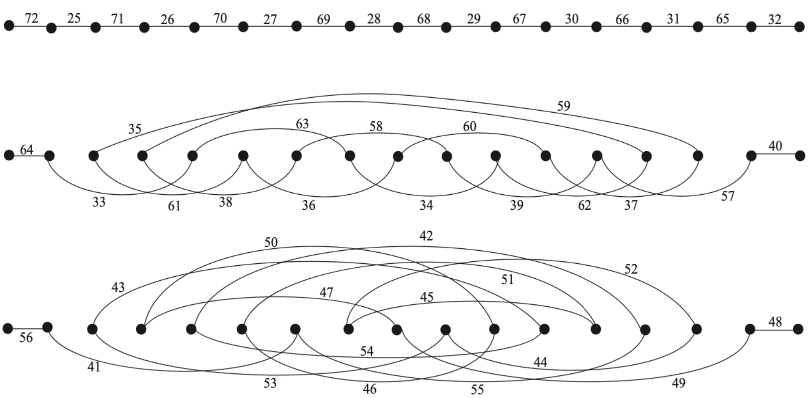

Figure 2.3: Paths $P_{2,1}$, $P_{2,2}$, $P_{2,3}$ that give the second component.

Example 2.3. Taking $t = 2$, $p = 15$, and $n = 3$, we can get various non-isomorphic 2-component graphs. One of them with the paths $P_{m,\phi}$ and the corresponding edge labels are given in Figures 2.2 and 2.3, respectively. The end vertices of each path are merged to one of the vertices with induced vertex label 97 bijectively, for example, the 3^{rd}, 5^{th}, and 7^{th} vertex (from the left) of degree 2, respectively. Each component of the resulting graph has two degree 8 vertices with induced vertex label 395, seven vertices of degree 6 with induced vertex label 291, and six vertices of degree 6 with induced vertex label 288.

For $9 \le p_1 \le p_2 \le \cdots \le p_t$, each p_i is odd for $1 \le i \le t$, $t \ge 2$, we can now define a family of more general graph, denoted

$$\mathcal{GL}_t\big((p_1, n_1), (p_2, n_2), \ldots, (p_t, n_t)\big),$$

$n_i \ge 2$, $1 \le i \le t$ (also denoted $\mathcal{GL}_t(p, n)$ if $p_i = p$, $n_i = n$) which is a family of disconnected graphs with t compoments such that the m-th component is $L_t(p, n_m | m, a_m^1, a_m^2, \ldots, a_m^{n_m})$ that has n_m vertices of degree $2n_m + 2$ and $p_m - n_m$ vertices of degree $2n_m$. Note that $\mathcal{GL}_t(p, n) = \mathcal{L}_t(p, n)$.

Observation 2.4. If $j_{i+1} - j_i$ is an odd constant $k \ge 3$ for all i, we may apply the concept of k-step Hamiltonian tour of cycles for odd k (see [5, Theorem 2.5]) that can give us an algorithmic approach to obtain the edges in $E_m^\phi, 2 \le \phi \le n_m$.

Example 2.5. Suppose $p_i = 23$ for $1 \le i \le t$, using the concept of k-step Hamiltonian tour, we can have a graph in $\mathcal{L}_t(23, n_i)$ for $k \in \{1\} \cup N$ such that $N \subseteq \{3, 5, 7, 9\}$ using $k \in \{1, 3, 5, 7, 9\}$. Note that $k \le \frac{p_i - 3}{2}$ and $2 \le n_i \le |N| + 1$. The corresponding paths of the m-th component can be as follows.

(i) For $k = 1$, the induced path is $u_{m,1}\, u_{m,2}\, \cdots\, u_{m,23}$.

(ii) For $k = 3$, the induced path is $u_{m,1}\, u_{m,4}\, u_{m,7}\, u_{m,10}\, u_{m,13}\, u_{m,16}\, u_{m,19}\, u_{m,22}\, u_{m,3}\, u_{m,6}\, u_{m,9}\, u_{m,12}\, u_{m,15}\, u_{m,18}\, u_{m,21}\, u_{m,2}\, u_{m,5}\, u_{m,8}\, u_{m,11}\, u_{m,14}\, u_{m,17}\, u_{m,20}\, u_{m,23}$.

(iii) For $k = 5$, the induced path is $u_{m,1}\, u_{m,6}\, u_{m,11}\, u_{m,16}\, u_{m,21}\, u_{m,4}\, u_{m,9}\, u_{m,14}\, u_{m,19}\, u_{m,2}\, u_{m,7}\, u_{m,12}\, u_{m,17}\, u_{m,22}\, u_{m,5}\, u_{m,10}\, u_{m,15}\, u_{m,20}\, u_{m,3}\, u_{m,8}\, u_{m,13}\, u_{m,18}\, u_{m,23}$.

(iv) For $k = 7$, the induced path is $u_{m,1}\, u_{m,8}\, u_{m,15}\, u_{m,22}\, u_{m,7}\, u_{m,14}\, u_{m,21}\, u_{m,6}\, u_{m,13}\, u_{m,20}\, u_{m,5}\, u_{m,12}\, u_{m,19}\, u_{m,4}\, u_{m,11}\, u_{m,18}\, u_{m,3}\, u_{m,10}\, u_{m,17}\, u_{m,2}\, u_{m,9}\, u_{m,16}\, u_{m,23}$.

(v) For $k = 9$, the induced path is $u_{m,1}\, u_{m,10}\, u_{m,19}\, u_{m,6}\, u_{m,15}\, u_{m,2}\, u_{m,11}$ $u_{m,20}\, u_{m,7}\, u_{m,16}\, u_{m,3}\, u_{m,12}\, u_{m,21}\, u_{m,8}\, u_{m,17}\, u_{m,4}\, u_{m,13}\, u_{m,22}\, u_{m,9}$ $u_{m,18}\, u_{m,5}\, u_{m,14}\, u_{m,23}$.

Each of the paths above will have both end-vertices joining to a single vertex in $\{u_{m,i} : i = 3, 5, 7, \ldots, 21\}$ bijectively.

Problem 2.6. Study $\chi_{la}(G)$ for

$$G \in \mathcal{GL}_t\big((p_1, n_1), (p_2, n_2), \ldots, (p_t, n_t)\big) \setminus \mathcal{GL}_t(p, n).$$

Observe that for graphs in $\mathcal{GL}_t(p, n)$, $t \geq 2$, $n \geq 2$, $p \geq 9$ is odd, and the defined local antimagic 3-coloring f, we may say every vertex of degree $2n$ is incident to $2n'$ edges (for $1 \leq n' < n$) with equal labels sum in $\big\{n'[tn(p+1)+1],\, n'[tn(p+1)]\big\}$ (respectively, of degree $2n+2$ is incident to $2n'$ edges (for $1 \leq n' \leq n$) with equal label sum $n'[(2tn^2 + 2tn + 1)(p + 1)/2 + n]$). We can now construct new families of connected graphs using an *edge-swap* process as follows:

(1) Choose 2 vertices, say x and y, of same induced vertex label from two different components.

(2) For each vertex, identify any $2n'$ $(1 \leq n' < n)$ incident edges with equal labels sum in $\big\{n'[tn(p + 1) + 1],\, n'[tn(p + 1)]\big\}$ (respectively, identify any $2(n' + n'')$ incident edges with equal labels sum $n'[tn(p+1)+1]+n''(2tn+1)(p+1)/2$ for $n'+n'' \leq n$, $1 \leq n' \leq n$, $n'' \in \{0, 1\}$).

(3) Redraw the $4n'$ edges (respectively, the $4(n' + n'')$ edges) so that the $2n'$ edges (respectively, the $2(n' + n'')$ edges) that are originally incident to x of a component are now incident to y, and vice versa.

(4) Repeat the above process for as many times as possible.

Consequently, by keeping all the edge labels, the new graph obtained is a connected graph that preserved the induced vertex labels. Note that all the graphs such obtained are still tripartite. Let the family of graphs such obtained using $t \geq 2$ components graph in $\mathcal{GL}_t(p, n)$ be denoted $\mathcal{CL}_t(p, n)$. We immediately have the following theorem with the proof omitted.

Theorem 2.7. *Every graph in $\mathcal{CL}_t(p, n)$ has local antimagic chromatic number 3.*

Example 2.8. Using the component $L_2(11, 2|1, 1, 4)$ in Figure 2.1, we can get various non-isomorphic graphs in $\mathcal{CL}_2(11, 2)$. Two of them are given in Figures 2.4 and 2.5.

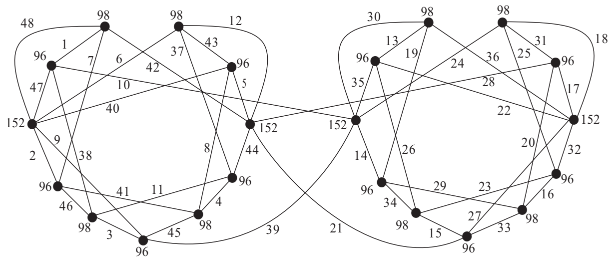

Figure 2.4: A graph in $\mathcal{CL}_2(11,2)$ by applying one time edge-swap.

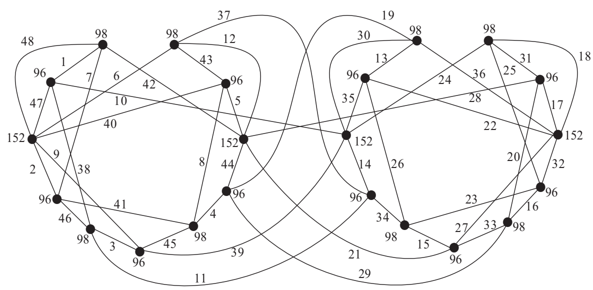

Figure 2.5: A graph in $\mathcal{CL}_2(11,2)$ by applying two times edge-swap.

More generally, we can also apply the idea of labeling graphs in $\mathcal{GL}_t(p,n)$ to define a familiy of graphs, denoted $\mathcal{GCL}_{s_1,\dots,s_t}(p,n)$ with $s_i \ge 2$ for at least an $i \in [1,t]$ and the i-th component is in $\mathcal{CL}_{s_i}(p,n)$, if $s_i \ge 2$, such that the graphs obtained also has local antimagic chromatic number 3. The theorem is stated below without proof.

Theorem 2.9. *Every graph $G \in \mathcal{GCL}_{s_1,\dots,s_t}(p,n)$ has $\chi_{la}(G) = 3$.*

Example 2.10. We give a 3-component example in Figure 2.6 with $s_1 = s_2 = 1$ and $s_3 = 2$ using the two graphs in Example 2.8.

Further observe that the local antimagic 3-coloring f defined in the proof of Theorem 2.1 induces a 3-independent partition with sizes tn, $t(p-2n+1)/2$, and $t(p-1)/2$ with vertices of degree $2n+2$, $2n$, and $2n$, respectively. Let $t = rs \ge 2$, $r \ge 1$, and $s \ge 2$. Partition the t components of each graph in $\mathcal{GL}_t(p,n) = \mathcal{L}_t(p,n)$ into r mutually disjoint set(s), say $D_1, D_2, \dots, D_r$, of s components. For each graph in $\mathcal{GL}_t(p,n)$, we now construct three families of graphs as follows. Consider $1 \le \rho \le r$.

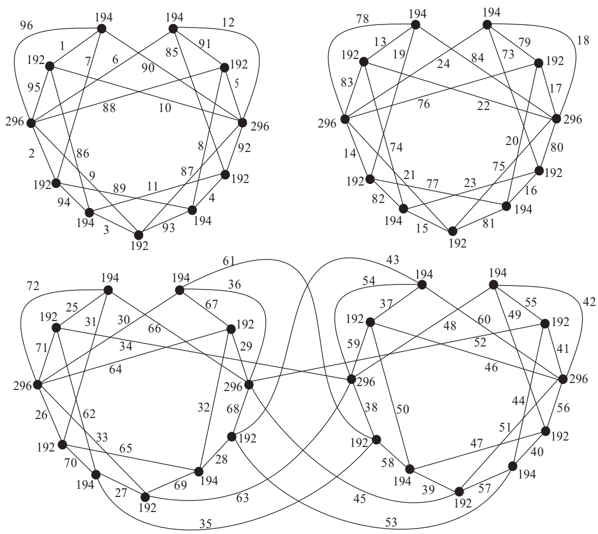

Figure 2.6: A 3-component graph in $\mathcal{GCL}_{1,1,2}(11,2)$.

(a) Let $\mathcal{GL}_r^1\big(s(p-n)+n,n\big)$ be the family of graph(s) with r component(s) such that the ρ-th component is of order $s(p-n)+n$ obtained by merging the k-th ($k\in[1,n]$) degree $2n+2$ vertex (with induced vertex label $(2tn^2+2tn+1)(p+1)/2+n$) of each component in D_ρ bijectively. Each component of the graph obtained has n vertices of degree $s(2n+2)$, $s(p-2n+1)/2$ vertices of degree $2n$, and another $s(p-1)/2$ vertices of degree $2n$.

(b) Let $\mathcal{GL}_r^2\big(s(p+2n-1)/2+(p-2n+1)/2,n\big)$ be the family of graph(s) with r component(s) such that the ρ-th component is of order $s(p+2n-1)/2+(p-2n+1)/2$ obtained by merging the k-th ($k\in[1,(p-2n+1)/2]$) degree $2n$ vertex (with induced vertex label $tn^2(p+1)+n$) of each component in D_ρ bijectively. Each component of the graph obtained has $(p-2n+1)/2$ vertices of degree $2sn$, sn vertices of degree $2n+2$, and another $s(p-1)/2$ vertices of degree $2n$.

(c) Let $\mathcal{GL}_r^3\big(s(p+1)/2+(p-1)/2,n\big)$ be the family of graph(s) with r component(s) such that the ρ-th component is of order $s(p+1)/2+(p-1)/2$ obtained by merging the k-th ($k\in[1,(p-1)/2]$) degree $2n$ vertex (with induced vertex label $tn^2(p+1)$) of each component in D_ρ bijectively. Each component has $(p-1)/2$ vertices of degree $2sn$, sn vertices of degree $2n+2$, and another $s(p-2n+1)/2$ vertices of degree $2n$.

Theorem 2.11. *If* $G \in \mathcal{GL}_r^1\big(s(p-n)+n,n\big) \cup \mathcal{GL}_r^2\big(s(p+2n-1)/2+(p-2n+1)/2,n\big) \cup \mathcal{GL}_r^3\big(s(p+1)/2+(p-1)/2,n\big)$*, then* $\chi_{la}(G)=3$*.*

Proof. Clearly, $\chi_{la}(G) \ge \chi(G) = 3$. Keeping the local antimagic 3-coloring for the corresponding graph as defined in the proof of Theorem 2.1. Suppose $G \in \mathcal{GL}_r^1\big(s(p-n)+n,n\big)$. Thus, G admits an edge labeling such that, in each component, there are n degree $s(2n+2)$ vertices with induced label $s\big((2tn^2+2tn+1)(p+1)/2+n\big)$ and the remaining degree $2n$ vertices still have induced vertex labels $tn^2(p+1)+n$ or $tn^2(p+1)$.

Suppose $G \in \mathcal{GL}_r^2\big(s(p+2n-1)/2+(p-2n+1)/2,n\big)$. Thus, G admits an edge labeling such that, in each component, there are $(p-2n+1)/2$ degree $2sn$ vertices with induced label $s\big(tn^2(p+1)+n\big)$; the sn degree $2n+2$ vertices and the remaining $s(p-1)/2$ degree $2n$ vertices still have induced vertex labels $(2tn^2+2tn+1)(p+1)/2+n$ and $tn^2(p+1)$, respectively.

Suppose $G \in \mathcal{GL}_r^3\big(s(p+1)/2+(p-1)/2,n\big)$. Thus, G admits an edge labeling such that, in each component, there are $(p-1)/2$ degree $2ns$ vertices with induced label $s\big(tn^2(p+1)\big)$; the sn degree $2n+2$ vertices and the remaining $s(p-2n+1)/2$ degree $2n$ vertices still have induced vertex labels $(2tn^2+2tn+1)(p+1)/2+n$ and $tn^2(p+1)+n$, respectively.

In each case, it is easy to check that all the induced vertex labels are distinct so that G admits a local antimagic 3-coloring. Thus, $\chi_{la}(G) \le 3$. The proof is complete. □

Note that when p is sufficiently large, it is possible that every component of a graph in $\mathcal{GL}_t(p,n)$ has the n degree $2n+2$ vertices, say $u_1,\dots,u_n$, that are not adjacent to nor have common neighbors with n degree $2n$ vertices of same induced vertex label, say $v_1,\dots,v_n$, under the local antimagic 3-coloring f as defined. Let G be the graph, necessarily without multiple edges nor loops, obtained by merging $u_1,\dots,u_n$ with $v_1,\dots,v_n$ bijectively. We now define three more families of graphs naturally as follows.

(a) Let $\mathcal{GL}_t^4(p-n,n)$ be the family of graphs with $t \ge 1$ component(s) such that the m-th component is of order $p-n$ obtained by merging the n degree $2n+2$ vertices bijectively with n degree $2n$ vertices that have induced label $tn^2(p+1)+n$.

(b) Let $\mathcal{GL}_t^5(p-n,n)$ be the family of graphs with $t \ge 1$ component(s) such that the m-th component is of order $p-n$ obtained by merging the n degree $2n+2$ vertices bijectively with n degree $2n$ vertices that have induced label $tn^2(p+1)$.

(c) Let $\mathcal{GL}_t^6(p-n+1,n)$ be the family of graphs with $t \geq 1$ component(s) such that the m-th component is of order $p-n+1$ obtained by merging all the n degree $2n+2$ vertices into a vertex of degree $n(2n+2)$.

Theorem 2.12. *If $G \in \mathcal{GL}_t^4(p-n,n) \cup \mathcal{GL}_t^5(p-n,n) \cup \mathcal{GL}_t^6(p-n+1,n)$, then $\chi_{la}(G) = 3$.*

Proof. By definition, G is still tripartite so that $\chi_{la}(G) \geq \chi(G) = 3$. (a) Note that u_i $(1 \leq i \leq n)$ has induced vertex label $(2tn^2+2tn+1)(p+1)/2+n$. If v_i $(1 \leq i \leq n)$ has induced vertex label $tn^2(p+1)+n$, after merging, the degree $4n+2$ vertices obtained have induced vertex labels $(4tn^2+2tn+1)(p+1)/2+2n$, which is larger than the remaining degree $2n$ vertices with induced vertex labels $tn^2(p+1)+n$ and $tn^2(p+1)$. (b) Similarly if v_i has induced vertex label $tn^2(p+1)$. (c) The degree $n(2n+2)$ vertex has induced vertex label $n(2tn^2+2tn+1)(p+1)/2+n^2$ while the other vertices have the same induced vertex labels. Thus, $\chi_{la}(G) \leq 3$. This completes the proof. $\square$

Example 2.13. A graph in $\mathcal{L}_1(17,2)$ that can give a graph in Theorem 2.12 is shown in Figure 2.7.

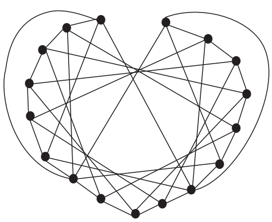

Figure 2.7: A graph in $\mathcal{L}_1(17,2)$.

Remark 2.14. We note that the construction of graphs in Theorems 2.11 and 2.12 using the graphs in $\mathcal{GL}_t(p,n)$ can be done by using the graphs in $\mathcal{CL}_t(p,n) \cup \mathcal{GCL}_{s_1,\ldots,s_t}(p,n)$ as well so that theorems similar to Theorems 2.11 and 2.12 can be obtained, too.

3 Conclusion and open problems

In this paper, we introduce algorithmic approaches to contruct various families of non-regular (possibly disconnected) tripartite graphs of even size and prove that all these graphs have local antimagic chromatic number 3.

In [7, Lemmas 2.3 & 2.4], the authors obtained sufficient conditions to have $\chi_{la}(G) = \chi_{la}(G-e)$. Let e be an edge of a graph G in any of the theorems in Section 2 with label 1 or $|E(G)|$ under the local antimagic 3-coloring f as defined. One may check the conditions in [7, Lemmas 2.3 & 2.4] to obtain the exact value of $\chi_{la}(G-e)$. In general, we can have the following problem.

Problem 3.1. If e is an edge of G, determine $\chi_{la}(G-e)$.

Observe that if we redefine graphs in $\mathcal{GL}_t(p,n)$ so that the degree $2n+2$ vertices are $u_{m,i}$ and $u_{m,i'}$ where i, i' not both odd for $1 \le m \le t$, then we have either a bipartite graph $B \in \mathcal{B}$ with distinct partite set size or else a tripartite graph $T \in \mathcal{T}$ that admits a local antimagic 3- or 4-coloring, respectively. Thus, $2 \le \chi_{la}(B) \le 3$ and $3 \le \chi_{la}(T) \le 4$.

Problem 3.2. Determine $\chi_{la}(B)$ and $\chi_{la}(T)$.

We end this paper with the following problems.

Problem 3.3. Determine the maximum value of n for each possible value of p.

Problem 3.4. Study the properties of the various graph polynomials (and uniqueness), and graph parameters of every graph in Section 2 such as (but not limited to) chromatic, domination, independent, star (or neighborhood) polynomials, magicness, antimagicness, Roman domination number, and Sudoku number.

4 Statements and declarations

No funds, grants, or other support was received. The author has no relevant financial or non-financial interests to disclose.

References

[1] S. Arumugam, K. Premalatha, M. Bača and A. Semaničová-Feňovčíková, Local antimagic vertex coloring of a graph, *Graphs Combin.*, **33** (2017), 275–285.

[2] M. Bača, A. Semaničová-Feňovčíková, and T. M. Wang, Local antimagic chromatic number for copies of graphs, *Mathematics*, **9**, 1230 (2021), `https://doi.org/10.3390/math9111230`.

[3] J. Bensmail, M. Senhaji and K. Szabo Lyngsie, On a combination of the 1-2-3 conjecture and the antimagic labelling conjecture, *Discrete Math. Theoret. Comput. Sci.*, **19**(1) (2017), Paper no. 21, 17 pp.

[4] T. L. Chan, G. C. Lau and W. C. Shiu, Complete solutions on local antimagic chromatic number of three families of disconnected graphs, *Commun. Comb. Optim.*, (2024), `https://doi.org/10.22049/CCO.2024.29032.1818`.

[5] G. C. Lau, S. M. Lee, K. Schaffer and S. M. Tong, On k-step Hamiltonian bipartite and tripartite graphs, *Malaya J. Mat.*, **2**(3) (2014), 180–187.

[6] G. C. Lau, H. K. Ng and W. C. Shiu, Affirmative solutions on local antimagic chromatic number, *Graphs Combin.*, **36** (2020), 1337–1354.

[7] G. C. Lau, W. C. Shiu and H. K. Ng, On local antimagic chromatic number of cycle-related join graphs, *Discuss. Math. Graph Theory*, **4(1)** (2021), 133–152, `https://doi.org/10.7151/dmgt.2177`.

[8] G. C. Lau and W. C. Shiu, On local antimagic chromatic number of lexicographic product graphs, *Acta Math. Hungar.*, **169**(1) (2023), 158–170.

[9] J. Haslegrave, Proof of a local antimagic conjecture, (2017), `https://doi.org/10.23638/DMTCS-20-1-18`.

Gee-Choon Lau
College of Computing, Informatics & Mathematics,
Universiti Teknologi MARA (Segamat Campus),
85000, Johor, Malaysia.
geeclau@yahoo.com