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Note on double bound multigraphs of posets

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Abstract. For a poset $P=(X, \leq_P)$, the *double bound multigraph* of P, denoted $\mathrm{DBM}(P)$, is the multigraph on X for which vertices u and v of $\mathrm{DBM}(P)$ are joined by multiple edges with multiplicity equal to the number of pairs of common upper bound elements and common lower bound elements of u and v in P. We obtain a characterization of double bound multigraphs.

1 Introduction

In this paper we consider finite graphs, finite multigraphs, finite posets, and finite digraphs. For a poset $P = (X, \leq_P)$ and $u \in X$, we define $U_P(u) := \{v \in X \mid u \leq_P v\}$ and $L_P(u) := \{v \in X \mid v \leq_P u\}$. For a poset $P = (X, \leq_P)$, we can consider $u \leq_P v$ as $u \to v$. Then a poset is considered as a reflexive, acyclic, transitive digraph.

For a multigraph M and two distinct vertices u and v, $m_M(u, v)$ is the number of multiple edges between u and v, which is the *multiplicity* of u and v.

For a poset $P = (X, \leq_P)$, the *double bound multigraph* (or DBM-*graph*) of P is the multigraph $\mathrm{DBM}(P)$ on X for which, given vertices u and v of $\mathrm{DBM}(P)$, there exist $m_M(u, v)$ multiple edges between u and v, where $m_M(u, v)$ is the number of pairs (x, y) such that $x \leq_P u \leq_P y$ and $x \leq_P v \leq_P y$. Note that for the double bound multigraph M of a poset P, $m_M(u, v) = |U_P(u) \cap U_P(v)| \times |L_P(u) \cap L_P(v)|$. In this paper we deal with double bound multigraphs and give a characterization of double bound multigraphs.

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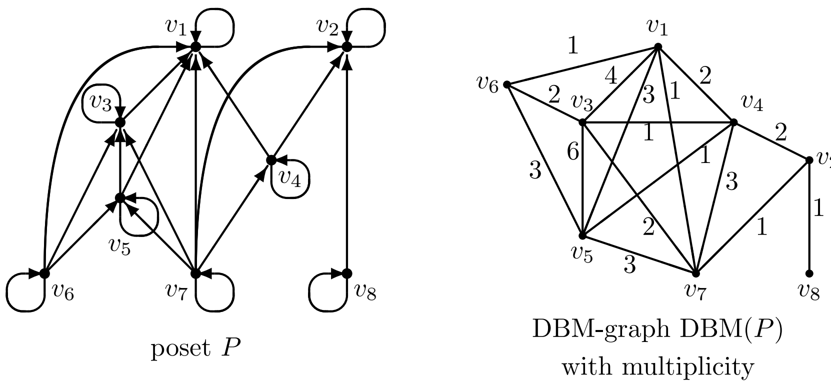

Figure 1.1: Example of a poset and a DBM-graph.

A *clique* in a graph G is the vertex set of a complete subgraph of G. For a multigraph M, let $M{\downarrow}$ be the underlying (simple) graph of M. A *clique* of the multigraph M is a simple subgraph of M whose vertex set induces a clique of $M{\downarrow}$. Hence, a clique of a multigraph contains exactly one edge from multiple edges joining two of its vertices. For a multigraph M, a family $\mathcal{F}$ of cliques of M is an *edge clique partition* of M if any two distinct vertices u and v are contained in exactly $m_M(u,v)$ cliques of the family $\mathcal{F}$.

McMorris and Zaslavsky [2] introduced the concepts of upper bound graphs and double bound graphs. They also gave a characterization of upper bound graphs. Diny [1] gave a characterization of double bound graphs. See [4], for results on upper bound graphs, double bound graphs, and so on. McKee [3] also introduced a concept of upper bound multigraphs and gave a characterization of upper bound multigraphs. In this paper we consider double bound multigraphs.

Park and Sano [5] dealt with the double competition multigraphs of digraphs. For a digraph D and $u \in V(D)$, $\mathrm{Out}_D(u) := \{v \in V(D) \mid u \to v \in A(D)\}$ and $\mathrm{In}_D(u) := \{v \in V(D) \mid v \to u \in A(D)\}$. Then $\mathrm{Out}_D(u)$ is called the *out-neighborhood* of u and $\mathrm{In}_D(u)$ is called the *in-neighborhood* of u. Note that if a digraph D is a reflexive, acyclic, transitive digraph (i.e., a poset P), then $\mathrm{Out}_D(u) = U_P(u)$ and $\mathrm{In}_D(u) = L_P(u)$. The *double competition multigraph* M of a digraph D is the multigraph that has the same vertex set as D and two vertices are joined by multiple edges with multiplicity equal to $|\,\mathrm{Out}_D(u) \cap \mathrm{Out}_D(v)| \times |\,\mathrm{In}_D(u) \cap \mathrm{In}_D(v)|$. In [5] Park and Sano gave a concept of double competition multigraphs of digraphs and some results on double competition multigraphs as follows. For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$.

Theorem 1.1 (Park and Sano [5]). *Let M be a multigraph with n vertices. Then M is the double competition multigraph of an arbitrary digraph if and only if there exist an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following condition holds:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$,*

where A_i and B_j are the sets defined by

(1) $A_i = S_{i*} \cup T_i^+$, $S_{i*} := \bigcup_{p \in [n]} S_{ip}$, $T_i^+ := \{v_b \mid a, b \in [n],\ v_i \in S_{ab}\}$,

(2) $B_j = S_{*j} \cup T_j^-$, $S_{*j} := \bigcup_{q \in [n]} S_{qj}$, $T_j^- := \{v_a \mid a, b \in [n],\ v_j \in S_{ab}\}$.

A digraph D is called *reflexive* if $v \to v \in A(D)$ holds for each vertex $v \in V(D)$.

Theorem 1.2 (Park and Sano [5]). *Let M be a multigraph with n vertices. Then M is the double competition multigraph of a reflexive digraph if and only if there exist an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of M and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that the following conditions hold:*

(I) *for any $i, j \in [n]$, if $|A_i \cap B_j| \geq 2$, then $A_i \cap B_j = S_{ij}$,*

(III) *for any $i \in [n]$, $v_i \in S_{i*} \cap S_{*j}$,*

where A_i and B_j are the sets defined by

(1) $A_i = S_{i*} \cup T_i^+$, $S_{i*} := \bigcup_{p \in [n]} S_{ip}$, $T_i^+ := \{v_b \mid a, b \in [n],\ v_i \in S_{ab}\}$,

(2) $B_j = S_{*j} \cup T_j^-$, $S_{*j} := \bigcup_{q \in [n]} S_{qj}$, $T_j^- := \{v_a \mid a, b \in [n],\ v_j \in S_{ab}\}$.

A digraph D is called *acyclic* if D has no directed cycles. An ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of D is called an *acyclic ordering* of D if $v_i \to v_j$ implies $i < j$. Note that a digraph D is acyclic if and only if D has an acyclic ordering.

Theorem 1.3 (Park and Sano [5]). *Let M be a multigraph with n vertices. Then M is the double competition multigraph of an acyclic digraph if and* *only if there exist an ordering* $(v_1, v_2, \dots, v_n)$ *of the vertices of* M *and a double indexed edge clique partition* $\{S_{ij} \mid i, j \in [n]\}$ *of* M *such that the following conditions hold:*

(I) *for any* $i, j \in [n]$, *if* $|A_i \cap B_j| \geq 2$, *then* $A_i \cap B_j = S_{ij}$,

(IV) *for any* $i, j, k \in [n]$, $v_k \in S_{ij}$ *implies* $i < k < j$,

where A_i *and* B_j *are the sets defined by*

(1) $A_i = S_{i*} \cup T_i^+$, $S_{i*} := \bigcup_{p \in [n]} S_{ip}$, $T_i^+ := \{v_b \mid a, b \in [n],\ v_i \in S_{ab}\}$,

(2) $B_j = S_{*j} \cup T_j^-$, $S_{*j} := \bigcup_{q \in [n]} S_{qj}$, $T_j^- := \{v_a \mid a, b \in [n],\ v_j \in S_{ab}\}$.

We consider posets in terms of digraphs. Posets are reflexive, acyclic, transitive digraphs, where a digraph D is said to be *transitive* if $u \to w \in A(D)$ and $w \to v \in A(D)$ imply $u \to v \in A(D)$. Thus the double bound multigraph is the double competition multigraph of a reflexive, acyclic, transitive digraph. From examination of the proofs of Theorems 1.1, 1.2, and 1.3 from Park and Sano [5], we obtain a characterization of double bound multigraphs.

2 A characterization of double bound multigraphs

We obtain a characterization of double bound multigraphs as follows:

Theorem 2.1. *Let* M *be a multigraphs with* n *vertices. Then* M *is the double competition multigraph of a reflexive, acyclic, transitive digraph—i.e.,* M *is the double bound multigraph—if and only if there exist an ordering* $(v_1, v_2, \dots, v_n)$ *of the vertices of* M *and a double indexed edge clique partition* $\{S_{ij} \mid i, j \in [n]\}$ *of* M *such that the following conditions hold:*

(I) *for any* $i, j \in [n]$, *if* $|A_i \cap B_j| \geq 2$, *then* $A_i \cap B_j = S_{ij}$,

(III) *for any* $i \in [n]$, $v_i \in S_{i*} \cap S_{*j}$,

(IV) *for any* $i, j, k \in [n]$, $v_k \in S_{ij}$ *implies* $i < k < j$,

(V) *for any* $i, j \in [n]$, $S_{i,j} \neq \varnothing$ *implies* $v_i \in S_{ij}$,

where A_i *and* B_j *are the sets defined by*

(1) $A_i = S_{i*} \cup T_i^+$, $S_{i*} := \bigcup_{p \in [n]} S_{ip}$, $T_i^+ := \{v_b \mid a, b \in [n],\ v_i \in S_{ab}\}$,

(2) $B_j = S_{*j} \cup T_j^-$, $S_{*j} := \bigcup_{q \in [n]} S_{qj}$, $T_j^- := \{v_a \mid a, b \in [n],\ v_j \in S_{ab}\}$.

Proof. In [5], Park and Sano consider the double competition multigraph M of a digraph D, $S_{ij} = \{v_k \mid v_i \to v_k, v_k \to v_j \in A(D)\}$ for any $i, j \in [n]$, and $\mathcal{F} = \{S_{ij} \mid i, j \in [n],\ |S_{ij}| \geq 2\}$. Then S_{ij} is a clique of M, and $\mathcal{F}$ is an edge clique partition of M. For M with these S_{ij} and $\mathcal{F}$, Park and Sano proved that condition (I) holds. And for the double competition multigraph M of a reflexive, acyclic digraph D with these S_{ij} and $\mathcal{F}$, they also proved that conditions (III) and (IV) hold.

We consider the double bound multigraph, i.e., the double competition multigraph M of a reflexive, acyclic, transitive digraph D. In the same way of the proofs of Park and Sano, we can prove that conditions (I), (III), and (IV) hold for M with these S_{ij} and $\mathcal{F}$. So we consider condition (V).

Then $S_{ij} \neq \varnothing$ implies that there exists v_k such that $v_i \to v_k$ and $v_k \to v_j \in A(D)$, and $v_i \to v_j \in A(D)$ because D is transitive. Since D is reflexive, $v_i \to v_i \in A(D)$. Thus, $v_i \to v_i$ and $v_i \to v_j$ imply $v_i \in S_{ij}$. Therefore condition (V) holds.

Conversely, in [5] Park and Sano dealt with a multigraph M with n vertices and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that conditions (I), (III), and (IV) hold. Park and Sano gave a digraph D with $V(D) = V(M)$ and

$$A(D) = \bigcup_{i,j \in [n]} \left(\bigcup_{v_k \in S_{ij}} \{v_i \to v_k,\ v_k \to v_j\} \right).$$

They prove that M is the double competition multigraph of D and that D is a reflexive, acyclic digraph.

We consider a multigraph M with n vertices and a double indexed edge clique partition $\{S_{ij} \mid i, j \in [n]\}$ of M such that conditions (I), (III), (IV), and (V) hold. Let D be a digraph with $V(D) = V(M)$ and

$$A(D) = \bigcup_{i,j \in [n]} \left(\bigcup_{v_k \in S_{ij}} \{v_i \to v_k,\ v_k \to v_j\} \right).$$

In the same way of the proofs of Park and Sano, we can show that M is the double competition multigraph of D and that D is a reflexive, acyclic digraph.

By the definition of D, $v_i \to v_k$ and $v_k \to v_j$ in $A(D)$ imply that there exists S_{ij} such that $v_k \in S_{ij}$, and $S_{ij} \neq \varnothing$. By condition (V), $S_{ij} \neq \varnothing$ implies that $v_i \in S_{ij}$. Thus there exist $v_i \to v_i$ and $v_i \to v_j$, because $v_i \in S_{ij}$. So, $v_i \in S_{ij}$ implies that there exists $v_i \to v_j$. Therefore $v_i \to v_k$ and $v_k \to v_j$ imply $v_i \to v_j$, i.e., D is a transitive digraph.

Therefore, D is a reflexive, acyclic, transitive digraph, and M is the double competition multigraph of a reflexive, acyclic, transitive digraph. In other words, D is a poset, and M is the double bound multigraph. $\square$

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