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Determining and distinguishing numbers of Praeger-Xu graphs

Sally Cockburn and Max Klivans

Abstract. Praeger-Xu graphs are a family of connected, symmetric, 4-regular graphs with unusually large automorphism groups relative to their order. Determining number and distinguishing number are parameters that measure the symmetry of a graph by investigating additional conditions that can be imposed on a graph to eliminate its nontrivial automorphisms. In this paper, we compute the values of these parameters for Praeger-Xu graphs. Most Praeger-Xu graphs are 2-distinguishable; for these graphs we also provide the cost of 2-distinguishing.

1 Introduction

A finite simple graph $G = (V, E)$ consists of a finite, nonempty set V of vertices and a set of 2-subsets of V, called edges. Some graphs have geometric representations that display visual symmetry. There are various ways to give a more rigorous mathematical characterization of symmetry.

An *automorphism* α of a graph $G = (V, E)$ is a permutation of V such that for all $u, v \in V$, $\{u, v\} \in E$ if and only if $\{\alpha(u), \alpha(v)\} \in E$. The set of automorphisms of G, denoted $\mathrm{Aut}(G)$, is a group under composition. For example, the automorphism group of the complete graph on n vertices is the entire group of permutations on n elements; that is, $\mathrm{Aut}(K_n) = S_n$. For $n \geq 3$, the automorphism group of the cycle C_n is the dihedral group D_n, consisting of rotations and reflections.

One way of characterizing the symmetry of a graph is to determine whether the vertices and/or edges play the same role, in the following sense. A graph G is *vertex-transitive* if for all $u, v \in V$ there is $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(v) = u$. Similarly, G is *edge-transitive* if for all $\{u, v\}, \{x, y\} \in E$

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there is $\alpha \in \text{Aut}(G)$ such that $\alpha(\{u,v\}) = \{\alpha(u), \alpha(v)\} = \{x,y\}$. More stringently, a graph is *arc-transitive* if for all $\{u,v\}, \{x,y\} \in E$ there exist $\alpha, \alpha' \in \text{Aut}(G)$ such that we have $\alpha(u) = x$ and $\alpha(v) = y$ and we have $\alpha'(u) = y$ and $\alpha'(v) = x$. Connected, arc-transitive graphs are automatically both vertex-transitive and edge-transitive and are simply called *symmetric* graphs. Complete graphs K_n and cycles C_n are examples of symmetric graphs.

Another way of characterizing the symmetry of a graph G is to quantify extra measures that can be taken to prevent the existence of nontrivial automorphisms of G. As one example, we could require that automorphisms of G fix point-wise a subset S of vertices. If the only automorphism doing so is the identity, then S is called a *determining set* of G. The *determining number* of G, denoted $\text{Det}(G)$, is the minimum size of a determining set of G. (Some authors use the term *fixing* instead of determining, for both sets and numbers.) Graphs with no nontrivial automorphisms, sometimes called asymmetric or rigid graphs, have determining number 0; at the opposite end of the spectrum, $\text{Det}(K_n) = n - 1$. A minimum determining set for C_n is any set of two non-antipodal vertices, so $\text{Det}(C_n) = 2$.

As another example, we could paint the vertices with different colors and require that automorphisms preserve set-wise the color classes. A graph G is *d-distinguishable* if the vertices can be colored with d colors in such a way that the only automorphism preserving the color classes is the identity. The *distinguishing number* of G, denoted $\text{Dist}(G)$, is the minimum number of colors required for a distinguishing coloring. For a discussion of elementary properties of determining numbers and distinguishing numbers, as well as the connections between them, see [2].

Remarkably, many infinite families of symmetric graphs have been found to have distinguishing number 2, including hypercubes [3], Cartesian powers $G^{\square n}$ of a connected graph where $G \notin \{K_2, K_3\}$ and $n \geq 2$ [1, 11, 14], and Kneser graphs $K_{n:k}$ with $n \geq 6$, $k \geq 2$ [2]. Boutin [4] introduced an additional invariant in such cases; the *cost of 2-distinguishing* G, denoted $\rho(G)$, is the minimum size of a color class in a 2-distinguishing coloring of G.

In this paper, we find these symmetry parameters for a family of symmetric graphs called Praeger-Xu graphs. They are remarkable among all connected, symmetric, 4-regular graphs for having very large automorphism groups relative to their order. Moreover, there is an infinite family of Praeger-Xu graphs with the property that the smallest subgroup of automorphisms that acts transitively on the vertices has an arbitrarily large

Table 1.1: Summary of symmetry parameters ($n \geq 3$).

Parameter	Value	Condition(s)
$\text{Det}\big(\text{PX}(n,k)\big)$	6	$(n,k) = (4,1)$
	$\lceil \frac{n}{k} \rceil$	$k \neq \frac{n}{2}$ but $(n,k) \neq (4,1)$
	$\lceil \frac{n}{k} \rceil + 1$	$k = \frac{n}{2}$
$\text{Dist}\big(\text{PX}(n,k)\big)$	5	$(n,k) = (4,1)$
	3	$n \neq 4$, $k = 1$
	2	$k \geq 2$
$\rho\big(\text{PX}(n,k)\big)$	5	$(n,k) = (4,2)$
$(k \geq 2)$	$\lceil \frac{n}{k} \rceil$	$5 \leq n < 2k$ or $2k < n$ and $n \notin \{0, -1 \bmod k\}$
	$\lceil \frac{n}{k} \rceil + 1$	otherwise

vertex stabilizer. For these and more results on Praeger-Xu graphs, see [12, 13, 15]. The large automorphism groups suggest that these graphs might have large determining and distinguishing numbers; the large vertex stabilizers suggest the opposite.

This paper is organized as follows. In Section 2, we provide a definition of the Praeger-Xu graphs and facts about their automorphism groups. In Section 3, we show that most Praeger-Xu graphs are twin-free; for those with twins, we use a quotient graph construction to find the determining and distinguishing number. In Section 4, we find the determining number for twin-free Praeger-Xu graphs. As a tool for computing distinguishing number, in Section 5 we characterize pairs of vertices in twin-free Praeger-Xu graphs that are interchangeable via an automorphism. Finally, in Section 6 we show that all twin-free Praeger-Xu graphs are 2-distinguishable and compute the cost of 2-distinguishing. Our results are summarized in Table 1.1.

2 Praeger-Xu graphs, $\mathbf{PX}(n, k)$

In 1989, Praeger and Xu [16] introduced a family of connected graphs they denoted by $C(m, r, s)$, where $m \geq 2$, $r \geq 3$, and $s \geq 1$, which are vertex-transitive for $r \geq s$ and arc-transitive, hence symmetric, for $r \geq s + 1$.

This was part of an investigation into connected, symmetric graphs whose automorphism groups have the property that for any vertex v, the subgroup of automorphisms fixing v (the stabilizer of v) does not act primitively on the set of neighbors of v. The Praeger-Xu graphs are those where $m = 2$; the notation $PX(n, k)$ denotes $C(2, n, k)$. There are several ways of describing Praeger-Xu graphs (see [9] and [10]); we use what is called the bitstring model.

Definition 2.1. Let $n \geq 3$ and $1 \leq k < n$. The corresponding Praeger-Xu graph is $\mathrm{PX}(n, k) = (V, E)$, where V is the set of all ordered pairs (i, x), where $i \in \mathbb{Z}_n$ and $x = x_0x_1 \cdots x_{k-1}$ is a bitstring of length k, and $\{(i, x), (j, y)\} \in E$ if and only if $j = i + 1$ and $x = az_1z_2 \cdots z_{k-1}$ and $y = z_1z_2 \cdots z_{k-1}b$ for some $z_1, \ldots, z_{k-1}, a, b \in \mathbb{Z}_2$.

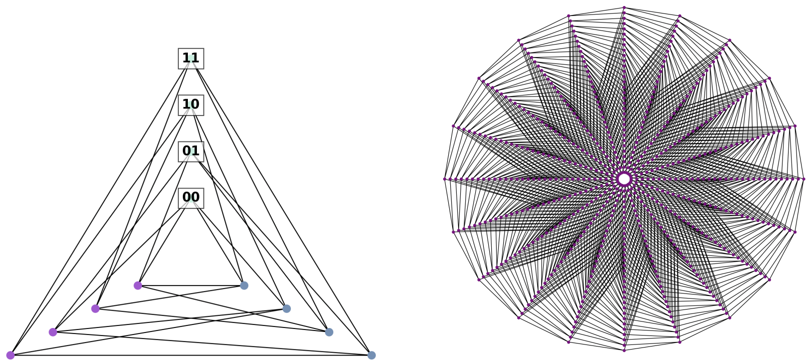
Throughout this paper, subscripts on bits will be considered elements of $\mathbb{Z}_k$. We say that the bit x_j in x is *flipped* if it is switched to $x_j + 1$ in $\mathbb{Z}_2$.

There is a natural partition of V into *fibres* $\mathcal{F}_i = \left\{(i, x) : x \in \mathbb{Z}_2^k\right\}$ for each $i \in \mathbb{Z}_n$. Each fibre is an independent set of 2^k vertices; every vertex in $\mathcal{F}_i$ is adjacent to exactly two vertices in each of $\mathcal{F}_{i+1}$ and $\mathcal{F}_{i-1}$, so $\mathrm{PX}(n, k)$ is 4-regular, or tetravalent. Two fibres $\mathcal{F}_i$ and $\mathcal{F}_j$ are *antipodal* if and only if n is even and $i - j = \frac{n}{2} \bmod n$.

Two Praeger-Xu graphs are illustrated in Figure 2.1. Figure 2.1a shows the smallest Praeger-Xu graph having $k > 1$, namely $\mathrm{PX}(3, 2)$, of order $3 \cdot 2^2 = 12$. Figure 2.1b shows the larger Praeger-Xu graph $\mathrm{PX}(20, 5)$ of order $20 \cdot 2^5 = 640$. In all our diagrams of Praeger-Xu graphs, $\mathcal{F}_0$ is the fibre in the 12 o'clock position, with remaining fibres labeled consecutively clockwise. The vertices in $\mathcal{F}_0$ on $\mathrm{PX}(3, 2)$ have been labeled with their bitstring components; the bitstring components of vertices in $\mathcal{F}_1$ and $\mathcal{F}_2$ follow the same pattern. More generally, the bitstring components are the binary representations of the integers 0 to $2^k - 1$, starting with the innermost vertex. Throughout this paper, we will be assuming that $n \geq 3$ and $1 \leq k < n$, unless otherwise explicitly indicated.

2.1 Automorphisms of $\mathrm{PX}(n, k)$

In [16], Praeger and Xu described the automorphism groups of all graphs in the family $C(p, r, s)$. We will adopt the notation used in [13] for automorphisms of the Praeger-Xu graphs, $PX(n, k)$. The automorphism group is generated by three different types of automorphisms.



(a) $\mathrm{PX}(3,2)$. (b) $\mathrm{PX}(20,5)$.

Figure 2.1: Two Praeger-Xu graphs.

The first is the rotation ρ, defined by $\rho\cdot(i,x) = (i+1, x)$. Composing ρ with itself s times corresponds to a rotation by s fibres: $\rho^s \cdot (i,x) = (i+s, x)$. If s is a multiple of n, then the resulting map is the identity, and so we can interpret s as an element of $\mathbb{Z}_n$.

The second type of automorphism is the reflection defined by $\mu \cdot (i,x) = (-i, x^-)$, where $x^- = (x_0x_1\cdots x_{k-1})^- = x_{k-1}\cdots x_1x_0$. It is easily verified that $\mu^2 = \mathrm{id}$ and $\mu\rho\mu = \rho^{-1}$, so the subgroup $\langle \rho, \mu\rangle$ of $\mathrm{Aut}\big(\mathrm{PX}(n,k)\big)$ is the dihedral group D_n.

Following [13], for each $s \in \mathbb{Z}_n$ we let $\mu_s = \rho^{s+1-k}\mu \in \langle\rho,\mu\rangle$, so that $\mu_s\cdot(i,x) = (s+1-k-i, x^-)$. With this notation, $\mu = \mu_{k-1}$; in particular, note that $\rho^0 = \mathrm{id}$ but $\mu_0 \neq \mathrm{id}$. We collect some elementary facts about the reflections μ_s in the following lemma.

Lemma 2.2. *Let $s, i, j \in \mathbb{Z}_n$.*

1. *The reflection μ_s interchanges fibres $\mathcal{F}_i$ and $\mathcal{F}_{s+1-k-i}$; equivalently, fibres $\mathcal{F}_i$ and $\mathcal{F}_j$ are interchanged by $\mu_{i+j+k-1}$.*
2. *If n is odd, then each μ_s preserves exactly one fibre. If n is even and $s = k \bmod 2$, then μ_s does not preserves any fibre, and if $s \neq k \bmod 2$, then μ_s preserves exactly two antipodal fibres.*

The proof of Lemma 2.2 is straightforward and left to the reader.

The third type of automorphism is, for each $s \in \mathbb{Z}_n$, defined by

$$\tau_s \cdot (i,x) = \begin{cases} (i, x^{s-i}), & \text{if } i \in \{s, s-1, s-2, \ldots, s-k+1\}, \\ (i,x), & \text{otherwise,} \end{cases}$$

where x^j denotes the bitstring x with bit x_j flipped. Thus τ_s flips bit x_{s-i} of the bitstring component of every vertex in $\mathcal{F}_i$ if $i \le s \le i+k-1$, and acts trivially on $\mathcal{F}_i$ otherwise. Equivalently, vertices in $\mathcal{F}_i$ have their bitstring components altered only by $\tau_i, \tau_{i+1}, \ldots, \tau_{i+k-1}$. Clearly each τ_s has order 2 and τ_s, τ_t commute for all $s, t \in \mathbb{Z}_n$. Hence the subgroup of $\mathrm{Aut}\big(\mathrm{PX}(n,k)\big)$ generated by these automorphisms satisfies $K = \langle \tau_0, \tau_1, \tau_2, \ldots, \tau_{n-1} \rangle \simeq \mathbb{Z}_2^n$. Each $\tau \in K$ can be represented by

$$\tau = \tau_0^{u_0} \tau_1^{u_1} \tau_2^{u_2} \cdots \tau_{n-1}^{u_{n-1}},$$

where $u_m \in \{0,1\}$ for each $m \in \mathbb{Z}_n$. It is easy to verify that $\rho^{-1}\tau_s\rho = \tau_{s+1}$ and $\mu\tau_s\mu = \tau_{k-1-s}$, so K is a normal subgroup of the group generated by ρ, μ, and $\tau_0, \ldots, \tau_{n-1}$.

Let $\mathcal{A} = K \rtimes \langle \rho, \mu \rangle = K \rtimes D_n$. Then if $\alpha \in \mathcal{A}$, $\alpha = \tau\delta$ for some $\tau \in K$ and $\delta \in \langle \rho, \mu \rangle = D_n$. Praeger and Xu showed in [16] that for all $n \neq 4$, $\mathcal{A} = \mathrm{Aut}\big(\mathrm{PX}(n,k)\big)$, while for $n = 4$, $\mathcal{A}$ is a proper subgroup of $\mathrm{Aut}\big(\mathrm{PX}(4,k)\big)$.

Note that, under any $\alpha \in \mathcal{A}$, vertices in the same fibre will be mapped to vertices in the same fibre. In other words, the fibres form a block system for the action of $\mathcal{A}$ on $\mathrm{PX}(n,k)$. From [13], the induced action of $\alpha = \tau\delta \in \mathcal{A}$ on the fibres of $\mathrm{PX}(n,k)$ is $\alpha(\mathcal{F}_i) = \mathcal{F}_{\delta(i)}$, where $\delta(i)$ is simply the action of the dihedral group element $\delta \in D_n$ on $i \in V(C_n) = \mathbb{Z}_n$. Since any $\tau = \tau_0^{u_0} \tau_1^{u_1} \tau_2^{u_2} \cdots \tau_{n-1}^{u_{n-1}} \in K$ preserves fibres, for any $\alpha = \tau\delta \in \mathcal{A}$ and $(i,x) \in V$, we have

$$\alpha \cdot (i,x) = \tau \cdot \big(\delta \cdot (i,x)\big) = \tau \cdot (\delta(i), y) = (\delta(i), z),$$

where $y = x$ if δ is a rotation ρ^s, $y = x^-$ if δ is a reflection μ_s, and for all $j \in \mathbb{Z}_k$, we have $z_j = y_j + 1$ if $u_{\delta(i)-j} = 1$ and $z_j = y_j$ otherwise.

3 Determining and distinguishing $\mathrm{PX}(n,1)$

For any vertex v in a graph $G = (V,E)$, the open *neighborhood* of v is $N(v) = \{u : \{u,v\} \in E\}$ and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. Distinct vertices x and y are *nonadjacent twins* if and only if $N(x) = N(y)$, and *adjacent twins* if and only if $N[u] = N[v]$. Twins are relevant to notions of graph symmetry because if x and y are nonadjacent or adjacent twins, then the map that interchanges x and y and fixes all other vertices is a graph automorphism. It is straightforward to verify that Praeger-Xu graphs have no adjacent twins; for the remainder of the paper, when we refer to twin vertices, we will always mean nonadjacent twin vertices.

Lemma 3.1. *For $k = 1$, two distinct vertices in* $\mathrm{PX}(n, 1)$ *are twins if and only if either they are in the same fibre, or $n = 4$ and they are in antipodal fibres. For $k \geq 2$, $PX(n, k)$ is twin-free.*

Proof. The case $k = 1$ is left as an exercise. So assume $k \geq 2$. Let u and v be distinct vertices in $\mathrm{PX}(n, k)$ such that $N(u) = N(v)$. Let $u = (i, axb)$ and $v = (j, cyd)$ for some $i, j \in \mathbb{Z}_n$, $a, b, c, d \in \{0, 1\}$, and $y, x \in \mathbb{Z}_2^{k-2}$ (where y and x are empty strings if $k = 2$). By definition, $N(u) = \big\{(i + 1,\, xb0),\, (i+1,\, xb1),\, (i-1,\, 0ax),\, (i-1,\, 1ax)\big\}$ and $N(v) = \big\{(j+1,\, yd0),\, (j+1,\, yd1),\, (j - 1,\, 0cy),\, (j - 1,\, 1cy)\big\}$. Since $N(u)$ consists of two vertices in each of $\mathcal{F}_{i+1}$ and $\mathcal{F}_{i-1}$, and $N(v)$ consists of two vertices in each of $\mathcal{F}_{j+1}$ and $\mathcal{F}_{j-1}$, we have $\{i + 1,\, i - 1\} = \{j + 1,\, j - 1\}$.

Suppose $i = j \bmod n$. Comparing neighbors in $\mathcal{F}_{i+1} = \mathcal{F}_{j+1}$ with the same final bit gives $xb0 = yd0$ and $xb1 = yd1$. Hence $xb = yd$ in $\mathbb{Z}_2^{k-1}$. An analogous argument can be used in $\mathcal{F}_{i-1}$ to show that $ax = cy$ in $\mathbb{Z}_2^{k-1}$. Thus $axb = cyd$ in $\mathbb{Z}_2^k$. Since $i = j$ in $\mathbb{Z}_n$, $(i, axb) = (j, cyd)$ and so $u = v$, contradicting the assumption that u and v are distinct.

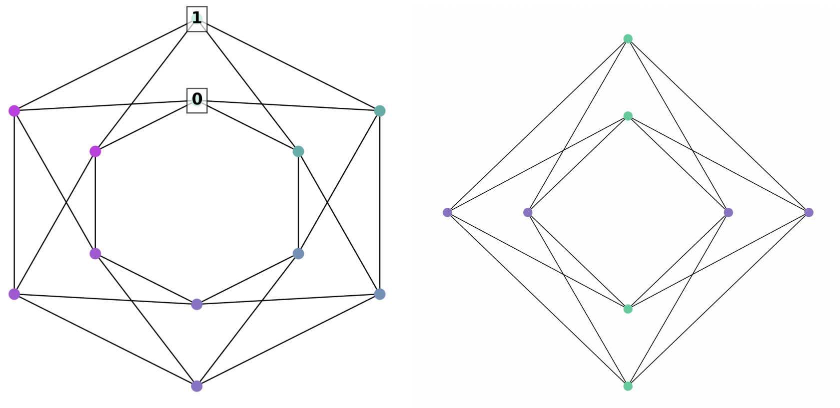
Alternatively, if $i \neq j \bmod n$, then as argued earlier in this proof, $n = 4$ and $i - 1 = j + 1 \bmod n$. Hence $N(u) \cap \mathcal{F}_{i-1} = N(v) \cap \mathcal{F}_{j+1}$, so

$$\big\{(i - 1,\, 0ax),\, (i - 1,\, 1ax)\big\} = \big\{(j + 1,\, yd0),\, (j + 1,\, yd1)\big\}.$$

Since x and y cannot be simultaneously both 0 and 1, this is impossible. $\square$

Figure 3.1 depicts two Praeger-Xu graphs with twins. Note that every vertex of $\mathrm{PX}(4, 1)$ is in a set of $t = 4$ mutual twins, while for any $n \geq 3$, $n \neq 4$, every vertex of $\mathrm{PX}(n, 1)$ is in a set of $t = 2$ mutual twins.

For any graph $G = (V, E)$ with twins, one can define an equivalence relation on V by $x \sim y$ if and only if x and y are twins. The corresponding *twin quotient graph* $\widetilde{G}$ has as its vertex set the set of equivalence classes $[x] = \{y \in V(G) : x \sim y\}$, with $\big\{[x], [z]\big\} \in E(\widetilde{G})$ if and only if there exist $p \in [x]$ and $q \in [z]$ such that $\{p, q\} \in E(G)$. (Note that by definition



(a) $\mathrm{PX}(6,1)$ (b) $\mathrm{PX}(4,1)$.

Figure 3.1: Two Praeger-Xu graphs with $k = 1$.

of the twin relation, $\big\{[x],[z]\big\} \in E(\widetilde{G})$ if and only if for all $p \in [x]$ and $q \in [z]$, $\{p,q\} \in E(G)$.) The symmetry parameters of the twin quotient graph $\widetilde{G}$ can be used to give information about the symmetry parameters of G.

In [7], Boutin et al. defined a *minimum twin cover* of G to be a subset $T \subseteq V(G)$ that contains all but one vertex from each equivalence class of twin vertices. For example, if $n \geq 3$ and $n \neq 4$, then any set of vertices that contains exactly one vertex in each fibre is a minimum twin cover of $\mathrm{PX}(n,1)$. In $\mathrm{PX}(4,1)$, any set of the form $V\big(\mathrm{PX}(4,1)\big) \setminus \{u,v\}$, where u and v are in adjacent fibres, is a minimum twin cover.

Clearly, any determining set must contain a minimum twin cover in order to break all twin-swapping automorphisms. In particular, if a minimum twin cover is a determining set, then it must be a minimum determining set.

For distinguishing number, we have the following result from Cockburn and Loeb.

Theorem 3.2 (Cockburn and Loeb [8, Theorem 2])**.** *Let G be a graph in which every vertex is in a set of t mutual twins. If* $\mathrm{Dist}(\widetilde{G}) = \widetilde{d}$*, then* $\mathrm{Dist}(G) = d$*, where d is the smallest positive integer such that* $\binom{d}{t} \geq \widetilde{d}$*.*

These results can be used to find the symmetry parameters of Praeger-Xu graphs with twins, $\mathrm{PX}(n,1)$.

Theorem 3.3. *For* $n=4$, $\mathrm{Det}\big(\mathrm{PX}(4,1)\big)=6$ *and* $\mathrm{Dist}\big(\mathrm{PX}(4,1)\big)=5$. *For all* $n\neq 4$, $\mathrm{Det}\big(\mathrm{PX}(n,1)\big)=n$ *and* $\mathrm{Dist}\big(\mathrm{PX}(n,1)\big)=3$.

Proof. As noted earlier, in $\mathrm{PX}(4,1)$, any set of the form $V\big(\mathrm{PX}(4,1)\big)\backslash\{u,v\}$, where u and v are in adjacent fibres, is a minimum twin cover. Because such u and v are not twins, such a set is also determining and hence a minimum determining set. Thus $\mathrm{Det}\big(\mathrm{PX}(4,1)\big)=6$. Since every vertex of $\mathrm{PX}(4,1)$ is in a set of $t=4$ mutual twins and $\mathrm{Dist}(K_2)=2$, by Theorem 3.2, $\mathrm{Dist}\big(\mathrm{PX}(4,1)\big)=5$.

Next assume $n\neq 4$. As noted earlier, a minimum twin cover contains exactly one vertex from each fibre. It is easy to verify that such a set is also a determining set, so $\mathrm{Det}\big(\mathrm{PX}(n,1)\big)=|T|=n$.

For distinguishing, since every fibre $\mathcal{F}_i$ in $\mathrm{PX}(n,1)$ is a vertex in $\widetilde{\mathrm{PX}}(n,1)$, and vertices in $\mathcal{F}_i$ are adjacent only to vertices in $\mathcal{F}_{i+1}$ and $\mathcal{F}_{i-1}$, we have $\widetilde{\mathrm{PX}}(n,1)=C_n$. Hence, $\widetilde{d}=\mathrm{Dist}\big(\widetilde{\mathrm{PX}}(n,1)\big)=\mathrm{Dist}(C_n)=3$ if $n\in\{3,5\}$, and $\widetilde{d}=2$ if $n\geq 6$. Since each vertex is in a set of $t=2$ twins, by Theorem 3.2, d is the smallest integer such that $\binom{d}{2}\geq 3$ if $n\in\{3,5\}$, and d is the smallest integer such that $\binom{d}{2}\geq 2$ if $n\geq 6$. In both cases, $d=3$. $\square$

4 Determining $\mathrm{PX}(n,k)$, $k\geq 2$

In this section, we find the determining number for twin-free Praeger-Xu graphs. Recall from Section 2 that for $n\neq 4$, we have $\mathcal{A}=K\rtimes\langle\rho,\mu\rangle=\mathrm{Aut}\big(\mathrm{PX}(n,k)\big)$, whereas for $n=4$, $\mathcal{A}$ is a proper subgroup of $\mathrm{Aut}\big(\mathrm{PX}(4,k)\big)$. Recall also that the induced action of $\alpha=\tau\delta\in\mathcal{A}$ on the fibres of $\mathrm{PX}(n,k)$ is $\alpha(\mathcal{F}_i)=\mathcal{F}_{\delta(i)}$, where $\delta\in\langle\rho,\mu\rangle$ is an element of the dihedral group. We begin with a lemma that applies to all Praeger-Xu graphs and apply it to the general case $n\neq 4$. We then consider the exceptional cases $\mathrm{PX}(4,2)$ and $\mathrm{PX}(4,3)$.

Lemma 4.1. *Let* $i\in\mathbb{Z}_n$ *and let* S_i *be a subset of the fibre* $\mathcal{F}_i\subset V\big(\mathrm{PX}(n,k)\big)$ *and let* $\tau=\tau_0^{u_0}\cdots\tau_{n-1}^{u_{n-1}}\in K$. *If* $\tau(S_i)=S_i$ *and* $|S_i|$ *is odd, then* τ *acts trivially on* $\mathcal{F}_i$; *equivalently,*

$$u_i=u_{i+1}=\cdots=u_{i+k-1}=0.$$

Proof. Assume $\tau(S_i) = S_i$ and that τ acts nontrivially on $\mathcal{F}_i$. Let $s \in S_i \subseteq \mathcal{F}_i$; by assumption, $\tau \cdot s \in S_i$. Since every element in $K \simeq \mathbb{Z}_2^n$ has order 2, $\tau \cdot (\tau \cdot s) = \tau^{-1} \cdot (\tau \cdot s) = s$. Additionally, since τ acts nontrivially on every vertex of $\mathcal{F}_i$, $s \neq \tau \cdot s$. Thus, S_i can be partitioned into pairs of the form $\{s, \tau \cdot s\}$, implying that S_i has an even number of vertices in total. □

Theorem 4.2. *For* $n \neq 4$,

$$\operatorname{Det}\big(\mathrm{PX}(n,k)\big) = \begin{cases} \left\lceil \frac{n}{k} \right\rceil, & \text{if } k \neq \frac{n}{2}, \\ \left\lceil \frac{n}{k} \right\rceil + 1, & \text{if } k = \frac{n}{2}. \end{cases}$$

Proof. First suppose $k \neq \frac{n}{2}$. Assume S is a determining set for $\mathrm{PX}(n,k)$ with $|S| = \left\lceil \frac{n}{k} \right\rceil - 1$. The set of indices of fibres containing elements of S is

$$I_S = \{i \in \mathbb{Z}_n \mid S \cap \mathcal{F}_i \neq \varnothing\} = \{i_1, i_2, \ldots, i_s\},$$

where $0 \leq i_1 < i_2 < \cdots < i_s \leq n-1$. The numbers of fibres in the gaps between these fibres are $i_2 - i_1 - 1, i_3 - i_2 - 1, \ldots, n + i_1 - i_s - 1$. If $i_{p+1} - i_p - 1 \geq k$ for some $i_p, i_{p+1} \in S$, then $\tau_{i_{p+1}-1}$ is a nontrivial automorphism fixing S, a contradiction. Thus each gap contains at most $k-1$ fibres. Since every fibre either contains a vertex in S or is in a gap between two such fibres, the total number of fibres satisfies

$$|I_S| + |I_S|(k-1) = |I_S|k \ \leq |S|k = \left(\left\lceil \tfrac{n}{k} \right\rceil - 1\right)k < n,$$

a contradiction. Thus, $\operatorname{Det}\big(\mathrm{PX}(n,k)\big) > \left\lceil \frac{n}{k} \right\rceil - 1$.

We claim $S = \left\{v_{ik} \in \mathcal{F}_{ik} : i \in \{0, 1, \ldots, \left\lceil \frac{n}{k} \right\rceil - 1\}\right\}$, where v_{ik} is any vertex in $\mathcal{F}_{ik}$, is a determining set for $\mathrm{PX}(n,k)$. Let $\alpha = \tau\delta \in \mathcal{A} = \operatorname{Aut}\big(\mathrm{PX}(n,k)\big)$ such that α fixes every vertex in S. Since $k \neq \frac{n}{2}$, $\mathcal{F}_0$ and $\mathcal{F}_k$ are non-antipodal fibres. Since the induced action of α on the fibres is an element of D_n that fixes non-antipodal vertices in C_n, $\delta = \mathrm{id}$.

Next we show that $\tau = \tau_0^{u_0}\tau_1^{u_1}\cdots\tau_{n-1}^{u_{n-1}} = \mathrm{id}$. Let $S_0 = S \cap \mathcal{F}_0$; note that $|S_0| = 1$ is odd. Since τ fixes every vertex in S, $\tau(S_0) = S_0$, and so by Lemma 4.1, $u_0 = u_1 = \cdots = u_{k+1} = 0$. Applying the same logic to $S_k = S \cap \mathcal{F}_k$, we get $u_k = u_{k+1} = \cdots = u_{2k-1} = 0$. Iterating this argument for S_{ik} for all $i \in \{0, 1, \ldots, \left\lceil \frac{n}{k} \right\rceil - 1\}$, we conclude $u_0 = \cdots = u_{n-1} = 0$. Thus $\tau = \mathrm{id}$. By definition, S is a determining set and so $\operatorname{Det}\big(\mathrm{PX}(n,k)\big) \leq |S| = \left\lceil \frac{n}{k} \right\rceil$. Thus $\operatorname{Det}\big(\mathrm{PX}(n,k)\big) = \left\lceil \frac{n}{k} \right\rceil$.

Now suppose $k = \frac{n}{2}$, so that $\left\lceil \frac{n}{k} \right\rceil + 1 = 3$. Assume S is a determining set of cardinality $\left\lceil \frac{n}{k} \right\rceil = 2$. Since $\mathrm{PX}(n,k)$ is vertex-transitive, we can assume without loss of generality that $S = \{z = (0, 00\cdots 0), v = (i, x)\}$. There are three cases: $i = 0$, $i = k = \frac{n}{2}$, or $i \neq 0$ and $i \neq k = \frac{n}{2}$.

For the first case, assume $i = 0$, so both $z, v \in \mathcal{F}_0$. Then since τ_{n-1} affects $\mathcal{F}_i$ if and only if $i \in \{k = \frac{n}{2}, \frac{n}{2}+1, \ldots, n-1\}$, and 0 is not in that set, τ_{n-1} is a nontrivial automorphism that fixes S, a contradiction.

For the second case, assume $i = k = \frac{n}{2}$. Then z and v are in antipodal fibres. If we apply the reflection μ to S, we get

$$\mu\big(S\big) = \big\{(0, (00\cdots 0)^-), (-k, x^-)\big\} = \big\{(0, 00\cdots 0), (k, x^-)\big\}.$$

For each m such that $x_m \neq (x^-)_m$, the automorphism $\tau_{k+m} \in K$ flips this bit in the bitstring component of every vertex in $\mathcal{F}_k$, but has no effect on the vertices in $\mathcal{F}_0$. Let $\tau = \tau_0^{u_0}\tau_1^{u_1}\cdots\tau_{n-1}^{u_{n-1}}$ where $u_{k+m} = 1$ if $x_m \neq (x^-)_m$ and 0 otherwise. Then $\zeta = \tau\mu$ fixes both z and v, contradicting our assumption that S is a determining set.

In the third case, $i \neq 0$ and $i \neq k = \frac{n}{2}$, so we can assume that in $\mathbb{Z}$, either $0 < i < k$ or $k < i < n$. In the first case, τ_{n-1} fixes both z and v, and in the second, τ_{i-1} fixes both z and v. Hence, S is not a determining set for $\mathrm{PX}(n,k)$. As we have covered all possible cases, we conclude $\mathrm{Det}\big(\mathrm{PX}(n,k)\big) > 2$.

Finally, let $S = \{v_0, v_1, v_k\}$ where $v_0 \in \mathcal{F}_0$, $v_1 \in \mathcal{F}_1$ and $v_k \in \mathcal{F}_k$ and assume $\alpha = \tau\delta$ fixes S. The induced action of α on the fibres corresponds to an element of the dihedral group that fixes non-antipodal vertices 0 and 1 in C_n, so $\delta = \mathrm{id}$. Next, $\tau = \tau_0^{u_0}\cdots\tau_{n-1}^{u_{n-1}}$ fixes one vertex in each of $\mathcal{F}_0$ and $\mathcal{F}_k$, so by Lemma 4.1, $u_0 = \cdots = u_{k-1} = u_k = \cdots = u_{2k-1} = u_{n-1} = 0$. Thus $\tau = \mathrm{id}$. By definition, S is a determining set for $\mathrm{PX}(n,k)$ so $\mathrm{Det}\big(\mathrm{PX}(n,k)\big) \leq |S| = 3$. Thus $\mathrm{Det}\big(\mathrm{PX}(n,k)\big) = 3 = \left\lceil \frac{n}{k} \right\rceil + 1$. $\square$

We now turn our attention to the exceptional cases $\mathrm{PX}(4,2)$ and $\mathrm{PX}(4,3)$. It is stated without proof in [16] that $Q_4 \cong \mathrm{PX}(4,2)$; we provide an explicit isomorphism geometrically. Figure 4.1 is a drawing of Q_4, with vertices positioned as they would be in a canonical drawing of $\mathrm{PX}(4,2)$, as explained at the beginning of Section 2.

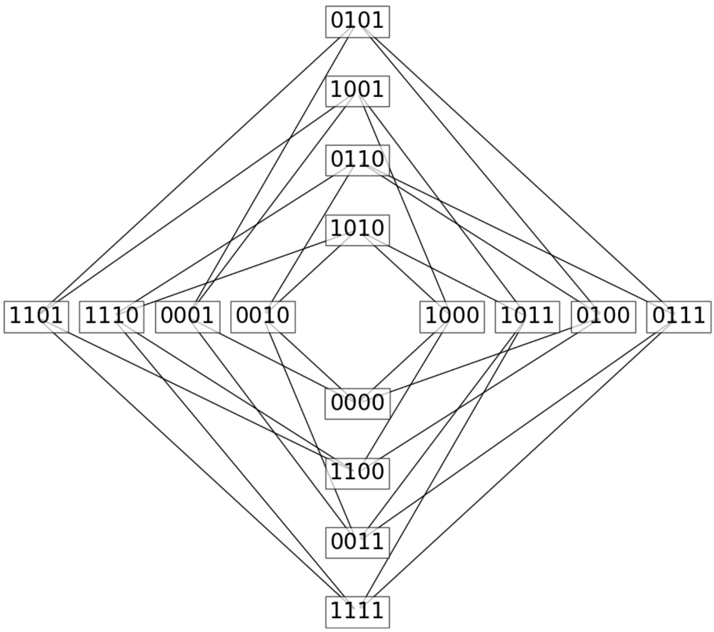


Figure 4.1: Q_4 is isomorphic to $\mathrm{PX}(4,2)$.

Proposition 4.3. *$\mathrm{Det}\big(\mathrm{PX}(4,2)\big)$=3, $\mathrm{Dist}\big(\mathrm{PX}(4,2)\big)$=2 and $\rho\big(\mathrm{PX}(4,2)\big)$=5.*

Proof. This follows immediately from previous work on the symmetry parameters of Q_4. By Theorem 3 from [5],

$$\mathrm{Det}\big(\mathrm{PX}(4,2)\big) = \mathrm{Det}(Q_4) = \lceil \log_2 4 \rceil + 1 = 2 + 1 = 3.$$

Notably, this expression agrees with the formula given in Theorem 4.2 because $\left\lceil \frac{4}{2} \right\rceil + 1 = 2 + 1 = 3$. By Theorem 5 from [3], $\mathrm{Dist}\big(\mathrm{PX}(4,2)\big) = \mathrm{Dist}(Q_4) = 2$; by Theorem 11 from [6], $\rho\big(\mathrm{PX}(4,2)\big) = \rho(Q_4) = 5$. □

Proposition 4.4. $\mathrm{Det}\big(\mathrm{PX}(4,3)\big) = 2 = \left\lceil \frac{4}{3} \right\rceil$.

Proof. From [16], $\mathcal{A} = K \rtimes \langle \rho, \mu \rangle$ is a proper subgroup of $\mathrm{Aut}\big(\mathrm{PX}(4,3)\big)$ of index 2. The proof of Theorem 4.2 can be used when $(n,k) = (4,3)$ to show that the only $\alpha \in \mathcal{A}$ that fixes two vertices from non-antipodal fibres is the identity. However, more care must be taken when choosing the two vertices. For example, there is a nontrivial automorphism $\xi \in \mathrm{Aut}(\mathrm{PX}(4,3))$ that fixes both elements of $S = \{(0,000),\ (3,000)\}$. More precisely, as a permutation ξ is the product of the disjoint 2-cycles in Table 4.1. Unlike for any $\alpha \in \mathcal{A}$, the fibres do not constitute a block system for ξ. However, if we partition each fibre into vertices whose bitstrings are palindromic ($x = x^-$) and vertices whose bitstrings are nonpalindromic ($x \neq x^-$), these half-fibres constitute a block system for ξ.

Table 4.1: Disjoint 2-cycles of $\xi \in \mathrm{Aut}(\mathrm{PX}(4,3))$.

within $\mathcal{F}_0$, $x = x^-$	$((0,010),\ (0,101))$
within $\mathcal{F}_2$, $x \neq x^-$	$((2,001),\ (2,110))$
between $\mathcal{F}_0$, $x \neq x^-$ and $\mathcal{F}_2$, $x = x^-$	$((0,001),\ (2,000))$, $((0,100),\ (2,010))$ $((0,011),\ (2,101))$, $((0,110),\ (2,111))$
within $\mathcal{F}_1$, $x \neq x^-$	$((1,100),\ (1,011))$
within $\mathcal{F}_3$, $x = x^-$	$((3,010),\ (3,101))$
between $\mathcal{F}_1$, $x = x^-$ and $\mathcal{F}_3$, $x \neq x^-$	$((1,000),\ (3,100))$, $((1,010),\ (3,001))$ $((1,101),\ (3,110))$, $((1,111),\ (3,011))$

Note that ξ has eight fixed points, with two in each fibre, namely $(0,000)$, $(0,111)$, $(1,001)$, $(1,110)$, $(2,100)$, $(2,001)$, $(3,000)$, and $(3,111)$. Since $\mathcal{A}$ has index 2 in $\mathrm{Aut}(\mathrm{PX}(4,3))$, every automorphism of $\mathrm{PX}(4,3)$ is in one of the two cosets, $\mathcal{A}$ and $\mathcal{A}\xi$.

Next we show that $S' = \{(0,000),\ (3,001)\}$ is a determining set. The two vertices in S' are from non-antipodal fibres, so no nontrivial automorphism in $\mathcal{A}$ fixes both vertices in S'. Table 4.1 shows that ξ clearly does not fix $(3,001)$; we must also show that no other automorphism in the coset $\mathcal{A}\xi$ fixes S'.

Assume there exists $\beta \in \mathcal{A}$ such that $\beta \circ \xi$ fixes S'. Then $(0,000) = \beta \circ \xi \cdot (0,000) = \beta \cdot (0,000)$ and $(3,001) = \beta \circ \xi \cdot (3,001) = \beta \cdot (1,010)$. The induced action of β on the fibres fixes $\mathcal{F}_0$ and takes $\mathcal{F}_1$ to $\mathcal{F}_3$, so by Lemma 2.2, $\beta = \tau \circ \mu = \tau_0^{u_0}\tau_1^{u_1}\tau_2^{u_2}\tau_3^{u_3}\mu$. Since β and μ both fix $(0,000)$, so must τ and by Lemma 4.1, $u_0 = u_1 = u_2 = 0$. However, because τ_3 can only affect the 0-th bit of the bitstring component of a vertex in $\mathcal{F}_3$, no value of u_3 satisfies $(3,001) = \beta \cdot (1,010) = \tau_3^{u_3}\mu \cdot (1,010) = \tau_3^{u_3} \cdot (3,010)$. □

The following theorem summarizes our results on the determining numbers of twin-free Praeger-Xu graphs.

Theorem 4.5. *For all* $n \geq 3$ *and* $2 \leq k < n$,

$$\operatorname{Det}\big(\mathrm{PX}(n,k)\big) = \begin{cases} \left\lceil \frac{n}{k} \right\rceil, & \text{if } k \neq \frac{n}{2}, \\ \left\lceil \frac{n}{k} \right\rceil + 1, & \text{if } k = \frac{n}{2}. \end{cases}$$

5 Interchangeable vertices in $\mathrm{PX}(n,k)$

As mentioned in Section 3, if two vertices in a graph are twins, then the map that interchanges them and leaves all other vertices fixed is a graph automorphism. By Lemma 3.1, if $k \geq 2$, then $\mathrm{PX}(n,k)$ is twin-free, but we will find it useful to identify when two vertices can be interchanged by an automorphism, regardless of its action on other vertices.

Definition 5.1. Distinct vertices u, v in a graph G are *interchangeable* if and only if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha \cdot u = v$ and $\alpha \cdot v = u$.

There are some situations where it is easy to find an automorphism interchanging vertices $u = (i,x)$ and $v = (j,y)$ in $\mathrm{PX}(n,k)$. If $i = j$, there exists $\tau = \tau_0^{u_0} \cdots \tau_{n-1}^{u_{n-1}} \in K$ that flips exactly the right bits in bitstring components of vertices in $\mathcal{F}_i$. More precisely, for each $t \in \{0,1,\ldots,k-1\}$, if $x_t \neq y_t$, set $u_{i+t} = 1$, and otherwise set $u_{i+t} = 0$. The values of u_m for any $m \in \mathbb{Z}_n$ not of the form $i+t$ do not affect the action of τ on u and v. If $i \neq j$, then we can find $\delta \in \langle \rho, \mu \rangle$ such that the induced action of δ on the fibres interchanges $\mathcal{F}_i$ and $\mathcal{F}_j$; we can then look for $\tau = \tau_0^{u_0} \cdots \tau_{n-1}^{u_{n-1}} \in K$ so that τ flips exactly the right bits in both $\mathcal{F}_i$ and $\mathcal{F}_j$ to ensure that $\tau\delta$ interchanges u and v. If $\mathcal{F}_i$ and $\mathcal{F}_j$ are far enough apart, then we can set the values $u_i, u_{i+1}, \ldots, u_{i+k-1}$ and $u_j, u_{j+1}, \ldots, u_{j+k-1}$ independently. However, if $M = \big\{i, i+1, i+2, \ldots, i+k-1\big\} \cap \big\{j, j+1, j+2, \ldots, j+k-1\big\} \neq \varnothing$, then for any $m \in M$, τ_m affects both vertices in $\mathcal{F}_i$ and $\mathcal{F}_j$ and there is potential for conflict.

Lemma 5.2. *Let* $u = (i,x), v = (j,y) \in V\big(\mathrm{PX}(n,k)\big)$ *and let*

$$M = \big\{i, i+1, i+2, \ldots, i+k-1\big\} \cap \big\{j, j+1, j+2, \ldots, j+k-1\big\} \subseteq \mathbb{Z}_n.$$

Then u *and* v *are interchangeable by some* $\alpha \in \mathcal{A}$ *if and only if one of the following holds:*

(1) $j = i$,

(2) $j \neq i$ *and for all* $m \in M$, $(x^-)_{m-j} = y_{m-j}$ *if and only if* $(y^-)_{m-i} = x_{m-i}$,

(3) $j = i + \frac{n}{2}$ *and for all* $m \in M$, $x_{m-j} = y_{m-j}$ *if and only if* $y_{m-i} = x_{m-i}$.

Proof. Assume u and v are interchangeable by $\alpha = \tau\delta \in \mathcal{A}$, but neither (1) nor (2) holds. Then $j \neq i$ and for some $m \in M$, either $(x^-)_{m-j} = y_{m-j}$ but $(y^-)_{m-i} \neq x_{m-i}$, or $(x^-)_{m-j} \neq y_{m-j}$ but $(y^-)_{m-i} = x_{m-i}$. As usual, either $\delta = \rho^s$ or $\delta = \mu_s$ for some $s \in \mathbb{Z}_n$.

If $\delta = \mu_s$, then $\mu_s \cdot u = (j, x^-)$ and $\mu_s \cdot v = (i, y^-)$, and hence $\tau \cdot (j, x^-) = (j, y)$ and $\tau \cdot (i, y^-) = (i, x)$, where $\tau = \tau_0^{u_0} \tau_1^{u_1} \cdots \tau_{n-1}^{u_{n-1}}$. If $(x^-)_{m-j} = y_{m-j}$ and $(y^-)_{m-i} \neq x_{m-i}$, then since τ_m flips the $(m-j)$-th bit of the bitstrings in $\mathcal{F}_j$, $u_m = 0$. However, τ_m flips the $(m-i)$-th bit of bitstrings in $\mathcal{F}_i$, so $u_m = 1$, a contradiction. A completely analogous argument works if $(x^-)_{m-j} \neq y_{m-j}$ and $(y^-)_{m-i} = x_{m-i}$. Thus, $\delta = \rho^s$.

Since $\rho^s \cdot u = (i+s, x)$ and $\rho^s \cdot v = (j+s, y)$, and τ fixes every fibre, $i + s = j$ and $j + s = i$. Since we are assuming $i \neq j$, $s = \frac{n}{2}$, so $j = i + \frac{n}{2}$. Let $m \in M$, and assume $x_{m-j} = y_{m-j}$. Since $\tau \cdot (j, x) = (j, y)$, τ must not flip the $(m-j)$-th bit of the bitstrings of $\mathcal{F}_j$, so $u_m = 0$. That means that τ must also not flip the $(m-i)$-th bit of the bitstrings of $\mathcal{F}_i$, so since $\tau \cdot (i, y) = (i, x)$, $y_{m-i} = x_{m-i}$. A completely analogous argument works if we assume $y_{m-i} = x_{m-i}$. Thus condition (3) holds.

Conversely, we will show that if one of (1), (2) or (3) holds, then u and v are interchangeable by some $\alpha \in \mathcal{A}$. First, assume (1) holds, so $j = i$. As noted in the paragraph before the statement of this lemma, there is some $\tau \in K \subset \mathcal{A}$ that interchanges u and v.

Next, assume (2) holds, so $j \neq i$ and for all $m \in M$, $(x^-)_{m-j} = y_{m-j}$ if and only if $(y^-)_{m-i} = x_{m-i}$. Let $\tau = \tau_0^{u_0} \tau_1^{u_1} \cdots \tau_{n-1}^{u_{n-1}} \in K$, where for each $s \in \mathbb{Z}_n$, $u_s = 1$ if $(x^-)_{s-j} \neq y_{s-j}$ and $(y^-)_{s-i} \neq x_{s-i}$ and $u_s = 0$ otherwise. Hence, in $\mathcal{F}_j$, τ flips the bits in every position that x^- and y differ and no others, and in $\mathcal{F}_i$, τ flips the bits in every position that y^- and x differ and no others. By Lemma 2.2, there exists $\mu_s \in \langle \rho, \mu \rangle$ such that $\mu_s \cdot u = (j, x^-)$ and $\mu_s \cdot v = (i, y^-)$. Let $\alpha = \tau\mu_s \in \mathcal{A}$. Then $\alpha \cdot u = \tau \cdot (\mu_s \cdot u) = \tau \cdot (j, x^-) = (j, y)$, and $\alpha \cdot v = \tau \cdot (\mu_s \cdot v) = \tau \cdot (i, y^-) = (i, x)$, so u and v are interchangeable by $\alpha \in \mathcal{A}$.

Lastly, assume (3) holds, so $j = i + \frac{n}{2}$ and for all $m \in M$, $x_{m-j} = y_{m-j}$ if and only if $y_{m-i} = x_{m-i}$. Let $\tau = \tau_0^{u_0} \tau_1^{u_1} \cdots \tau_{n-1}^{u_{n-1}} \in K$, where for each $s \in \mathbb{Z}_n$, $u_s = 1$ if $x_{s-j} \neq y_{s-j}$ and $u_s = 0$ otherwise. By assumption, that also means that $u_s = 1$ if $y_{s-i} \neq x_{s-i}$ and $u_s = 0$ otherwise. Hence, in both $\mathcal{F}_i$ and $\mathcal{F}_j$, τ flips the bits in every position that x and y differ, and no others. It is straighforward to verify that $\alpha = \tau\rho^{n/2} \in \mathcal{A}$ interchanges u and v. $\square$

For example, let $u = (i, x) = (0, 101), v = (j, y) = (1, 001) \in V\big(\mathrm{PX}(5, 3)\big)$. Then $M = \{0, 1, 2\} \cap \{1, 2, 3\} = \{1, 2\}$. Since $x^- = 101$ and $y^- = 100$, for $m = 1$, $(x^-)_{m-j} = 1 \neq 0 = y_{m-j}$ and $(y^-)_{m-i} = 0 = x_{m-i}$. Hence, by Lemma 5.2, u and v are not interchangeable.

Note that it is possible for a pair of vertices to be interchangeable by two different automorphisms. For example, $z = (0, 000)$ and $v = (5, 000)$ in $\mathrm{PX}(10, 3)$ satisfy both conditions (2) and (3), and so can be interchanged using either a rotation or a reflection.

We will find it useful in the next section to identify which vertices of $\mathrm{PX}(n, k)$ are interchangeable with $z = (0, 00 \cdots 0)$.

Corollary 5.3. *Let $v = (j, y) \in V\big(\mathrm{PX}(n, k)\big)$ and let*

$$M = \big\{0, 1, 2, \ldots, k-1\big\} \cap \big\{j, j+1, j+2, \ldots, j+k-1\big\}.$$

Then $z = (0, 00 \cdots 0)$ and v are interchangeable by some $\alpha \in \mathcal{A}$ if and only if one of the following holds:

(1) $j = 0$,

(2) $j \neq 0$ *and for all* $m \in M$, $y_{m-j} = (y^-)_m = y_{k-1-m}$,

(3) $j = \frac{n}{2}$ *and for all* $m \in M$, $y_{m-j} = y_m$.

To illustrate how vertex interchangeability can be used to compute symmetry parameters, we consider the smallest twin-free Praeger-Xu graph.

Lemma 5.4. *Any two distinct vertices of* $\mathrm{PX}(3, 2)$ *are interchangeable.*

Proof. By vertex-transitivity, it suffices to show that $z = (0, 00)$ is interchangeable with any vertex $v = (j, y)$. Since $n = 3$ is odd, we need only check that condition (2) of Corollary 5.3 holds. If $j = 1$, then $M = \{1\}$ and $y_{m-j} = y_{k-1-m} = y_0$. If $j = 2$, then $M = \{0\}$ and $y_{m-j} = y_{k-1-m} = y_1$. $\square$

Theorem 5.5. $\mathrm{Dist}\big(\mathrm{PX}(3, 2)\big) = 2$ *and* $\rho\big(\mathrm{PX}(3, 2)\big) = 3$.

Proof. Color the vertices in $R = \{(0,00),(1,01),(2,00)\}$ red and every other vertex blue. Assume $\alpha = \tau\delta \in \mathcal{A} = \mathrm{Aut}\big(\mathrm{PX}(3,2)\big)$ preserves colors, where $\tau = \tau_0^{u_0}\tau_1^{u_1}\tau_2^{u_2}$.

Suppose $\delta = \rho^s$ for some $s \in \{1,2\}$. If $s = 1$, then $\delta \cdot (1,01) = (2,01)$. Since α preserves colors, $\tau \cdot (2,01) = (2,00)$, which implies $u_0 = 1$ and $u_2 = 0$. However, then $\tau\delta \cdot (2,00) = \tau \cdot (0,00) = (0,1a)$ for some $a \in \mathbb{Z}_2$. This contradicts our assumption that α preserves colors. If $s = 2$, then $\delta \cdot (1,01) = (0,01)$, and hence $u_1 = 1$ and $u_0 = 0$. A similar contradiction arises because $\delta \cdot (2,00) = (1,00)$, and $u_1 = 1$ means $\tau \cdot (1,00) = (1,1a)$ for some $a \in \mathbb{Z}_2$. Thus δ cannot be a nontrivial rotation.

Suppose $\delta = \mu_s$ for some $s \in \mathbb{Z}_3$. By Lemma 2.2, δ preserves one fibre and interchanges the other two. If δ preserves $\mathcal{F}_0$ and interchanges $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\delta \cdot (1,01) = (2,10)$. Since α preserves colors, $\tau \cdot (2,10) = (2,00)$, so $u_2 = 1$. However, $\delta \cdot (2,00) = (1,00)$, which implies $\tau \cdot (1,00) = (1,1a)$ for some $a \in \mathbb{Z}_2$, a contradiction. Similar arguments apply to the remaining two cases. Since δ is neither a nontrivial rotation nor a reflection, $\delta = \mathrm{id}$.

Since α preserves colors and fibres, τ preserves the one red vertex in each fibre, so by Lemma 4.1, $u_0 = u_1 = u_2 = 0$. Hence, $\alpha = \tau = \mathrm{id}$, so this is a 2-distinguishing coloring with smaller color class of size 3. Thus, $\mathrm{Dist}\big(\mathrm{PX}(3,2)\big) = 2$ and $\rho\big(\mathrm{PX}(3,2)\big) \le 3$.

To show $\rho\big(\mathrm{PX}(3,2)\big) > 2$, let $R = \{u,v\} \subset V\big(\mathrm{PX}(3,2)\big)$. Color the vertices in R red and every other vertex blue. By Lemma 5.4, u and v are interchangeable; any automorphism interchanging them is a nontrivial color-preserving automorphism. Thus, $\rho\big(\mathrm{PX}(3,2)\big) > 2$, so $\rho\big(\mathrm{PX}(3,2)\big) = 3$. □

6 Distinguishing $\mathrm{PX}(n,k)$, $k \ge 2$

We have already found the distinguishing parameters for a number of Praeger-Xu graphs. Theorem 3.3 covers the case $k = 1$; Theorem 5.5 covers $\mathrm{PX}(3,2)$ and Proposition 4.3 covers $\mathrm{PX}(4,2)$. The next result covers the exceptional case $\mathrm{PX}(4,3)$. The remainder of this section covers the case $n \ge 5$ and $k \ge 2$.

Theorem 6.1. $\mathrm{Dist}\big(\mathrm{PX}(4,3)\big) = 2$ *and* $\rho\big(\mathrm{PX}(4,3)\big) = 3 = \lceil \frac{4}{3} \rceil + 1$.

Proof. Color the vertices in $R = \{(0,000),(2,000),(3,001)\}$ red and all other vertices blue. Suppose $\beta \in \mathrm{Aut}\big(\mathrm{PX}(4,3)\big)$ preserves this coloring.

Recall that $\mathrm{Aut}\big(\mathrm{PX}(4,3)\big)$ can be partitioned into the cosets $\mathcal{A}$ and $\mathcal{A}\xi$. First assume $\beta = \alpha\xi$ for some $\alpha \in \mathcal{A}$. Then by assumption,

$$R = \beta\big(\{(0,000),(2,000),(3,001)\}\big) = \alpha(\{(0,000),(0,001),(1,010)\}).$$

Note that R contains vertices in three different fibres, but since the fibres form a block system for any $\alpha \in \mathcal{A}$, $\alpha(\{(0,000),(0,001),(1,010)\})$ contains two vertices in one fibre and a third vertex in a different fibre. So these two sets cannot be equal. Hence $\beta \notin \mathcal{A}\xi$.

Thus $\beta \in \mathcal{A}$, so $\beta = \tau\delta$ for some $\delta \in \langle \rho, \mu \rangle$. Note that $(2,000)$ and $(3,001)$ are adjacent, but neither is adjacent to $(0,000)$. Thus β fixes $(0,000)$. If the induced action of β on the fibres fixes $\mathcal{F}_0$, then either $\delta = \mathrm{id}$ or $\delta = \mu$. Since μ does not interchange fibres $\mathcal{F}_2$ and $\mathcal{F}_3$, $\delta = \mathrm{id}$. Thus β fixes every vertex in R, which contains $\{(0,000),(3,001)\}$, the determining set for $\mathrm{PX}(4,3)$ found in Proposition 4.4. Hence $\beta = \mathrm{id}$. Thus this is a 2-distinguishing coloring, proving that $\mathrm{Dist}\big(\mathrm{PX}(4,3)\big) = 2$.

Next we show that we cannot create a 2-distinguishing coloring with fewer red vertices. If $R = \{(i,x)\}$, then τ_{i-1} is a nontrivial automorphism preserving the coloring. To show that no two-element set of red vertices provides a distinguishing coloring, it suffices, by vertex transitivity, to show that every vertex in $\mathrm{PX}(4,3)$ is interchangeable with $z = (0,000)$. Corollary 5.3 shows that $z = (0,000)$ is interchangeable with every vertex in $\mathrm{PX}(4,3)$ by some $\alpha \in \mathcal{A}$ except those listed below:

$$(1,010),(1,011),(1,100),(1,101),(3,010),(3,110),(3,001),(3,101). \quad (*)$$

For each vertex in (*), we can find $\alpha \in \mathcal{A}$ such that $\alpha\xi$ interchanges it with $(0,000)$. For $(1,010)$, we seek $\alpha \in \mathcal{A}$ that satisfies $\alpha\big(\xi \cdot (0,000)\big) = (1,010)$ and $\alpha\big(\xi \cdot (1,010)\big) = (0,000)$. Referring to Table 4.1 for the action of ξ, we seek $\alpha \in \mathcal{A}$ such that $\alpha \cdot (0,000) = (1,010)$ and $\alpha \cdot (3,001) = (0,000)$. It is easy to verify that $\alpha = \tau_2\rho$ satisfies this condition. For each vertex v in (*), Table 6.1 gives an α satisfying $\alpha\xi \cdot (0,000) = \alpha \cdot (0,000) = v$ and $\alpha\xi \cdot v = (0,000)$. □

Theorem 6.2. *Let $n \geq 5$ and $k \geq 2$. Then* $\mathrm{Dist}\big(\mathrm{PX}(n,k)\big) = 2$ *and* $\lceil \frac{n}{k} \rceil \leq \rho\big(\mathrm{PX}(n,k)\big) \leq \lceil \frac{n}{k} \rceil + 1$.

Proof. Let $x = 00\cdots0, y = 11\cdots1 \in \mathbb{Z}_2^k$. Then let

$$R = \big\{(ik,x) : i \in \{0,1,\ldots,\lceil \tfrac{n}{k} \rceil - 1\}\big\} \cup \big\{(1,y)\big\}.$$

Table 6.1: $\alpha \in \mathcal{A}$ such that $\alpha\xi$ interchanges $(0, 000)$ and v in $(*)$.

v	$\xi \cdot v$	α	v	$\xi \cdot v$	α
$(1, 010)$	$(3, 001)$	$\tau_2\rho$	$(3, 010)$	$(3, 101)$	$\tau_0\tau_2\mu_1$
$(1, 011)$	$(1, 100)$	$\tau_2\tau_3\mu_3$	$(3, 110)$	$(1, 101)$	$\tau_0\tau_2\tau_3\rho^3$
$(1, 100)$	$(1, 011)$	$\tau_0\tau_1\mu_3$	$(3, 001)$	$(1, 010)$	$\tau_1\rho^3$
$(1, 101)$	$(3, 110)$	$\tau_0\tau_1\tau_3\rho$	$(3, 101)$	$(3, 010)$	$\tau_1\tau_3\mu_1$

Color the vertices in R red and all other vertices blue. Assume $\alpha = \tau\delta \in \operatorname{Aut}\bigl(\mathrm{PX}(n, k)\bigr)$ preserves these color classes. Then the induced action of δ on the fibres must preserve the set $I = \bigl\{0, 1, k, 2k, \ldots, \bigl(\lceil \frac{n}{k} \rceil - 1\bigr)k\bigr\} \subset \mathbb{Z}_n$. Note that $|R| = |I| = \lceil \frac{n}{k} \rceil + 1 < n$. Interpreting $\mathbb{Z}_n$ as the vertex set of the cycle C_n, the (non-spanning) subgraph of C_n induced by I consists of a path containing at least the vertices 0 and 1, and possibly some isolated vertices. Let $F \subset \mathbb{Z}_n$ denote the set of vertices in the path; these will be the indices corresponding to a set of adjacent fibres of $\mathrm{PX}(n, k)$ containing red vertices. Note that the action of δ on C_n must preserve F. Since no nontrivial rotation preserves a proper subpath of C_n, $\delta \neq \rho^s$ for any $0 \neq s \in \mathbb{Z}_n$. So assume $\delta = \mu_s$ for some $s \in \mathbb{Z}_n$, and as usual, $\tau = \tau_0^{u_o} \cdots \tau_{n-1}^{u_{n-1}}$.

First assume $k = 2$. If $n \geq 5$ is odd, then $I = \{0, 1, 2, 4, \ldots, n-1\}$ and $F = \{n-1, 0, 1, 2\}$. For F to be preserved under reflection, μ_s must interchange the vertex pairs $\{0, 1\}$ and $\{n-1, 2\}$ in C_n. By Lemma 2.2, $s = 2$. In $\mathrm{PX}(n, 2)$, $\alpha = \tau\mu_2$ must interchange the vertex pairs $\{(0, 00), (1, 11)\}$ and $\{(n-1, 00), (2, 00)\}$. Thus $\tau \cdot (0, 11) = (0, 00)$ and $\tau \cdot (1, 11) = (1, 00)$, which implies $u_0 = u_1 = u_2 = 1$. However, it must also be the case that $\tau \cdot (n-1, 00) = (n-1, 00)$, so $u_0 = 0$, a contradiction.

If instead $n \geq 5$ is even, then $I = \{0, 1, 2, 4, \ldots, n-2\}$ and $F = \{0, 1, 2\}$. In this case, μ_s must fix 1 and interchange 0 and 2, so by Lemma 2.2, $s = 3$. In this case, as an element of $\operatorname{Aut}\bigl(\mathrm{PX}(n, 2)\bigr)$, μ_3 is a nontrivial automorphism preserving R, so this does not define a 2-distinguishing coloring. However, let

$$R' = \{(0, y)\} \cup \bigl\{(ik, x) : i \in \{1, \ldots, \lceil \tfrac{n}{k} \rceil - 1\}\bigr\} \cup \bigl\{(1, x)\bigr\}.$$

Then $I' = I = \{0, 1, 2, 4, \ldots, n-2\}$ and $F' = F = \{0, 1, 2\}$. The only reflection preserving F' is still μ_3. If $\alpha = \tau\mu_3$ preserves R', then $\tau\mu_3 \cdot (0, 11) = \tau \cdot (2, 11) = (2, 00)$, which means $u_2 = u_3 = 1$. Also, $\tau\mu_3 \cdot (2, 00) = \tau \cdot (0, 00) = (0, 11)$, so $u_0 = u_1 = 1$. This creates a contradiction because $\tau\mu_3 \cdot (1, 00) = \tau(1, 00) = (1, 00)$, which implies $u_1 = u_2 = 0$.

Now assume $k > 2$. If $n \neq 1 \bmod k$, then $F = \{0, 1\}$. Then μ_s must interchange 0 and 1, so by Lemma 2.2, $s = k$. Since $\alpha = \tau\mu_k$ preserves R, $\tau \cdot (0, y) = (0, x)$ and $\tau \cdot (1, x) = (1, y)$. Then $u_0 = u_1 = \cdots = u_k = 1$. Since $k \in I$ and μ_k preserves I, $\mu_k(j) = k$ for some $j \in I \setminus \{0, 1\}$. Then $\alpha(j, x) = \tau\mu_s(j, x) = \tau \cdot (k, x) = (k, x)$, so $u_k = 0$, a contradiction.

If instead $n = 1 \bmod k$, then $F = \{n - 1, 0, 1\}$. Then μ_s fixes 0 and interchanges $n - 1$ and 1, so $s = k - 1$. Then $\tau \cdot (n - 1, y) = (n - 1, x)$ and $\tau \cdot (1, x) = (1, y)$. Hence $u_{n-1} = u_0 = \cdots = u_k = 1$. However, $\tau \cdot (0, x) = (0, x)$, so $u_0 = u_1 = \cdots = u_{k-1} = 0$, a contradiction.

Thus $\delta \neq \mu_s$ for any $s \in \mathbb{Z}_n$, so $\delta = \mathrm{id}$ and hence $\alpha = \tau \in K$. For every $0 \leq t \leq \lceil \frac{n}{k} \rceil - 1$, fibre $\mathcal{F}_{tk}$ contains exactly one red vertex that is fixed by τ, so by Lemma 4.1, $u_{tk} = u_{tk+1} = \cdots = u_{tk+k-1} = 0$. Hence $u_0 = \cdots = u_{n-1} = 0$ and so $\tau = \mathrm{id}$. Thus, this is a 2-distinguishing coloring of $\mathrm{PX}(n, k)$ with a color class of size $\lceil \frac{n}{k} \rceil + 1$, so $\mathrm{Dist}\big(\mathrm{PX}(n, k)\big) = 2$ and $\rho\big(\mathrm{PX}(n, k)\big) \leq \lceil \frac{n}{k} \rceil + 1$.

To establish the lower bound on cost, assume there exists a set of vertices $R = \{u_1, u_2, \ldots, u_r\}$ with $r < \lceil \frac{n}{k} \rceil$ such that coloring the vertices of R red and all other vertices blue defines a 2-distinguishing coloring of $\mathrm{PX}(n, k)$. If $\alpha \in \mathrm{Aut}\big(\mathrm{PX}(n, k)\big)$ fixes every vertex in R, then certainly α preserves the color classes and so by assumption $\alpha = \mathrm{id}$. Hence, R is a determining set of size $r < \lceil \frac{n}{k} \rceil$, a contradiction of Theorem 4.5. Thus, $\rho\big(\mathrm{PX}(n, k)\big) \geq \lceil \frac{n}{k} \rceil$. □

The remaining theorems indicate which Praeger-Xu graphs (for $n \geq 5$ and $k \geq 2$) have cost $\lceil \frac{n}{k} \rceil$ and which have cost $\lceil \frac{n}{k} \rceil + 1$.

Theorem 6.3. *Let $n \geq 5$ and $2 \leq k < n$. If k divides n, then*

$$\rho\big(\mathrm{PX}(n, k)\big) = \left\lceil \frac{n}{k} \right\rceil + 1 = \frac{n}{k} + 1.$$

Proof. Let $R \subset V$ be any set of $\frac{n}{k}$ vertices. Color every vertex in R red and every other vertex blue. It suffices to show we can always find a nontrivial automorphism preserving R.

Let $I = \{i_1, i_2, \ldots, i_r\} \subseteq \mathbb{Z}_n$ be the set of indices of fibres containing red vertices, where we assume that as integers, $0 \leq i_1 < i_2 < \cdots < i_r < n$. Then the gaps between these fibres contain $i_2 - i_1 - 1, i_3 - i_2 - 1, \ldots,$ $n + i_1 - i_r - 1$ fibres, respectively. If there exists $i_p \in I$ such that the gap between i_p and i_{p+1} contains at least k fibres, then $\tau_{i_{p+1}-1}$ is a nontrivial

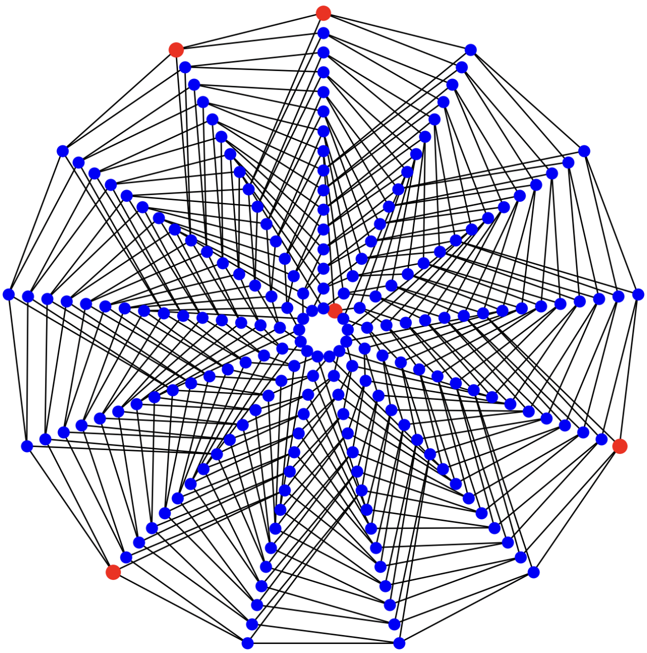


Figure 6.1: PX(13, 4) with a 2-distinguishing coloring of cost $5 = \lceil \frac{13}{4} \rceil + 1$.

automorphism that preserves colors. So assume that for all $i_p \in I$, the gap between i_p and i_{p+1} contains fewer than k fibres.

Suppose there exists $i_p \in I$ such that the gap between i_p and i_{p+1} contains fewer than $k-1$ fibres. Since $r = |I| \leq |R| = \frac{n}{k}$, the total number of fibres is strictly less than $r + (k-1)r = kr \leq k \cdot \frac{n}{k} = n$, a contradiction. Thus for all $i \in \mathbb{Z}_n$, $i \in I$ if and only if $i + k \in I$, so the induced action of ρ^k preserves I as a subset of $V(C_n)$.

Since the fibres containing red vertices are separated by $k-1$ fibres, we can define $\tau \in K$ such that τ adjusts the bitstring components of vertices in these fibres independently. More precisely, let $\tau = \tau_0^{u_0}\tau_1^{u_1}\cdots\tau_{n-1}^{u_{n-1}}$, where $u_m = 1$ if and only if there exist $(i_p, x), (i_{p+1}, y) \in R$ such that $x_{m-i_p} \neq y_{m-i_{p+1}}$. Then for all $(i_p, x) \in R$, we have $\tau\rho^k \cdot (i_p, x) = \tau \cdot (i_{p+1}, x) = (i_{p+1}, y) \in R$. Thus, $\tau\rho^k$ is a nontrivial automorphism that preserves colors. $\square$

Theorem 6.4. *If $5 \le n < 2k$, then $\rho\big(\mathrm{PX}(n,k)\big) = \lceil \frac{n}{k} \rceil = 2$.*

Proof. Let $j = \lfloor \frac{n}{2} \rfloor - 1$. Then $5 \le n < 2k$ implies $0 < j < k-1$. Next, let $R = \{z, v\}$, where $z = (0, 000\cdots 0)$ and $v = (j, y) = (j, 011\cdots 1)$.

Color all the vertices in R red and all other vertices blue; assume $\alpha \in \mathrm{Aut}\big(\mathrm{PX}(n,k)\big)$ preserves these color classes. Let $M = \big\{0, 1, \ldots, k-1\big\} \cap \big\{j, j+1, \ldots, j+k-1\big\}$. Since $0 < j < k-1$, $j \in M$. For $m = j$, $y_{m-j} = y_0 = 0$, but $y_{k-1-m} = y_{k-1-j} = 1$. By Corollary 5.3, z and v are not interchangeable, so α can only preserve R by fixing z and v. Because fibres $\mathcal{F}_0$ and $\mathcal{F}_j$ are not antipodal, R is a determining set by Theorem 4.2. By definition, α is the identity. Thus we have defined a 2-distinguishing coloring in which the smaller color class has size 2. □

Theorem 6.5. *Let $k \ge 2$ and $n > 2k$ such that k does not divide n. Then*

$$\rho\big(\mathrm{PX}(n,k)\big) = \begin{cases} \lceil \frac{n}{k} \rceil + 1, & \text{if } n = -1 \bmod k, \\ \lceil \frac{n}{k} \rceil, & \text{if } n \neq -1 \bmod k. \end{cases}$$

Proof. First assume $n = -1 \bmod k$, so $n = \lceil \frac{n}{k} \rceil k - 1$. Let R be any set of $\lceil \frac{n}{k} \rceil$ vertices. Color every vertex in R red and every other vertex blue. Let $I = \{i_1, i_2, \ldots, i_r\} \subset \mathbb{Z}_n$ be the set of indices of the fibres containing red vertices, where as integers $0 \le i_1 < i_2 < \cdots < i_r < n$. We will show that there is a nontrivial automorphism preserving R.

If there exists $i_p \in I$ such that the gap between i_p and i_{p+1} contains at least k fibres, then τ_{i_p+k} is a nontrivial automorphism that preserves colors. So assume that every gap has at most $k-1$ fibres. Suppose there exist at least two gaps that contain at most $k-2$ fibres. Then the total number of fibres is at most

$$r + 2(k-2) + (r-2)(k-1) = rk - 2 \le \lceil \tfrac{n}{k} \rceil k - 2 < \lceil \tfrac{n}{k} \rceil k - 1 = n,$$

a contradiction. Thus at most one gap contains at most $k-2$ fibres and the others contain exactly $k-1$ fibres. If two vertices $u, v \in R$ are in the same fibre, then $r < \lceil \frac{n}{k} \rceil$ and then the total number of fibres is

$$r + (r-1)(k-1) + k - 2 = rk - 1 < \lceil \tfrac{n}{k} \rceil k - 1 = n,$$

a contradiction. Thus $r = \lceil \frac{n}{k} \rceil$, every gap except one contains $k-1$ fibres, and the remaining gap contains $k-2$ fibres. By vertex-transitivity, we can assume $I = \big\{0, k, 2k, \ldots, \big(\lceil \frac{n}{k} \rceil - 1\big)k\big\}$.

Let $j = \left(\lceil \frac{n}{k} \rceil - 1\right)k = n - (k-1)$. The gap between $\mathcal{F}_j$ and $\mathcal{F}_0$ is the one containing exactly $k-2$ fibres; all other gaps contain $k-1$ fibres. Let $u = (0,x), v = (j,y) \in R$ be the red vertices in $\mathcal{F}_0$ and $\mathcal{F}_j$, respectively. Then, as defined in Lemma 5.2, let

$$M = \{0, 1, \dots, k-1\} \cap \{j, j+1, \dots, j+k-1\} = \{0\}.$$

For the only $m \in M$, we have $m - j = k - 1$ and $m - i = 0$. Then $(x^-)_{m-j} = x_0$, $y_{m-j} = y_{k-1}$, $(y^-)_{m-i} = (y^-)_0 = y_{k-1}$, and $x_{m-i} = x_0$. Of course, $x_0 = y_{k-1}$ if and only if $y_{k-1} = x_0$. By Lemma 5.2, u and v are interchangeable by an automorphism of the form by $\alpha = \tau\mu_s \in \mathcal{A}$, where $s = n = 0 \bmod n$ and $\tau = \tau_0^{u_0}\tau_1^{u_1}\cdots\tau_{n-1}^{u_{n-1}}$ is designed to flip exactly the right bits of the bitstring components of vertices in $\mathcal{F}_0$ and $\mathcal{F}_j$. More precisely, let $t \in \{0, 1, \dots, k-1\}$. If $y_t \neq (x^-)_t = x_{k-1-t}$, then $u_{j+t} = u_{k-1-t} = 1$; and if $y_t = (x^-)_t = x_{k-1-t}$, then $u_{j+t} = u_{k-1-t} = 0$. Note that this prescribes the value of u_m for all $m \in \{0, 1, \dots, k-1\} \cup \{j, j+1, \dots, j+k-1\}$; vertices in $\mathcal{F}_0$ and $\mathcal{F}_j$ are unaffected by the value u_ℓ for any $\ell \in \{k, k+1, \dots, j-1\}$.

We claim that we can set the value of u_ℓ for all $\ell \in \{k, k+1, \dots, j-1\}$ in such a way that $\tau\mu_0$ preserves R. Note that

$$\{k, k+1, \dots, j-1\} = \bigsqcup_{a=1}^{\lceil \frac{n}{k} \rceil - 2} \{ak, ak+1, ak+2, \dots, ak+k-1\}.$$

Let $b \in \left\{1, \dots, \lceil \frac{n}{k} \rceil - 2\right\}$ and let (bk, w) be the red vertex in $\mathcal{F}_{bk}$. Then $\tau\mu_n \cdot (bk, w) = \tau \cdot (ak, w^-)$, for some $a \in \left\{1, \dots, \lceil \frac{n}{k} \rceil - 2\right\} \setminus \{b\}$. We can arrange to have $\tau \cdot (ak, w^-)$ equal the red vertex in $\mathcal{F}_{ak}$ by flipping bits in w^- as necessary; this can be achieved by appropriately setting the values of $u_{ak}, u_{ak+1}, \dots, u_{ak+k-1}$. These values won't affect vertices in any of the other fibres containing red vertices.

Now assume $n \neq -1 \bmod k$; because we are also assuming that $k \nmid n$, the remainder after n is divided by k satisfies $0 < n - \left(\lceil \frac{n}{k} \rceil - 1\right)k < k - 1$. Again, let $I = \left\{0, k, 2k, \dots, \left(\lceil \frac{n}{k} \rceil - 1\right)k\right\}$. To simplify notation, again let $j = \left(\lceil \frac{n}{k} \rceil - 1\right)k$. Then let

$$R = \left\{(i, 00\cdots00) : i \in I \setminus \{j\}\right\} \cup \left\{(j, 00\cdots01)\right\}.$$

Note that $|R| = |I| = \lceil \frac{n}{k} \rceil$. Color every vertex in R red and all other vertices blue. Let $\alpha = \tau\delta \in \mathcal{A} = \mathrm{Aut}\big(\mathrm{PX}(n,k)\big)$ such that α preserves these color classes. The induced action of α on the fibres must preserve the set I, interpreted as a subset of $V(C_n)$. The distance between 0 and

$j = \left(\lceil \frac{n}{k} \rceil - 1\right)k$ in C_n is strictly less than $k+1$, whereas the distance between any other two consecutive elements of I in C_n is exactly $k+1$. So no nontrivial rotation preserves I.

Thus $\delta = \mu_s$, where μ_s interchanges 0 and j. Then $\tau\mu_s$ interchanges the red vertices in $\mathcal{F}_0$ and $\mathcal{F}_j$, so $\tau\delta \cdot (0, 00\cdots 0) = \tau(j, 00\cdots 0) = (j, 00\cdots 01)$, which implies $u_j = u_{j+1} = \cdots = u_{j+k-2} = 0$ and $u_{j+k-1} = 1$. Additionally, $\tau\delta \cdot (j, 00\cdots 01) = \tau \cdot (0, 10\cdots 00) = (0, 00\cdots 00)$, which implies $u_0 = 1$ and $u_2 = u_3 = \cdots = u_{k-1} = 0$. A contradiction arises because $0 < n-j < k-1$ implies that in $\mathbb{Z}_n$, $0 = n = j+m$ for some $m \in \{1, 2, \ldots, k-2\}$. Thus, $\delta = \mathrm{id}$ and so $\alpha = \tau = \tau_0^{u_0}\tau_1^{u_1}\cdots\tau_{n-1}^{u_{n-1}} \in K$.

For every $0 \leq t \leq \lceil \frac{n}{k} \rceil - 1$, fibre $\mathcal{F}_{tk}$ contains exactly one red vertex that is fixed by τ, so by Lemma 4.1, $u_{tk} = u_{tk+1} = \cdots = u_{tk+k-1} = 0$. Hence $u_0 = \cdots = u_{n-1} = 0$ and so $\tau = \mathrm{id}$. Thus, this is a 2-distinguishing coloring of $\mathrm{PX}(n,k)$ with a color class of size $\lceil \frac{n}{k} \rceil$. By Theorem 6.2, $\rho\big(\mathrm{PX}(n,k)\big) = \lceil \frac{n}{k} \rceil$. □

Our results on distinguishing number and cost are summarized below.

Theorem 6.6. *Let $n \geq 3$ and $2 \leq k < n$. Then* $\mathrm{Dist}\big(\mathrm{PX}(n,k)\big) = 2$ *and*

$$\rho\big(\mathrm{PX}(n,k)\big) = \begin{cases} 5, & \text{if } (n,k) = (4,2), \\ \lceil \frac{n}{k} \rceil, & \text{if } 5 \leq n < 2k, \\ \lceil \frac{n}{k} \rceil, & \text{if } 2k < n \text{ and } n \notin \{0, -1 \bmod k\}, \\ \lceil \frac{n}{k} \rceil + 1, & \text{otherwise.} \end{cases}$$

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Sally Cockburn*and Max Klivans
Mathematics and Statistics Department, Hamilton College,
Clinton, NY 13323.
scockbur@hamilton.edu, mklivans@hamilton.edu