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Γ-supermagic labeling of 4-regular Archimedean graphs with dihedral groups

Dalibor Froncek

Abstract. A Γ-supermagic labeling of a graph $G = (V, E)$ is a bijection from E to a group Γ of order $|E|$ such that for every vertex $x \in V$ a product of labels of all edges incident with x is equal to the same element $\mu \in \Gamma$. D_{2k}-supermagic labelings of the Cartesian, direct, and strong product of cycles C_m and C_n by dihedral group D_{2k} for any $m, n \geq 3$ were found recently. In this paper we present D_{2k}-supermagic labelings of the four 4-regular Archimedean graphs, antiprisms, and their non-planar generalizations, j-antiprisms.

1 Introduction

A *supermagic labeling* (sometimes called *vertex-magic edge labeling*) of a graph $G = (V, E)$ is a bijection from the edge set E to the set of first $|E|$ positive integers such that the sum of labels of all edges incident with each vertex (called the *weight* of x) is equal to the same constant c.

When the set of labels is instead a group Γ of order $|E|$, we speak about Γ-*supermagic labelings*. Of course, when Γ is an Abelian group, then the order in which the edge labels are used in the weight of x does not matter. When Γ is non-Abelian, we require that some product of the incident edges is equal to the same element $\mu \in \Gamma$. Because of this, the problems seems to be more complicated for general graphs, even if we restrict ourselves to regular or even vertex-transitive graphs.

So far, the only results for Γ-supermagic labelings with non-Abelian groups were obtained by the author for Cartesian, direct, and strong products of cycles C_m and C_n by dihedral group D_{2k} [2].

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A more detailed overview of the above mentioned results is given in Section 3.

In this paper we restrict our investigation to one infinite class of 4-regular graphs and four graphs of Archimedean solids. We consider antiprisms, their non-planar generalization, and four Archimedean graphs: cuboctahedron, rhombicuboctahedron, icosidodecahedron, and rhombicosidodecahedron.

The Archimedean solids are convex polyhedra with similar arrangement of non-intersecting regular plane convex polygons with unit side length of two or more different types at each vertex.

Disclaimer. The topic of this paper is very similar to the topics of [2], [5], and [6]. Most of the known results cited in this paper have also been cited in these three papers, and the statements of the cited theorems here are therefore identical. Also, some text in Sections 2 and 3 may be taken directly from [2], [5], or [6].

2 Definitions

For the sake of completeness, we start with rigorous definitions of relevant labelings. The notion of *supermagic labeling* was also studied under the name of *vertex-magic edge labeling.*

Definition 2.1. A *supermagic labeling* of a graph $G = (V, E)$ with $|E| = q$ is a bijection f from E to the set $\{1, 2, \ldots, q\}$ such that the sum of labels of all incident edges of every vertex $x \in V$, called the *weight* of x and denoted $w(x)$, is equal to the same positive constant c, called the *magic constant*. That is,

$$w(x) = \sum_{xy \in E} f(xy) = c$$

for every vertex $x \in V$. A graph that admits a supermagic labeling is called a *supermagic graph.*

There were also some more general forms of edge labelings studied by Sedláček [10] and by Stanley [12, 13]. Stewart [14] introduced the notion of supermagic labelings, where the set of labels consisted of $|E|$ consecutive integers. When a supermagic graph is regular, then the edge labels can start with any positive integer and therefore are always considered to be $1, 2, \ldots, |E|$.

Definition 2.2. A Γ-*supermagic labeling* of a graph $G = (V, E)$ with $|E| = q$ is a bijection f from E to a group Γ of order q such that for every vertex $x \in V$ and its incident edges $e_1, e_2, \ldots, e_r$ there exists an ordering $e_{i_1}, e_{i_2}, \ldots, e_{i_r}$ for which the *weight* of x, denoted $w(x)$ and defined as

$$w(x) = f(e_{i_r})f(e_{i_{r-1}}) \ldots f(e_{i_1}),$$

is equal to the same element $\mu \in \Gamma$, called the *magic constant*.

When the edges incident with each vertex can be ordered in the same way according to some well defined rule, we say that the labeling is *uniform*.

A graph that admits a Γ-supermagic labeling is called a Γ-*supermagic graph*.

While for Abelian groups the order in which the edge labels are considered is irrelevant, for non-Abelian groups different orders may produce different weights. It is indeed desirable that the order for every vertex is in some way predictable or uniform. Although for general graphs it may be hard to achieve, for some classes of graphs it can be done in a uniform way. For instance, when the graph is drawn in the plane or on the torus. Examples of such labelings are presented in [2].

The dihedral group D_{2k} of order $2k$ (sometimes also denoted by D_k) is the group consisting of k rotations r_i and k reflections s_i, where the rotations form a cyclic group of order k and each reflection generates a subgroup of order 2. A more formal definition is below.

Definition 2.3. The *dihedral group* of order $2k$ where $k \geq 3$, denoted by D_{2k}, is defined on the set of elements $\{r_0, r_1, \ldots, r_{k-1}, s_0, s_1, \ldots, s_{k-1}\}$ where $r_0 = e$, $r_i = r^i$, $s_0 = s$, $s_i = r^i s$, $s_i^2 = e$, and $r^i s = sr^{-i}$ for $i = 0, 1, \ldots, k-1$. The elements r_i are called *rotations*, and the elements s_i are called *reflections*.

An important property of D_{2k} is used in our constructions. If follows directly from the definition.

Proposition 2.4. *In any dihedral group D_{2k}, we have $sr^i s = r^{-i}$ for every $i = 0, 1, \ldots, k-1$.*

3 Known results

The study of 4-regular supermagic graphs was initiated by Ivančo [7] who investigated labelings with positive integers. He proved two results.

Theorem 3.1 (Ivančo, [7]). *Let $n \geq 3$. Then the Cartesian product $C_n \square C_n$ has a supermagic labeling.*

Theorem 3.2 (Ivančo, [7]). *Let $m, n \geq 4$ be even integers. Then $C_m \square C_n$ has a supermagic labeling.*

Ivančo also conjectured that there exists a supermagic labeling for all Cartesian products $C_m \square C_n$.

Conjecture 3.3 (Ivančo, [7]). The Cartesian product $C_m \square C_n$ allows a supermagic labeling for any $m, n \geq 3$.

Froncek in an unpublished manuscript [1] verified that the conjecture is true also when m, n are both odd and not relatively prime.

Theorem 3.4 (Froncek, [1]). *Let $m, n \geq 3$ be odd integers and* $\gcd(m, n) > 1$. *Then $C_m \square C_n$ has a supermagic labeling.*

Froncek, McKeown, McKeown, and McKeown [3] and later Froncek and McKeown [4] (using a different labeling) proved a result similar to Theorems 3.2 and 3.4 for the cyclic group Z_{2mn}.

Theorem 3.5 (Froncek et al.,[3, 4]). *The Cartesian product $C_m \square C_n$ admits a Z_{2mn}-supermagic labeling for all $m, n \geq 3$.*

The construction from [4] was then used by Sorensen [11] and Paananen [9] to obtain a slightly more general result.[1] Notice that when mn is even, the group used in the theorem is not cyclic.

Theorem 3.6 (Paananen [9] and Sorensen [11]). *For any $m, n \geq 3$, the Cartesian product $C_m \square C_n$ admits a Γ-supermagic labeling for $\Gamma = Z_{mn} \oplus Z_2$.*

[1] Paananen [9] and Sorensen [11] worked on a joint project for their MS theses. While all results cited here are their joint work, their theses were written and defended independently. Both theses contain Theorem 3.6.

Paananen [9] and Sorensen [11] also proved some more partial results that were later generalized by Froncek, Paananen, and Sorensen [5, 6].

Theorem 3.7 (Froncek et al., [5, 6]). *Let $m, n \geq 3$ and $m \equiv n \pmod 2$. Then the Cartesian product $C_m \square C_n$ admits a Γ-supermagic labeling by any Abelian group Γ of order $2mn$.*

The case of $m \equiv n + 1 \pmod 2$ remains open except for the groups Z_{2mn} and $Z_{mn} \oplus Z_2$.

4 Antiprisms

An *antiprism* A_{2n} of order $2n$ is a graph consisting of two cycles C_n (called *rims*) on vertices $x_0, x_1, \ldots, x_{n-1}$ (*upper rim*) and $y_0, y_1, \ldots, y_{n-1}$ (*lower rim*) with edges $a_i = x_i x_{i+1}$ and $b_i = y_i y_{i+1}$, respectively, and a set of edges (called *spokes*) forming two perfect matchings with edges $d_i = x_i y_i$ for $i = 0, 1, \ldots, n-1$ (called *vertical spokes*) and $t_i = y_i x_{i+1}$ for $i = 0, 1, \ldots, n-1$ (called *tilt spokes*). Addition in the subscripts is performed modulo n. The spokes induce a cycle of length $2n$ on vertices $x_0, y_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}$ with edges $d_0, t_0, d_1, t_1, \ldots, d_{n-1}, t_{n-1}$. It is usually assumed that $n \geq 4$; for $n = 3$ the graph is one of the five Platonic graphs, namely the *octahedron*.

We label the edges of A_{2n} with D_{4n} as follows.

Construction 4.1 (Uniform D_{4n}-supermagic labeling of A_{2n}). We label the spokes with the subgroup of all rotations and the rims with two cosets of reflections. In particular, we set

$$f(a_i) = r^{2i}s, \qquad f(b_i) = r^{2i+1}s$$

and

$$f(d_i) = r^{2i}, \qquad f(t_i) = r^{2i+1}.$$

Now for the upper rim, we calculate the weights as

$$w(x_i) = f(t_{i-1})f(a_i)f(d_i)f(a_{i-1}),$$

and obtain

$$\begin{aligned} w(x_i) &= r^{2i-1}(r^{2i}s)r^{2i}(r^{2i-2}s) \\ &= r^{2i-1}r^{2i}(sr^{2i}r^{2i-2}s) \\ &= r^{2i-1}r^{2i}r^{-2i}r^{-(2i-2)} \\ &= r. \end{aligned}$$

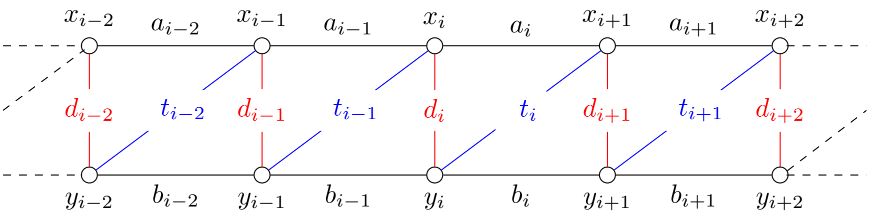

Figure 4.1: Antiprism in natural position.

For the lower rim, we calculate the weights as

$$w(y_i) = f(d_i)f(b_i)f(t_i)f(b_{i-1}).$$

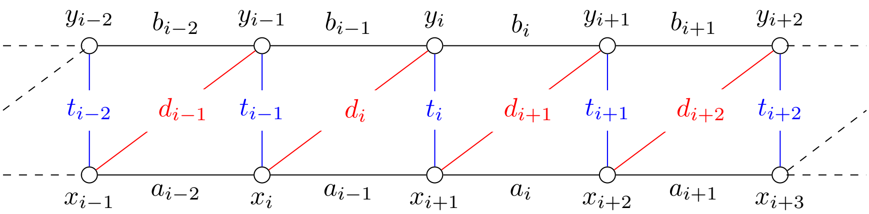

Figure 4.2: Upside-down shifted antiprism

While the order of edges in the weight may seem different from $w(x_i)$, it is in fact following the same pattern as can be observed in Figures 4.1–4.2. We turned the antiprism upside-down and then shifted the vertices x_i by one position counter-clockwise. This way the (originally tilt) spoke t_i becomes vertical and d_i (originally vertical) becomes tilt. The weight then is

$$\begin{aligned}
w(y_i) &= r^{2i}(r^{2i+1}s)r^{2i+1}(r^{2i-1}s) \\
&= r^{2i}r^{2i+1}(sr^{2i+1}r^{2i-1}s) \\
&= r^{2i}r^{2i+1}r^{-(2i+1)}r^{-(2i-1)} \\
&= r.
\end{aligned}$$

Our first result follows immediately.

Theorem 4.2. *The antiprism A_{2n} is uniformly D_{4n}-supermagic for every $n \geq 3$.*

The notion of antiprism can be generalized into a non-planar 4-regular graph with similar structure. A *j-antiprism* j-A_{2n} of order $2n$ is a graph consisting of two cycles C_n (called *rims*) on vertices $x_0, x_1, \dots, x_{n-1}$ (*upper rim*) and $y_0, y_1, \dots, y_{n-1}$ (*lower rim*) with edges $a_i = x_i x_{i+1}$ and $b_i = y_i y_{i+1}$, respectively, and a set of edges (called *spokes*) forming two perfect matchings with edges $d_i = x_i y_i$ for $i = 0, 1, \dots, n-1$ (called *vertical spokes*) and $t_i = y_i x_{i+j}$ for $i = 0, 1, \dots, n-1$ (called *j-tilt spokes*). Addition in the subscripts is performed modulo n. The spokes induce k cycles of length $m = 2n/k$ where $k = \gcd(n, j)$ on vertices $x_i, y_i, x_{i+j}, y_{i+j}, \dots, x_{i-j}, y_{i-j}$ with edges $d_i, t_i, d_{i+j}, t_{i+j}, \dots, d_{i-j}, t_{i-j}$. Notice that, because of symmetry, we can always assume that $j \leq n/2$, and therefore we can always set $0 \leq i \leq m-1$. Of course, a 1-antiprism 1-A_{2n} is just the usual antiprism A_{2n}.

Construction 4.3 (Uniform D_{4n}-supermagic labeling of j-A_{2n})**.** The rim edges are labeled exactly as in Construction 4.1. That is,

$$f(a_i) = r^{2i}s, \qquad f(b_i) = r^{2i+1}s.$$

The spokes are labeled as follows. Let $i = uj + v$, where $0 \leq u < 2n$ and $0 \leq v < j$. Then

$$f(d_i) = f(d_{uj+v}) = r^{2uj+v}, \qquad f(t_i) = f(t_{uj+v}) = r^{(2u+1)j+v}.$$

For the upper rim, we define

$$w(x_i) = f(t_{i-j}) f(a_i) f(d_i) f(a_{i-1}),$$

and obtain

$$\begin{aligned} w(x_i) = w(x_{uj+v}) &= r^{(2(u-1)+1)j+v}(r^{2i}s)r^{2uj+v}(r^{2i-2}s) \\ &= r^{(2u-1)j+v}r^{2i}(sr^{2uj+v}r^{2i-2}s) \\ &= r^{2uj-j+v}r^{2i}r^{-(2uj+v)}r^{-(2i-2)} \\ &= r^{-j+2}. \end{aligned}$$

For the lower rim, we calculate the weights as

$$w(y_i) = f(d_i) f(b_i) f(t_i) f(b_{i-1}).$$

To see that we follow the same pattern as for the upper rim, we again perform the same operations as in Construction 4.1. That is, turn the j-antiprism upside-down and then shift the vertices x_i by j positions counter-clockwise. This way the (originally j-tilt) spoke t_i becomes vertical and d_i (originally vertical) becomes j-tilt.

The weight is then

$$\begin{aligned} w(y_i) = w(y_{uj+v}) &= r^{2uj+v}(r^{2i+1}s)r^{(2u+1)j+v}(r^{2i-1}s) \\ &= r^{2uj+v}r^{2i+1}(sr^{2uj+j+v}r^{2i-1}s) \\ &= r^{2uj+v}r^{2i+1}r^{-(2uj+j+v)}r^{-(2i-1)} \\ &= r^{-j+2}. \end{aligned}$$

Thus we proved the following.

Theorem 4.4. *The j-antiprism j-A_{2n} is uniformly D_{4n}-supermagic for every $n \geq 3$ and every $1 \leq j \leq n-1$.*

5 Cuboctahedron

The cuboctahedron graph has 12 vertices, 24 edges, 8 triangular, and 6 rectangular faces. We denote the triangles as $T^0, T^1, \ldots, T^7$ with the provision that T^0, T^1, T^2, and T^3 share edges with the top rectangle consecutively. That is, T^i has a common vertex with T^{i-1} and T^{i+1}, where the superscripts are calculated modulo 4. Similarly, T^4, T^5, T^6 and T^7 share edges with the bottom rectangle consecutively. Here T^i has a common vertex with T^{i-1} and T^{i+1}, and in particular T^7 shares a vertex with T^6 and T^4. Moreover, T^0 shares a vertex with T^7, T^1 shares a vertex with T^4, T^2 shares a vertex with T^5, and T^3 shares a vertex with T^6. Therefore, T^i and T^j share a vertex only when i and j have different parity.

It should be obvious that the four odd-numbered triangles form one 2-factor $4C_3$ while the even-numbered ones form another 2-factor $4C_3$. We label each triangle with a coset of the subgroup $\langle r^8 \rangle$: the odd ones with reflections and the even ones with rotations.

Construction 5.1 (Uniform D_{24}-supermagic labeling of cuboctahedron)**.** We denote the edges in T^i as t_0^i, t_1^i, t_2^i, rotating counter-clockwise.

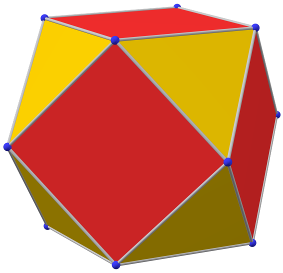

Figure 5.1: Cuboctahedron.

We label the edges of the odd triangles by the cosets of reflections of the subgroup $\langle r^4 \rangle$ as

$$f(t_k^i) = r^{4k+i}s,$$

that is, each T^i is labeled by the coset $\langle r^4 \rangle r^i s$.

The edges of the even triangles are labeled by the cosets of rotations of the subgroup $\langle r^4 \rangle$ as

$$f(t_m^j) = r^{4m+j},$$

which means that each T^j is labeled by the coset $\langle r^4 \rangle r^j$.

Every vertex x is then incident with edges t_k^i, t_{k+1}^i, t_m^j, and t_{m+1}^j for some i, j, k, m in the relevant range, where i is odd and j is even. We define the weight of each vertex as

$$w(x) = f(t_{m+1}^j) f(t_{k+1}^i) f(t_m^j) f(t_k^i).$$

Therefore, for each x we have

$$\begin{aligned} w(x) &= f(t_{m+1}^j) f(t_{k+1}^i) f(t_m^j) f(t_k^i) \\ &= r^{4(m+1)+j}\big(r^{4(k+1)+i}s\big) r^{4m+j}\big(r^{4k+i}s\big) \\ &= r^{4(m+1)+j} r^{4(k+1)+i}\big(s r^{4m+j} r^{4k+i} s\big) \\ &= r^{4(m+1)+j} r^{4(k+1)+i} r^{-(4m+j)} r^{-(4k+i)} \\ &= r^8 \end{aligned}$$

for each vertex x regardless of its location. Therefore, the labeling is uniformly D_{24}-supermagic.

Thus we proved the following.

Theorem 5.2. *The cuboctahedron graph is uniformly D_{24}-supermagic.*

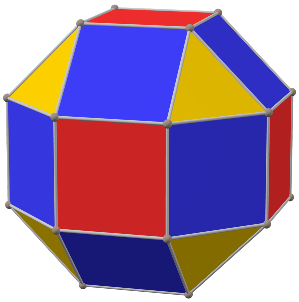

Figure 6.1: Rhombicuboctahedron.

6 Rhombicuboctahedron

The rhombicuboctahedron graph has 24 vertices, 48 edges, 8 triangles, and 18 squares. It can be decomposed into two uniform 2-factors. One consisting of the 8 triangles and one consisting of 6 squares, depicted in red in Figure 6.1.

We label the triangular factor with cosets of reflections of the subgroup $\langle r^8 \rangle$ and the squares with cosets of rotations of the subgroup $\langle r^6 \rangle$.

Construction 6.1 (Uniform D_{24}-supermagic labeling of rhombicosidodecahedron)**.** We denote the triangles in the factor $8C_3$ as $T^1, T^2, \ldots, T^8$ and the squares in $6C_4$ as $Q^1, Q^2, \ldots, Q^6$. The exact location of each triangle or square in the graph is irrelevant. Denote the edges in each T^i by t_0^i, t_1^i, t_2^i and in each Q^j by $q_0^j, q_1^j, q_2^j, q_3^j$, rotating in both cases counter-clockwise.

We label the edges of the triangles by the cosets of reflections of the subgroup $\langle r^8 \rangle$ as

$$f(t_k^i) = r^{8k+i}s,$$

that is, each T^i is labeled by the coset $\langle r^8 \rangle r^i s$.

The edges of the squares are labeled by the cosets of rotations of the subgroup $\langle r^6 \rangle$ as

$$f(q_m^j) = r^{6m+j},$$

which means that each Q^j is labeled by the coset $\langle r^6 \rangle r^j$.

Now every vertex x is incident with edges t_k^i, t_{k+1}^i, q_m^j, and q_{m+1}^j for some i, j, k, m in the relevant range. We define the weight of each vertex as

$$w(x) = f(q_{m+1}^j) f(t_{k+1}^i) f(q_m^j) f(t_k^i).$$

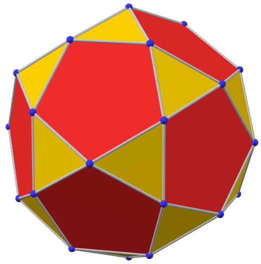

Figure 7.1: Icosidodecahedron.

Then for each x we have

$$\begin{aligned} w(x) &= f(q^j_{m+1})f(t^i_{k+1})f(q^j_m)f(t^i_k) \\ &= r^{6(m+1)+j}(r^{8(k+1)+i}s)r^{6m+j}(r^{8k+i}s) \\ &= r^{6(m+1)+j}r^{8(k+1)+i}(sr^{6m+j}r^{8k+i}s) \\ &= r^{6(m+1)+j}r^{8(k+1)+i}r^{-(6m+j)}r^{-(8k+i)} \\ &= r^{14} \end{aligned}$$

regardless of the location of vertex x, and therefore the labeling is uniformly D_{24}-supermagic.

We thus proved our next result.

Theorem 6.2. *The rhombicuboctahedron graph is uniformly D_{24}-supermagic.*

7 Icosidodecahedron

The icosidodecahedron graph has 60 vertices, 120 edges, 20 triangles, and 12 pentagons. It can be decomposed into two Hamiltonian cycles as shown in Figure 7.2. We label the red cycle with the coset of all reflections of the subgroup $\langle r\rangle$ and the blue one with the subgroup of all rotations $\langle r\rangle$.

Construction 7.1 (Uniform D_{120}-supermagic labeling of rhombicosidodecahedron)**.** We denote the edges in the red cycle C_{60} as $a_0, a_1, \ldots, a_{59}$ and in the blue C_{60} as $b_0, b_1, \ldots, b_{59}$.

We label the edges of the red cycle by the coset of reflections of the subgroup $\langle r\rangle$ as

$$f(a_i) = r^i s$$

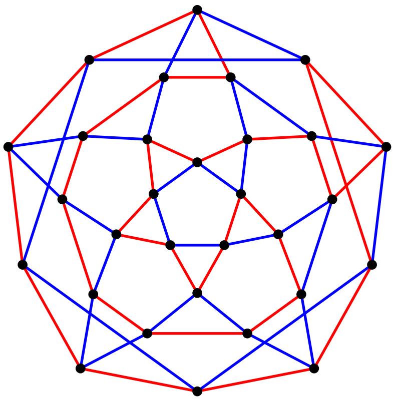

Figure 7.2: Hamiltonian cycles in an icosidodecahedron.

and the edges of the blue cycle by the subgroup of all rotations $\langle r \rangle$ as

$$f(b_j) = r^j.$$

Every vertex x is incident with edges a_i, a_{i+1}, b_j, and b_{j+1} for some $0 \leq i, j \leq 59$. We define the weight of each vertex as

$$w(x) = f(b_{j+1})f(a_{i+1})f(b_j)f(a_i).$$

This yields

$$\begin{aligned} w(x) &= f(b_{j+1})f(a_{i+1})f(b_j)f(a_i) \\ &= r^{j+1}(r^{i+1}s)r^j(r^i s) \\ &= r^{j+1}r^{i+1}(sr^j r^i s) \\ &= r^{j+1}r^{i+1}r^{-j}r^{-i} \\ &= r^2 \end{aligned}$$

regardless of the location of vertex x, and therefore the labeling is D_{120}-supermagic.

The following then holds.

Theorem 7.2. *The icosidodecahedron graph is D_{120}-supermagic.*

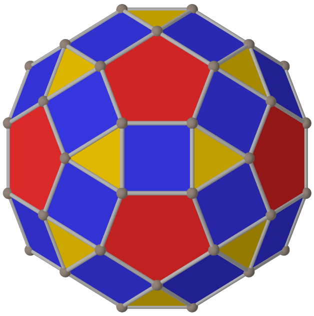

Figure 8.1: Rhombicosidodecahedron.

8 Rhombicosidodecahedron

The rhombicosidodecahedron graph has 60 vertices, 120 edges, 20 triangles, 30 squares, and 12 pentagons. It can be decomposed into two uniform 2-factors: one consisting of the 20 triangles and one consisting of the 12 pentagons. We use the factorization in our labeling as follows.

Construction 8.1 (Uniform D_{120}-supermagic labeling of rhombicosidodecahedron)**.** We denote the triangles in the factor $20C_3$ as $T^1, T^2, \dots, T^{20}$ and the pentagons in $12C_5$ as $P^1, P^2, \dots, P^{12}$. The exact location of each triangle or pentagon in the graph is irrelevant. We denote the edges in each T^i by t_0^i, t_1^i, t_2^i and in each P^i by $p_0^i, p_1^i, \dots, p_4^i$, rotating in both cases counter-clockwise.

We label the edges of the triangles by the cosets of reflections of the subgroup $\langle r^{20} \rangle$ as

$$f(t_k^i) = r^{20k+i}s,$$

that is, each T^i is labeled by the coset $\langle r^{20} \rangle r^i s$.

The edges of the pentagons are labeled by the cosets of rotations of the subgroup $\langle r^{12} \rangle$ as

$$f(p_m^j) = r^{12m+j},$$

which means that each P^j is labeled by the coset $\langle r^{12} \rangle r^j$.

Now every vertex x is incident with edges t_k^i, t_{k+1}^i, p_m^j, and p_{m+1}^j for some i, j, k, m in the relevant range. We define the weight of each vertex as

$$w(x) = f(p_{m+1}^j) f(t_{k+1}^i) f(p_m^j) f(t_k^i).$$

Therefore, for each x we have

$$\begin{aligned} w(x) &= f(p^j_{m+1})f(t^i_{k+1})f(p^j_m)f(t^i_k) \\ &= r^{12(m+1)+j}(r^{20(k+1)+i}s)r^{12m+j}(r^{20k+i}s) \\ &= r^{12(m+1)+j}r^{20(k+1)+i}(sr^{12m+j}r^{20k+i}s) \\ &= r^{12(m+1)+j}r^{20(k+1)+i}r^{-(12m+j)}r^{-(20k+i)} \\ &= r^{32} \end{aligned}$$

regardless of the location of vertex x, and therefore the labeling is uniformly D_{120}-supermagic.

The following then holds.

Theorem 8.2. *The rhombicosidodecahedron graph is uniformly D_{120}-supermagic.*

9 Concluding remarks

We have found D_{2k}-supermagic labelings of the infinite classes of antiprisms and j-antiprisms as well as of the four 4-regular graphs of Archimedean solids:

- cuboctahedron,
- rhombicuboctahedron,
- icosidodecahedron, and
- rhombicosidodecahedra.

A natural next step would be to find D_{2k}-supermagic labelings of infinite classes of other 4-regular (vertex-transitive) graphs.

The remaining nine Archimedean solids are 3- or 5-regular. We are not aware of any odd-regular D_{2k}-supermagic graphs and currently do not know if any such graphs exist.

Attribution. The Hamiltonian decomposition of the icosidodecahedron shown in Figure 7.2 and the figure itself were provided by Don Kreher [8]. The remaining figures were created by Tilman Piesk and taken from Wikipedia.org under the Creative Commons Attribution 4.0 International license.

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D. Froncek
University of Minnesota Duluth
1117 University Drive
Duluth, MN 55812-3000, U.S.A.
dalibor@d.umn.edu