BULLETIN OF THE ICA
Volume **104** (2025), 95–107

Maximum stack size differences of period-2 configurations on paths and cycles in parallel diffusion

C. David Leach

Abstract. Parallel diffusion is a chip-firing process on graphs introduced by Duffy et al. in 2018. In parallel diffusion each vertex in a graph has a stack of chips, and at each step, every vertex gives one chip to each of its poorer neighbors. A vertex is allowed to have a negative number of chips. Long and Narayanan in 2019 showed that the process is ultimately periodic with period 1 or 2. In this paper we define a *ground configuration* as a graph together with an assignment of chips in which the poorest vertex has zero chips. We then determine the maximum possible stack sizes in ground configurations on paths and cycles.

1 Introduction

Parallel diffusion is a variation of chip-firing on graphs that was first described by Duffy, Lidbetter, Messinger, and Nowakowski [2]. A *configuration* $\mathcal{C}$ on a graph G having n vertices is a map $\mathcal{C}\colon V(G) \to \mathbb{Z}^n$, where $\mathcal{C}(v)$ is the number of chips, or *stack size* at vertex v. We will use the notation $|v|_t$ to denote the value of $\mathcal{C}(v)$ at time t, and when necessary we will use $\mathcal{C}_t$ to denote the configuration $\mathcal{C}$ at time t. A vertex is richer than another if it has more chips, and poorer if it has fewer. When fired, a vertex sends one chip to each of its poorer neighbors. As this is *parallel* diffusion, at each step all vertices fire simultaneously, resulting in a new configuration. Chips are neither created nor destroyed in the firing process, so the sum of the stack sizes (i.e., the total number of chips) remains constant. It is possible for a vertex to have a negative stack size.

Repeatedly firing the vertices gives a dynamical system consisting of a sequence of configurations. The authors of [2] conjectured that for any

Key words and phrases: chip-firing, discrete-time graph processes, diffusion
Mathematics Subject Classifications: 05C57, 37B15, 05C38

Received: 18 April 2023
Accepted: 15 June 2024

initial configuration on any finite graph, this sequence must be ultimately periodic with period length 1 or 2. Long and Narayanan in [3] proved this to be true. A configuration that is part of a period of length 2 is called a p_2 configuration.

In [1], Carlotti and Herrman showed that, on a graph with n vertices, a sequence of configurations can be guaranteed to be non-negative by initially placing at least $n-2$ chips at each vertex.

Note that the movement of chips along an edge uv is determined only by the difference $|v|-|u|$, and not by the actual values of $|u|$ and $|v|$. Therefore, adding or subtracting a constant number of chips to every vertex will not change the dynamics of the system. Given a configuration with smallest stack size k, subtracting k from each stack gives a configuration with the same dynamics, but with smallest stack size 0. We call such a configuration a *ground configuration*, borrowing the usage of "ground" from electrical networks.

In [4], Mullen et al. show that the number of p_2 ground configurations of P_n is given by the recurrence

$$T_{n+4} = 3T_{n+3} + 2T_{n+2} + T_{n-1} - T_n$$

with initial conditions $T_1 = 0$, $T_2 = 2$, $T_3 = 8$, and $T_4 = 26$.

The question we focus on in the remainder of this paper is to find the largest possible difference among stack sizes in p_2 configurations on paths and cycles. We can now restate this as finding the largest possible stack size in p_2 ground configurations on paths and cycles.

2 Paths

2.1 Small cases

For small n, the p_2 ground configurations of P_n can be enumerated by hand. Note that for any graph, the configuration $\{0, 0, \ldots, 0\}$ is a ground configuration, but not a p_2 ground configuration.

For P_1, there's only one vertex. Thus the only ground configuration is $\{0\}$, which is a fixed configuration. Thus it has no p_2 configuration.

For P_2, the p_2 ground configurations are $\{1, 0\}$ and $\{0, 1\}$. So the maximum stack size is 1.

For P_3, the p_2 ground configurations are below. The maximum stack size is 2. Note that it is possible for a ground configuration to fire resulting in a non-ground configuration. On P_3 this happens with configurations A and H.

A	B	C	D
$\{0,1,0\}$	$\{0,2,0\}$	$\{0,2,1\}$	$\{1,0,1\}$
E	F	G	H
$\{1,0,2\}$	$\{1,2,0\}$	$\{2,0,1\}$	$\{2,0,2\}$

From this scant evidence, one might guess that the maximum stack size for ground configurations on P_n is $n-1$. We will show that this is true.

2.2 Limitations on differences between neighbors

Denote the path graph on n vertices as $P_n = v_1e_1v_2e_2\dots e_{n-1}v_n$, and denote the stack size of vertex v_i by $|v_i|$. As in [4], we consider v_1 to be the rightmost vertex and v_n to be the leftmost. We first observe that in a p_2 configuration on a path, the stack sizes of adjacent vertices can't differ very much. To quantify this, we define the **delta** of edge e_i to by $\delta(e_i) = |v_{i+1}| - |v_i|$ for $1 \leq i \leq n-1$. When dealing with path graphs, we can shorten $\delta(e_i)$ to δ_i with no ambiguity. These delta values are essentially the same as the multipliers used in [4]. We also observe that consecutive delta values cannot differ by very much.

A configuration on a graph induces a partial orientation on the edges. If both vertices incident to an edge have the same stack size, we call the edge *flat*, and leave it undirected. If the incident vertices have different stack sizes, we direct the edge toward the vertex with smaller stack size. The directed edges indicate the direction a chip will move during the subsequent firing. The following lemmas will be useful:

Lemma 2.1 (J. Long and B. Narayanan, [3])**.** *Let $\mathcal{C}$ be a p_2 configuration on a graph, and let $\mathcal{C}'$ be the other p_2 configuration in the period. Then every undirected edge in $\mathcal{C}$ is also undirected in $\mathcal{C}'$, and every directed edge in $\mathcal{C}$ has the opposite orientation in $\mathcal{C}'$.*

Lemma 2.2. *For any edge e_i, $1 \leq i \leq n-1$, in a p_2 configuration on a path, $-3 \leq \delta_i \leq 3$.*

Proof. Let $\mathcal{C}_t$ be a configuration on a path at time t, and for some $0 \leq i \leq n-1$ let v_i and v_{i+1} be adjacent vertices on the path. Now suppose that $|v_i|_t - |v_{i+1}|_t > 3$, making the edge directed from v_i to v_{i+1} in $\mathcal{C}_t$. In $\mathcal{C}_{t+1}$, $|v_i|_{t+1}$ must be $|v_i|_t$, $|v_i|_t - 1$, or $|v_i|_t - 2$. Similarly $|v_{i+1}|_{t+1}$ must be $|v_{i+1}|_t$, $|v_{i+1}|_t + 1$, or $|v_{i+1}|_t + 2$. In all cases we have that $|v_i|_{t+1} \geq |v_{i+1}|_{t+1}$, which leaves the orientation of the edge unchanged. Thus $\mathcal{C}_t$ cannot be a p_2 configuration. □

Lemma 2.3. *Let $\mathcal{C}_t$ be a p_2 orientation on a path at time t. Then for any two consecutive edges e_i and e_{i+1}, the ordered pair (δ_i, δ_{i+1}) cannot be any of the following:*

$$\pm(3,3),\ \pm(2,3),\ \pm(3,2),\ \pm(2,2),\ \pm(3,1),\ \pm(1,3),$$
$$\pm(0,3),\ \pm(3,0),\ \pm(2,1),\ \pm(1,2),\ (0,0).$$

Proof. The ordered pair $(0,0)$ is excluded by Theorem 2a of [4]. We will show that if any of the other ordered pairs exist in a configuration $\mathcal{C}_t$, then there is at least one edge whose orientation is not reversed in $\mathcal{C}_{t+1}$. It suffices to show this for only the positive ordered pairs; the argument for the negative ordered pairs is equivalent but with all of the orientations reversed.

$(3,3)$: Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+3$, and $|v_{i+2}|_t = a+6$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+3$, and $|v_{i+2}|_{t+1} \in \{a+4, a+5, a+6\}$. In all cases, both edges are still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

$(2,3)$: Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+2$, and $|v_{i+2}|_t = a+5$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+2$, and $|v_{i+2}|_{t+1} \in \{a+3, a+4, a+5\}$. In all cases, the edge e_{i+1} is still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

$(3,2)$: Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+3$, and $|v_{i+2}|_t = a+5$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+3$, and $|v_{i+2}|_{t+1} \in \{a+3, a+4, a+5\}$. In all cases, the edge e_i is still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

$(2,2)$: Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+2$, and $|v_{i+2}|_t = a+4$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+2$, and $|v_{i+2}|_{t+1} \in \{a+2, a+3, a+4\}$. In none of these is either edge directed left in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

(3, 1): Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+3$, and $|v_{i+2}|_t = a+4$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+3$, and $|v_{i+2}|_{t+1} \in \{a+2, a+3, a+4\}$. In all cases, e_i is still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

(1, 3): Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+1$, and $|v_{i+2}|_t = a+3$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+1$, and $|v_{i+2}|_{t+1} \in \{a+2, a+3, a+4\}$. In all cases, e_{i+1} is still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

(0, 3): Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a$, and $|v_{i+2}|_t = a + 3$. Now e_{i+1} is directed to the right and e_i is flat in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a-1, a, a+1\}$, $|v_{i+1}|_{t+1} = a+1$, and $|v_{i+2}|_{t+1} \in \{a+1, a+2, a+3\}$. In all cases, e_{i+1} is either flat or still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

(3, 0): Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+3$, and $|v_{i+2}|_t = a+3$. Now e_i is directed to the right and e_{i+1} is flat in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+2$, and $|v_{i+2}|_{t+1} \in \{a+2, a+3, a+4\}$. In all cases, e_i is either flat or still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

(2, 1): Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+2$, and $|v_{i+2}|_t = a+3$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+2$, and $|v_{i+2}|_{t+1} \in \{a+1, a+2, a+3\}$. In all cases, e_i is either flat or still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration.

(1, 2): Suppose that for some i, $|v_i|_t = a$, $|v_{i+1}|_t = a+1$, and $|v_{i+2}|_t = a+3$. Both e_i and e_{i+1} are directed to the right in $\mathcal{C}_t$. Then $|v_i|_{t+1} \in \{a, a+1, a+2\}$, $|v_{i+1}|_{t+1} = a+1$, and $|v_{i+2}|_{t+1} \in \{a+1, a+2, a+3\}$. In all cases, e_{i+1} is either flat or still directed to the right in $\mathcal{C}_{t+1}$. Thus $\mathcal{C}_t$ is not a p_2 configuration. □

Corollary 2.4. *Let $\mathcal{C}$ be a p_2 configuration on a path. Then for any two consecutive edges e_i and e_{i+1}, $-2 \leq \delta_i + \delta_{i+1} \leq 2$.*

Proof. Suppose that $\mathcal{C}$ is a p_2 configuration on a path. By Lemma 2.2, the delta values all come from the set $\{-3, -2, -1, 0, 1, 2, 3\}$. The only ordered pairs from this set that have a sum of 3 or greater are

$$(3,3),\ (3,2),\ (2,3),\ (2,2),\ (3,1),\ (1,3),\ (2,1),\ (1,2),\ (0,3),\ (3,0)$$

all of which are excluded from p_2 configurations by Lemma 2.3. Thus we have $\delta_i + \delta_{i+1} \le 2$.

A symmetric argument shows that $\delta_i + \delta_{i+1} \ge -2$. □

Theorem 2 of [4], restated specifically for paths, gives us the following lemmas:

Lemma 2.5. *Let $\mathcal{C}$ be a p_2 configuration on a P_n. No two consecutive delta values are* 0*.*

Lemma 2.6. *Let $\mathcal{C}$ be a p_2 configuration on a P_n. Then $\delta_1 \neq 0$ and $\delta_{n-1} \neq 0$.*

Lemma 2.7. *Let $\mathcal{C}$ be a p_2 configuration on a path. Then no three consecutive deltas have the same sign.*

Lemma 2.8. *Let $\mathcal{C}$ be a p_2 configuration on a path. If $\delta_i = 0$, then δ_{i-1} and δ_{i+1} cannot have the same sign.*

Lemma 2.9. *Let $\mathcal{C}$ be a p_2 configuration on P_n. Then δ_1 and δ_2 cannot have the same sign, and δ_{n-2} and δ_{n-1} cannot have the same sign.*

We now prove a lemma similar to Lemma 2.6.

Lemma 2.10. *Let $\mathcal{C}$ be a p_2 configuration on a path. Then $\delta_1 \neq \pm 3$ and $\delta_{n-1} \neq \pm 3$.*

Proof. Suppose that $\mathcal{C}_t$ is a p_2 configuration on a P_n with $|v_1|_t = 0$. First suppose that $\delta_1 = 3$. Then we have $|v_2|_t = 3$, and e_1 is directed to the right. In $\mathcal{C}_{t+1}$, $|v_0|_{t+1} = 1$ and $|v_2|_t \in \{1, 2, 3\}$. Thus in $\mathcal{C}_{t+1}$, e_1 is either flat or still directed to the right. Thus $\mathcal{C}_t$ is not a p_2 configuration.

The cases for $\delta_1 = -3$, $\delta_{n-1} = 3$, and $\delta_{n-1} = -3$ can be proved similarly. □

2.3 Configurations with maximum stack size

Now we turn to constructing a p_2 ground configuration with the largest possible stack size. We will assume that $|v_1| = 0$ and describe the configuration in terms of its delta values. To maximize the stack size of a vertex, we'd want the sum of the delta values before it to be as large as possible so that, as we progress toward v_n, the stack sizes grow. However, our choices of delta values are limited by Lemmas 2.2–2.9. The table below shows all (δ_i, δ_{i+1}) pairs, with those eliminated by Lemma 2.4 and Lemma 2.3 grayed out.

$(-3,-3)$	$(-3,-2)$	$(-3,-1)$	$(-3,0)$	$(-3,1)$	$(-3,2)$	$(-3,3)$
$(-2,-3)$	$(-2,-2)$	$(-2,-1)$	$(-2,0)$	$(-2,1)$	$(-2,2)$	$(-2,3)$
$(-1,-3)$	$(-1,-2)$	$(-1,-1)$	$(-1,0)$	$(-1,1)$	$(-1,2)$	$(-1,3)$
$(0,-3)$	$(0,-2)$	$(0,-1)$	$(0,0)$	$(0,1)$	$(0,2)$	$(0,3)$
$(1,-3)$	$(1,-2)$	$(1,-1)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$
$(2,-3)$	$(2,-2)$	$(2,-1)$	$(2,0)$	$(2,1)$	$(2,2)$	$(2,3)$
$(3,-3)$	$(3,-2)$	$(3,-1)$	$(3,0)$	$(3,1)$	$(3,2)$	$(3,3)$

The stack size of vertex v_k is given by

$$|v_k| = \sum_{i=1}^{k-1} \delta_i.$$

We need to find the maximum possible value of this sum, subject to the restrictions imposed by Lemmas 2.2–2.9. In Theorem 2.11 we construct a p_2 ground configuration with stack size $n-1$ for all n, and in Theorem 2.12 we show that a ground configuration with a stack size of n or larger is impossible.

Theorem 2.11. *Let n be a positive integer. Then there exists a p_2 ground configuration on P_n with a stack size of $n-1$.*

Proof. We give direct constructions for the even and odd cases. For n even, we let $|v_1|_0 = 0$ and let

$$\delta_i = \begin{cases} -2, & \text{if } i = 1, \\ 3, & \text{if } i \text{ is even}, \\ -1, & \text{if } i > 1 \text{ and } i \text{ is odd}. \end{cases} \tag{1}$$

These delta values induce a p_2 configuration in which v_2 has the lowest stack size of -2 and v_{n-1} has the highest stack size of $n-3$. Adding 2 chips to each stack makes this a p_2 ground configuration in which v_{n-1} has stack size $n-1$.

For n odd, we let $|v_1|_0 = 0$ and let

$$\delta_i = \begin{cases} -2, & \text{if } i = 1, \\ 3, & \text{if } i \text{ is even and } i < n-1, \\ -1, & \text{if } i > 1 \text{ and } i \text{ is odd}, \\ 2, & \text{if } i = n-1. \end{cases} \tag{2}$$

These delta values induce a p_2 configuration in which v_2 has the lowest stack size of -2 and v_n has the highest stack size of $n-3$. Adding 2 chips to each stack gives a p_2 ground configuration in which v_n has stack size $n-1$. □

Theorem 2.12. *Let n be a positive integer. There is no p_2 ground configuration on P_n with a stack size of n.*

Proof. When n is odd, the number of edges (i.e., the number of delta values) is $n-1$, which is even. We partition them into

$$\{\delta_1, \delta_2\},\ \{\delta_3, \delta_4\},\ \ldots,\ \{\delta_{n-2}, \delta_{n-1}\}.$$

By Lemma 2.4, each of these pairs has a sum that is at most 2. Since there are $(n-1)/2$ pairs, the sum of all deltas is

$$\sum_{i=1}^{n-1} \delta_i \leq 2\left(\frac{n-1}{2}\right) = n-1.$$

For even n, assume to the contrary that we have a p_2 ground configuration with $|v_n| \geq n$. We partition the $n-1$ delta values into $(n-4)/2$ consecutive pairs and a triple at the end:

$$\{\delta_1, \delta_2\},\ \{\delta_3, \delta_4\},\ \ldots,\ \{\delta_{n-5}, \delta_{n-4}\},\ \{\delta_{n-3}, \delta_{n-2}, \delta_{n-1}\}.$$

In order for the delta sum to equal or exceed n and given that each pair sums to at most 2, the sum of the triple must be at least $n - 2\left(\frac{n-4}{2}\right) = 4$. The only possible triples of consecutive deltas with a sum of 4 or greater are

$$\{2,-1,3\},\ \{2,0,2\},\ \{3,-2,3\},\ \{3,-1,3\},\ \text{and } \{3,-1,2\}.$$

The first four triples listed are impossible by Lemma 2.8 and Lemma 2.10.

The triple $\{3,-1,2\}$ is also impossible. Since this triple sums to 4, all of the pairs must sum to at least $n-4$. Since two consecutive deltas sum to at most 2, it follows that every pair must sum to exactly 2. Thus each pair must be one of the following:

$$\{-1,3\},\ \{0,2\},\ \{1,1\},\ \{2,0\},\ \{3,-1\}.$$

We are restricted in which pair can be used first:

1. $\{\delta_1,\delta_2\}$ can't be $\{3,-1\}$ by Lemma 2.10.
2. $\{\delta_1,\delta_2\}$ can't be $\{0,2\}$ by Lemma 2.6.
3. $\{\delta_1,\delta_2\}$ can't be $\{1,1\}$ by Lemma 2.9.

If $\{\delta_1,\delta_2\}=\{2,0\}$, then $\{2,0\}$ cannot be followed by $\{0,2\}$, $\{1,1\}$, $\{2,0\}$, $\{3,-1\}$, or $\{3,-1,2\}$ by Lemmas 2.5 and 2.8. This leaves only $\{-1,3\}$ as possible values for $\{\delta_3,\delta_4\}$.

So it must be the case that either $\{\delta_1,\delta_2\}=\{-1,3\}$ or $\{\delta_3,\delta_4\}=\{-1,3\}$. This $\{-1,3\}$ pair cannot be followed by anything other than another $\{-1,3\}$ pair. It can't be followed by $\{0,2\}$, by Lemma 2.8; it can't be followed by the other pairs or the triple $\{3,-1,2\}$, by Corollary 2.4. Thus it is impossible to construct a p_2 ground configuration on P_n with a vertex having stack size n or greater. □

Theorems 2.11 and 2.12 together give the main result of this section:

Theorem 2.13. *Let $\mathcal{C}$ be a ground configuration on P_n. Then the largest possible stack size in $\mathcal{C}$ is $n-1$.*

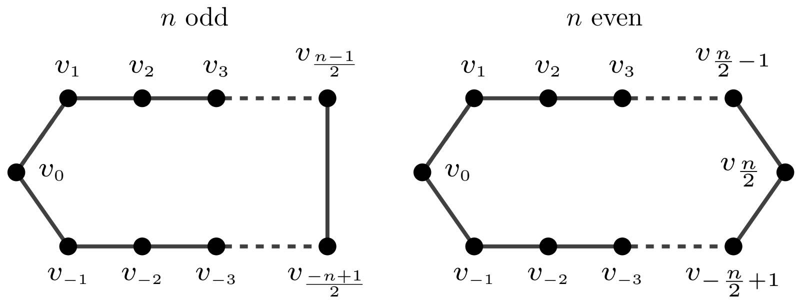
3 Cycles

We now look at the maximum stack sizes possible in a ground p_2 configuration on the cycle graph C_n. For small values of n, this is feasible to determine by computer (or even by hand for $n \leq 4$). The following table lists the distinct p_2 configurations on C_n for $n \leq 4$.

n	$\#p_2$ GCs	# non-isomorphic p_2 GCs	non-isomorphic GCs
3	18	5	$\{0,0,1\},\{0,0,2\},\{0,1,1\},\{0,1,2\},\{0,2,2\}$
4	46	12	$\{0,0,1,1\},\{0,1,0,1\},\{0,1,0,2\},\{0,1,0,3\}$ $\{0,1,2,1\},\{0,2,0,2\},\{0,2,0,3\},\{0,2,1,2\}$ $\{0,2,1,3\},\{0,3,0,3\},\{0,3,1,3\},\{0,3,2,3\}$

The numbers in the second column of the table are for a labeled graph; the numbers in the third column are for unlabeled graphs (i.e., the configurations $\{0,1,1\}$, $\{1,0,1\}$ and $\{1,1,0\}$ on C_3 are counted as three configurations in the second column and a single configuration in the third column).

For the remainder of this section, we will assume that $n \geq 5$. We number the vertices of C_n as shown in the diagrams below, using $\{0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}\}$ if n is odd and $\{0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}, \frac{n}{2}\}$ if n is even.



We call the vertices and edges with positive indices the *top* of the cycle and those with negative indices the *bottom*. For odd n, we call the edge between $v_{\frac{n-1}{2}}$ and $v_{\frac{-n+1}{2}}$ the *middle edge*; for even n we call vertex $v_{\frac{n}{2}}$ the *middle vertex*. We label the edges (other than the middle edge) as e_i, where i is the index of the incident vertex further from v_0.

As with paths, two adjacent vertices cannot have stack sizes that differ by more than 3:

Lemma 3.1. *Let $\mathcal{C}$ be a p_2 configuration on a cycle C_n with $n \geq 5$. Then for any two adjacent vertices, the stack sizes differ by at most* 3.

Proof. Let C_n be a cycle of length at least 5, and let u and v be adjacent vertices in C_n. The subgraph induced by u, v, and their two other neighbors is a path on four vertices. By Lemma 2.2 we have that $\big||u| - |v|\big| \leq 3$. □

The construction of a p_2 configuration with maximum stack size on a cycle is very similar to that of paths. We start with vertex v_0 having zero chips, and choose delta values that will make the stack sizes increase as we move away from v_0. But we can't do this all the way around the cycle. If we did, starting along the top, then the difference between $|v_{-1}|$ and $|v_0|$ would be greater than 3. The best we can do is to start at v_0 and along both the top and the bottom choose delta values that make the stack sizes increase, meeting in the middle, and giving a maximum stack size that is roughly $n/2$.

A similar argument gives us the following lemma:

Lemma 3.2. *Let $\mathcal{C}$ be a p_2 configuration on a cycle C_n with $n \geq 5$. Let e_i and e_{i+1} be adjacent edges that are either both on the top or both on the bottom of the cycle. Then $|\delta_i + \delta_{i+1}| \leq 2$.*

Proof. Let e_i and e_{i+1} be adjacent edges along either the top or bottom of the cycle. The subgraph induced by these two edges is a path on three vertices, and by Lemma 2.3 and Corollary 2.4 the result follows. □

Theorem 3.3. *Let $\mathcal{C}$ be a ground configuration on C_n. Then the maximum possible stack size for a vertex in $\mathcal{C}$ is given by*

$$\begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd,} \\ \frac{n+2}{2}, & \text{if } n = 4k, \\ \frac{n+4}{2}, & \text{if } n = 4k+2. \end{cases}$$

Proof. From the listing of p_2 configurations above, the result holds for $n = 3$ and $n = 4$. When $n \geq 5$ we break into four cases depending on the value of n modulo 4.

Case 1: $n = 4k + 1$

By Lemma 3.2, the delta values of two adjacent edges is at most 2. If $n = 4k + 1$, we construct a p_2 configuration by defining the delta values of the edges as

$$\delta_i = \begin{cases} 3, & \text{if } |i| \text{ is odd and } i \leq \frac{n-1}{2}, \\ -1, & \text{if } |i| \text{ is even,} \\ 0, & \text{for the middle edge.} \end{cases} \tag{3}$$

This results in each pair of adjacent edges along the top (and bottom) summing to 2, which is best possible. Since there are an even number of edges along the top, the last top edge has delta value -1, and hence the penultimate vertex on the top, $v_{\frac{n-3}{2}}$, has stack size

$$3\left(\tfrac{n-1}{4}\right) - 1\left(\tfrac{n-1}{4} - 1\right) = \tfrac{n+1}{2},$$

which is the maximum possible.

Case 2: $n = 4k + 3$

If $n = 4k + 3$, we construct a p_2 configuration by defining the delta values of the edges as

$$\delta_i = \begin{cases} 3, & \text{if } |i| \text{ is odd and } i < \frac{n-1}{2}, \\ -1, & \text{if } |i| \text{ is even and } i < \frac{n-3}{2}, \\ 2, & \text{if } |i| = \frac{n-1}{2}, \\ 0, & \text{if } |i| = \frac{n+1}{2}. \end{cases} \tag{4}$$

The edges along the top have indices $1, 2, \ldots, \frac{n-1}{2}$. This is an odd number. The maximum possible stack size for the penultimate vertex along the top would be $2(\frac{n-3}{4}) = \frac{n-3}{2}$. Since delta values can't exceed 3, the stack size of the last vertex on top is at most $\frac{n-3}{2} + 3$. However, using $\delta_{\frac{n-1}{2}} = 3$ results in a non-period configuration. Using $\delta_{\frac{n-1}{2}} = 2$ instead gives the last vertex on top a stack size of

$$2\left(\tfrac{n-3}{4}\right) + 2 = \tfrac{n+1}{2},$$

which is the maximum possible.

Case 3: $n = 4k$

If $n = 4k$, we construct a p_2 configuration by defining the delta values of the edges as

$$\delta_i = \begin{cases} 3, & \text{if } |i| \text{ is odd}, \\ -1, & \text{if } |i| \text{ is even}. \end{cases} \tag{5}$$

This results in each pair of adjacent edges along the top (and bottom) summing to 2, which is best possible. Since there are an even number of edges along the top, the last top edge has delta value -1, and hence the penultimate vertex on the top, $v_{\frac{n-2}{2}}$, has stack size

$$3\left(\tfrac{n}{4}\right) - 1\left(\tfrac{n}{4} - 1\right) = \tfrac{n+2}{2},$$

which is the maximum possible.

Case 4: $n = 4k + 2$

If $n = 4k + 2$, we construct a p_2 configuration by defining the delta values of the edges as

$$\delta_i = \begin{cases} 3, & \text{if } |i| \text{ is odd and } i < \frac{n-1}{2}, \\ -1, & \text{if } |i| \text{ is even and } i < \frac{n-3}{2}. \end{cases} \tag{6}$$

This results in each pair of adjacent edges along the top (and bottom) summing to 2, which is best possible. Since there is an odd number of edges along the top, the last top edge has delta value 3, and hence the last vertex is the middle vertex $v_{\frac{n}{2}}$, which has stack size

$$3\left(\tfrac{n+2}{4}\right) - 1\left(\tfrac{n-2}{4}\right) = \tfrac{n+4}{2}$$

which is the maximum possible. □

4 Open problems

It remains to determine the maximum stack sizes for graphs other than paths and cycles. This will require more advanced machinery to deal with graphs with higher-degree vertices. Another interesting open question would be to determine the maximum total number of chips in a ground configuration.

References

[1] A. Carlotti and R. Herrman, *Uniform bounds for non-negativity of the diffusion game*, (2018), `https://doi.org/10.48550/arXiv.1805.05932`, accessed 2024-09-5.

[2] C. Duffy, T. F. Lidbetter, M. E. Messinger, and R. J. Nowakowski, A variation on chip-firing: The diffusion game, *Discrete Math. Theor. Comput. Sci.* **20**(1) (2018), Paper no. 4.

[3] J. Long and B. Narayanan, Diffusion on graphs is eventually periodic, *J. Comb.* **10**(2) (2019), 235–241.

[4] T. Mullen, R. J. Nowakowski, and D. Cox, *Counting path configurations in parallel diffusion*, (2020), `https://doi.org/10.48550/arXiv.2010.04750`, accessed 2024-09-5.

C. David Leach
Department of Computing and Mathematics
University of West Georgia
cleach@westga.edu