BULLETIN OF THE ICA
Volume **103** (2025), 47–56

Classroom note: A note on weaving cubes

Robert S. D. Thomas

Abstract. This note recalls a sequence of papers on woven fabrics, based on work of Branko Grünbaum and Geoffrey Shephard in the 1980s and Richard Roth in the 1990s, and on woven cubes by Jean Pedersen in the 1980s. With no new theory, it shows examples of cubes woven in more than two colours based on isonemal fabrics and having strong symmetry properties.

1 Introduction

As this note concerns the mathematical idealization of woven fabrics covering both the plane and cubes, it will be helpful to set out what the idealizations conventionally are, based on the pioneering work of Branko Grünbaum and Geoffrey Shephard [1]. The simplest mathematical weaving concerns strips running in perpendicular directions in two vertical parallel planes and bounded by evenly spaced parallel lines. These strips are the foundation of *strands*, which are almost coincident with the strips but slightly narrower so as not to touch. It is convenient also to consider a third parallel *reference* plane between the others. If the strands remain in their distinct planes, there is no weaving. If they rise and fall to form a structure from which no subset can be pulled off, the result is called a *fabric*. It is nearly planar and can be represented as planar. Cubes can be woven with the same interleaving behaviour but of short lengths of strand with ends identified. A variety of pictorial representations of fabrics and woven cubes are possible. The conventional one for fabrics is to picture a fragment with the strands running top to bottom dark and those running left to right pale as in Figure 1.1, illustrating plain weave twice with nine *cells*, and Figure 2.1.

Key words and phrases: weaving, fabric, isonemal, cube, colouring
AMS (MOS) Subject Classifications: 05B30, 05B45, 52C20

Received: 17 March 2024
Accepted: 31 May 2024

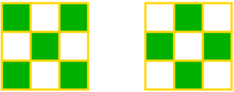


Figure 1.1: (a) Standard diagram of fragment of plain weave. (b) Standard diagram of fragment of Figure 1.1(a) rotated about its centre.

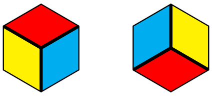


Figure 1.2: (a) Simplest cube covering with three strands. (b) Rear as seen in a mirror behind cube. The diagram is ambiguous. Red can be behind yellow and so yellow behind blue, or red can be behind blue and so blue behind yellow.

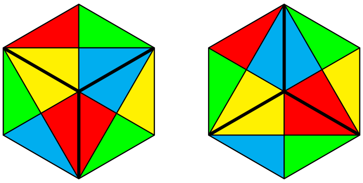


Figure 1.3: Covering by four woven strands. (a) Front. (b) Rear of cube in mirror. On the front, red passes beneath yellow and then green, yellow behind blue, blue behind red and then green. Green passes behind blue on the left, yellow on the right, and red on top.

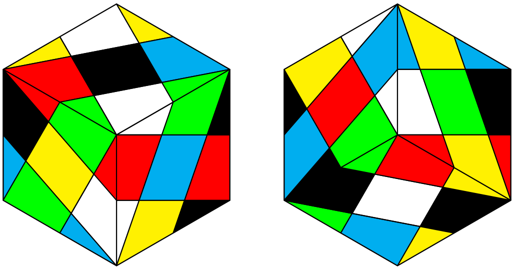


Figure 1.4: (a) Cube woven with six strands. (b) Rear of cube in mirror.

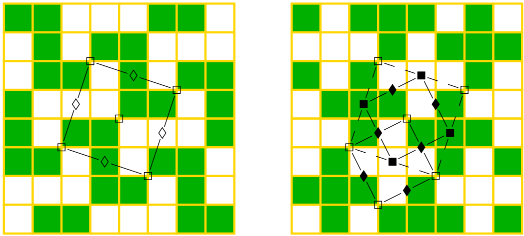
No colouring convention has been proposed for cubes. The simplest way to indicate the pattern of over and under is to colour the different strands different colours as in the examples in Figures 1.2, 1.3, and 1.4. What does need a convention is how the hidden faces of a cube are to be displayed. I have chosen to show the back side of the cube as it would appear in a mirror behind the cube as a dentist looks at the hidden side of a tooth. Seeing around the edges is easy, if not at first.[1]

2 Symmetry

As symmetry plays so important a part in this discussion, it will be as well to indicate some aspects of it. The plane weaving patterns studied are all periodic, so that translations in various directions are always symmetries. If the full symmetry group G_1 of a fabric is transitive on its strands, then it is called *isonemal*. Three papers [11–13], based on [8], discuss the symmetry groups of isonemal fabrics in full detail. The little required here concerns rotations. If the fragment of the most symmetric of fabrics, plain weave, in Figure 1.1(a) is rotated in its plane a quarter turn about its centre, the result is that shown in Figure 1.1(b). Because it has been thought desirable to consider this a symmetry of the weave, another operation is introduced, reflection in the reference plane. This reflection reverses dark and pale; so combined with the rotation it restores the diagram of Figure 1.1(a) by reversing the *sides* of the fabric. This operation is never a symmetry, but it can be combined with a lot of rigid motions to make symmetries in G_1. The subgroup of symmetries not involving this reflection—like translations and the central half turns above—is called the side-preserving subgroup H_1, and it is all we are concerned with here because no natural operation on a cube corresponds to reflection in the reference plane.

We need to consider the periodicity of the weaving. I use the term in its two-dimensional sense as explained by Doris Schattschneider in her exposition of plane symmetry groups [10]. There is a non-unique finite region, and two linearly independent translations, such that the set of all images of the region under the group generated by these translations reproduces the original configuration, assumed to be infinite in all directions for convenience. She gives the name *unit* to any such smallest region of the plane. Such units are all of the same area, the *period*, but in general can be of a

[1]Figures 1.2 and 1.3 are redrawn from [7], where they appear without colour. They reappear as Figures 8.5, 9.7, 14.3, 14.6, and 14.8 in [4], which is mainly concerned with more complex polyhedra. Figure 1.4 does not appear in either place, only instructions for weaving the cube [7, fn. 3], credited to Geoffrey Shephard.



(a) (b)

Figure 2.1: (a) Standard diagram of fabric 10-107-1, with lattice unit of G_1 marked, coloured in Figure 2.3(g). [18, Fig. 4a]. (b) Standard diagram of fabric 10-93-1, with alternative lattice units of G_1 and lattice unit (dashed) of H_1 marked, coloured in Figure 2.3(h). [18, Fig. 13a]. The marked points are centres of rotational symmetry.

variety of shapes. Units whose vertices are all images of a single point under the action of the translation group are called *lattice units*. Lattice units of isonemal fabrics can be either rectangular or rhombic; those here are both, square. The standard diagrams of two fabrics are shown in Figure 2.1 with their lattice units.[2]

One other aspect of symmetry needs to be described. An ordinary symmetry, an operation comprised by G_1, acts on a fabric but leaves its structure (diagrammatic appearance) unchanged. A weaker class of action has already been indicated for weaving diagrams, an action like the quarter turn in Figure 1.1 that leaves the diagram changed in colour consistently but only in colour. Such a non-symmetry is called a *colour symmetry*. For standard weaving diagrams, colour symmetries are the actions made into symmetries by reflection in the reference plane. This special case has always been a feature of the weaving literature. When more than two colours are used and for cubes, where there is no reference plane, colour symmetries are of just as much interest as full symmetries. Note the full colour symmetry of the cube colouring in Figure 1.2. Each of the 24 rigid-body symmetries of the cube is a colour symmetry producing a permutation of the three colours: fourfold rotations about the three axes joining opposite face centres, threefold rotations about the four axes joining diagonally opposite vertices, and the twofold rotations about six axes joining diagonally opposite mid-points of edges. In contrast, the number of full symmetries is very small, just the

[2] Catalogue numbers of fabrics refer to the catalogues of Grünbaum and Shephard [2, 3] and extensions.

(a) (b)

Figure 2.2: (a) Cube coloured by the thick striping catalogued as 20-8367-2* [15, Fig. 14]. (b) Back of the cube of (a) reflected in a mirror [15, Fig. 15].

three half turns about the fourfold axes. I believe that the cube of Figure 1.3 is also fully colour symmetric but has no full symmetry. The cube of Figure 1.4 also has no full symmetry and has as colour symmetries only the half turns about the fourfold axes and the threefold rotations. One feature of the colouring of these three cubes that is significantly distinct from those to appear below is that no strand crosses itself.

3 Results

This note provides the aesthetic conclusion of a long project whose mathematical high point was published in [14, 15] before the intended conclusion was reached. Having studied weaving with W.D. Hoskins [6] following Grünbaum and Shephard [1], I wondered what weaving with more than the two colours conventionally used to code the topology would look like. With J.A. Hoskins, I found that the case of just two colours not used conventionally was interesting [5], but we did not know how to deal with more colours. Inspired, he said, by our work, Richard Roth studied the symmetry groups of all isonemal fabrics [8] and perfect colourings with two colours [9]. Thinking that his classification allowed for the multi-coloured study, I again found mathematically interesting results [14, 15] with only two colours, including application to woven cubes. In particular, I found that fully colour-symmetric cubes woven with two colours from fabrics of the smallest lattice units were from a family represented by the cube of Figure 2.2.[3]

[3] I had not yet discovered TikZ, and so Figures 2.2 and 3.1 are photos of models kindly taken by Allen Patterson working for the University of Manitoba.

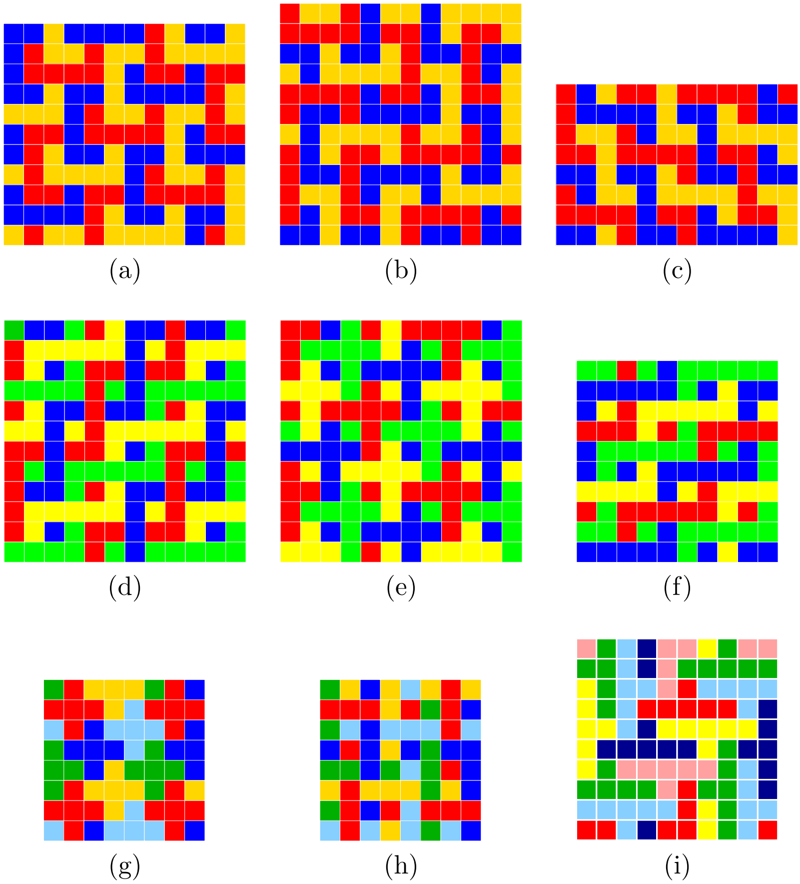
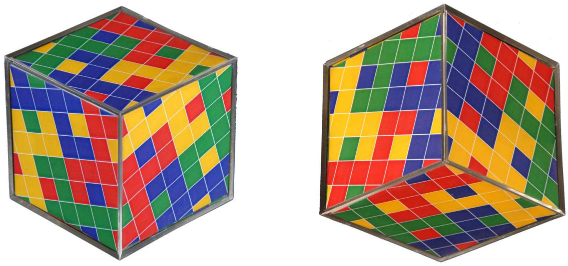


Figure 2.3: (a) Fabric 12-183-1 3-coloured [17, Fig. 4b]. (b) 12-315-4 3-coloured [17, Fig. 7b]. (c) 12-189-1 3-coloured [17, Fig. 19b]. (d) 8-11-1 4-coloured [17, Fig. 34b]. (e) 8-11-1 4-coloured differently [17, Fig. 34c]. (f) 8-5-1 4-coloured [17, Fig. 43b]. (g) 10-107-1 5-coloured [18, Fig. 4b]. (h) 10-93-1 5-coloured [18, Fig. 13b]. (i) 12-111-2 6-coloured [18, Fig. 31a].

I moved on to more colours [16, 17], losing interest beyond six colours for lack of visual interest or mathematical significance. It is much easier to have colour symmetries with more colours: if all the strands of a fabric are of different colours, then every symmetry is a colour symmetry. As Figure 1.4 illustrates, that is not true of cubes. Patterns can be isonemally woven with adjacent pairs of strands of the same colour, called *thick striping*. The weaving of the cube in Figure 2.2 is thickly striped. Independent striping is called *thin*. Analysis similar to that for two colours can be done, and the visual results are as interesting and attractive as I hoped they would be (Figure 2.3).

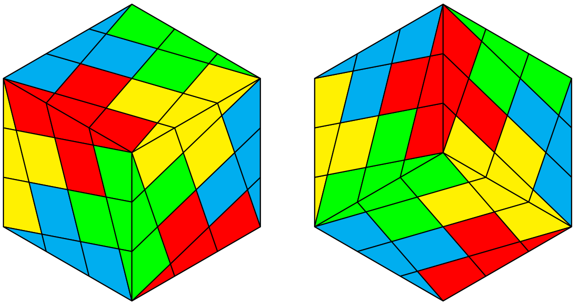
In Figure 2.2, crossing pairs of strands (thick stripes) of the same colour can be readily identified in the squares of four cells of that colour. In Figure 2.2 a thick stripe cannot be traced like each strand because each such pair, when it reaches a vertex, is split with the two strands crossing each other and going their perpendicular ways. It is not quite meaningful to ask how many stripes of each colour there are for this reason, but there is a sense in which one can say that there are two of each. Colouring each strand differently, which takes eight colours, is again a colour-symmetric colouring. However, a point of this note is that one can colour pairs of adjacent strands with four colours and preserves the colour symmetry of the rotations (Figure 3.1) [18]. The stripes so coloured are not the stripes of Figure 2.2 and cannot be. For their colours to be permuted by quarter turns at the centre of faces, all four thick stripes must pass by the centre, whereas the cube of Figure 2.2 has two thick stripes crossing *at* those centres. To the extent that there is a theory of thick and thin striping, it involves these places where a colour crosses itself, called *redundant* cells because—as far as colour is concerned—it does not matter how they are woven. The distribution of redundant cells, singly or in groups of four, determines the distribution of stripes.

A natural question to ask is, since Figure 2.2 is a cube woven from a fabric of minimal lattice with two colours and is also colourable with four colours (easier), how small can a lattice unit of a fabric be and be the face of a cube coloured with four colours in the normal way with redundant cells. A candidate that suggests itself is the cube woven from strands arranged like the stripes in Figure 3.1, shown in Figure 3.2. It has the right symmetry conditions, but Figure 3.2 does not indicate how it comes from a woven fabric. A woven fabric from which it comes is 10-107-1, displayed in Figure 2.1(a). It is 5-coloured in Figure 2.3(g), where the fifth colour (red) required for the plane, is not needed because it lies outside the cube face outlined in Figure 2.1(a) as the G_1 lattice unit.



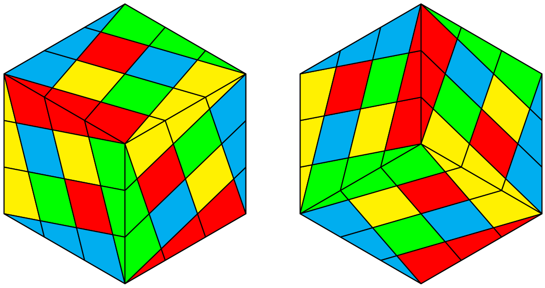
(a) (b)

Figure 3.1: (a) Four-colouring of the cube of Figure 2.2. (b) The back as reflected in a mirror.



(a) (b)

Figure 3.2: (a) 4-coloured cube inspired by Figure 3.1. (b) Back as reflected in a mirror.



(a) (b)

Figure 3.3: (a) Second 4-coloured cube. (b) Back as reflected in a mirror.

That cube is not unique. Similar to it is another cube that has the right symmetry conditions, shown in Figure 3.3. A woven fabric from which it comes is 10-93-1, displayed in Figure 2.1(b). It is 5-coloured in Figure 2.3(h), where the fifth colour (red) required for the plane, is not needed because it lies outside the cube face with dashed outline in Figure 2.1(b). I identify these cubes as + and ÷ respectively.

The symmetry group G_1 of each fabric in Figure 2.1 has a lattice unit with solid outline; in the case of Figure 2.1(b) there are alternatives. In Figures 2.1(a) and 2.3(g) the G_1 lattice unit is used as the face of the + cube. In Figures 2.1(b) and 2.3(h) it is lattice unit of the side-preserving subgroup H_1 of the fabric that is used as the face of the ÷ cube. As in the + cube, the fifth colour required for the plane in Figure 2.3(h) (red) is not needed for the ÷ cube.

Finally, there is a handedness to the weaving of all of the cubes shown here, Figures 1.2–1.4, 2.2 and 3.1–3.3. Their mirror images have the opposite handedness, and they can be seen, following the same conventions, by looking at the given diagrams upside down.

Acknowledgments

I am particularly grateful to photographer Allen Patterson for producing Figures 2.2 and 3.1 from my models when working for the University of Manitoba.

References

[1] B. Grünbaum and G. C. Shephard, Satins and twills: An introduction to the geometry of fabrics, *Math. Mag.*, **53** (1980), 139–161.

[2] B. Grünbaum and G. C. Shephard, A catalogue of isonemal fabrics, in *Discrete Geometry and Convexity*, J. E. Goodman et al., eds. *Ann. N. Y. Acad. Sci.*, **440** (1985), 279–298.

[3] B. Grünbaum and G. C. Shephard, An extension to the catalogue of isonemal fabrics, *Discrete Math.*, **60** (1986), 155–192.

[4] P. J. Hilton, J. Pedersen and S. Donmoyer, *A mathematical tapestry: Demonstrating the beautiful unity of mathematics*, Cambridge: Cambridge University Press, 2010.

[5] J. A. Hoskins and R. S. D. Thomas, The patterns of the isonemal two-colour two-way two-fold fabrics, *Bull. Aust. Math. Soc.*, **44** (1991), 33–43.

[6] W. D. Hoskins and R. S. D. Thomas, Conditions for isonemal arrays on a Cartesian grid, *Linear Algebra Appl.*, **57** (1984), 87–103.

[7] J. Pedersen, Some isonemal fabrics on polyhedral surfaces, in *The geometric vein*, pp. 99–122. C. Davis, B. Grünbaum, and F. A. Sherk, eds. New York: Springer, 1981.

[8] R. L. Roth, The symmetry groups of periodic isonemal fabrics, *Geom. Dedicata*, **48** (1993), 191–210.

[9] R. L. Roth, Perfect colorings of isonemal fabrics using two colors, *Geom. Dedicata*, **56** (1995), 307–326.

[10] D. Schattschneider, The plane symmetry groups: Their recognition and notation, *Amer. Math. Monthly*, **85** (1978), 439–450.

[11] R. S. D. Thomas, Isonemal prefabrics with only parallel axes of symmetry, *Discrete Math.*, **309** (2009), 2696–2711. `https://doi.org/10.1016/j.disc.2008.06.028`

[12] R. S. D. Thomas, Isonemal prefabrics with perpendicular axes of symmetry, *Util. Math.*, **82** (2010), 33–70. Online: `http://arxiv.org/abs/0805.3791v1`

[13] R. S. D. Thomas, Isonemal prefabrics with no axes of symmetry, *Discrete Math.* **310**, (2010), 1307–1324. `http://doi.org/10.1016/j.disc.2009.12.015`

[14] R. S. D. Thomas, Perfect colourings of isonemal fabrics by thin striping, *Bull. Aust. Math. Soc.*, **83** (2011), 63–86. `http://doi.org/10.1017/S0004972710001632`

[15] R. S. D. Thomas, Perfect colourings of isonemal fabrics by thick striping, *Bull. Aust. Math. Soc.*, **85** (2012), 325–349. `http://doi.org/10.1017/S0004972711002899`

[16] R. S. D. Thomas, Colouring isonemal fabrics with more than two colours by thick striping, *Contrib. Discrete Math.*, **8** (2013), 38–65. Only online at `https://doi.org/10.11575/cdm.v8i1.62180`

[17] R. S. D. Thomas, Colouring isonemal fabrics with more than two colours by thin striping, *Contrib. Discrete Math.*, **9** (2014), 57–90. Only online at `https://doi.org/10.11575/cdm.v9i2.62210`

[18] R. S. D. Thomas, Colouring isonemal fabrics with more than two colours and non-twilly redundancy. Online: `https://arxiv.org/abs/2406.07810`

Robert S.D. Thomas
St John's College and Department of Mathematics,
University of Manitoba, Winnipeg, Manitoba R3T 2N2 Canada
robert.thomas@umanitoba.ca