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Super total local antimagic coloring of graphs

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Abstract. Let $G = (V, E)$ be a finite, simple, undirected graph without isolated vertices. A bijective map $f: V \cup E \to \{1, 2, \dots, |V| + |E|\}$ gives a *labeling* of the vertices and edges of G. With each vertex v, we associate a weight $w(v)$ as the sum of all labels of vertices that are neighbors of v (not including v), together with the labels of edges incident at v. The labeling given by f is called *total local antimagic* if adjacent vertices have distinct weights. Furthermore, f is called a *super vertex total local antimagic labeling* if vertices have labels $1, 2, \dots, |V|$. Similarly, f is called a *super edge total local antimagic labeling* if the edges have labels $1, 2, \dots, |E|$. The labeling f induces a proper vertex coloring of G. The *super vertex* (*edge*) *total local antimagic chromatic number* of a graph G is the minimum number of colors used over all colorings of G induced by the super vertex (edge) total local antimagic labeling of G. In this paper, we discuss these parameters for some families of graphs.

1 Introduction

Throughout the paper, we make the assumption that $G = (V, E)$ is a simple, finite, undirected graph without isolated vertices. The term *labeling* in the context of a graph refers to a one-to-one correspondence established between the labeling set and the elements of a graph, which could be vertices, edges, or both. Consequently, numerous variations exist in graph labeling, including vertex labeling, edge labeling, and super vertex (edge) labeling. For graph theoretic terminology and notations, we refer readers to West [14].

Hartsfield and Ringel [8] introduced the concept of *antimagic labeling* for a graph G. They defined antimagic labeling as follows: A graph G is

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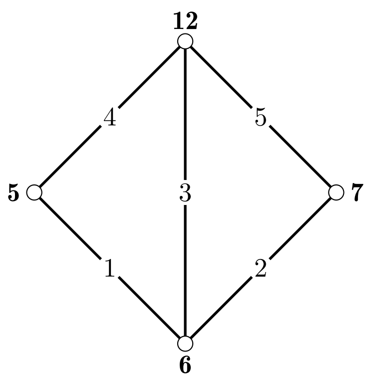
antimagic if there is bijection $f\colon E \to \{1, 2, \ldots, |E|\}$ such that weights of any two vertices are distinct, where the weight of a vertex v is defined as $w(v) := \sum_{uv \in E} f(uv)$. They put forth a conjecture that states, "any connected graph G other than K_2 is antimagic."

This conjecture has been proven true for many families of graphs, such as trees, dense graphs, regular graphs, and cartesian products of graphs (for more details, see [1,4,5,10,11]). However, the general case of the conjecture remains open to this day.

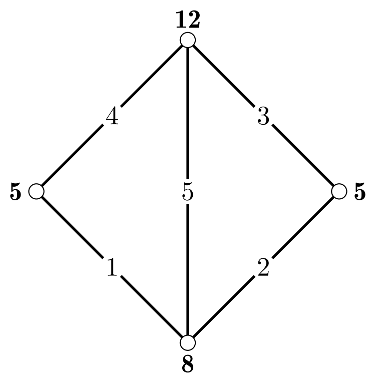
Arumugam et al. [2] and Bensmail et al. [3] independently introduced a local version of antimagic labeling of a graph G. The edge labeling

$$f\colon E \to \{1, 2, \ldots, |E|\}$$

of G is said to be a local antimagic labeling if $w(u) \neq w(v)$, whenever uv is an edge in G. In the same paper, they conjectured that "every graph without K_2 as a component is local antimagic." Haslegrave [9] proved this conjecture. The concepts of antimagic and local antimagic labelings for the graph $K_4 - e$ are illustrated in Figure 1.1a and Figure 1.1b, respectively.



(a) Antimagic labeling.



(b) Local antimagic, but not antimagic labeling.

Figure 1.1: Antimagic and local antimagic labelings of $K_4 - e$.

Once we treat vertex weights as colors, the local antimagic labeling naturally induces proper vertex coloring. The *local antimagic chromatic number* of a graph G, denoted by $\chi_{la}(G)$ is the minimum number of colors used over all colorings of G induced by local antimagic labeling of G.

Recently researchers extended the notion of local antimagic labeling. Instead of labeling only edges, one can label the edges as well as vertices and calculate the weights in various manners so that adjacent vertices receive different weights.

Putri et al. [12] defined *local vertex antimagic total coloring of a graph* $G = G(V, E)$ as a bijective function $f\colon V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that for any two adjacent vertices u and v in G, $w(u) \neq w(v)$, where

$$w(x) = f(x) + \sum_{xy \in E} f(xy)$$

for any $x \in V(G)$. A local vertex antimagic total coloring of C_4 is shown in Figure 1.2.

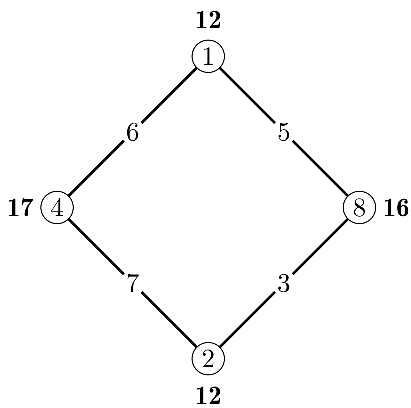


Figure 1.2: A local vertex antimagic total coloring of C_4.

In all of the variants of *total labelings* (both vertices as well as edges are labeled) given in [6, 12, 13], the weight of a vertex is the sum of the label of the vertex and the labels of all edges incident to it. The natural way to extend the definition of a local antimagic graph is to label both the vertices and edges and to calculate the weight of a given vertex by summing the labels of its adjacent vertices and incident edges. We define this formally as follows:

Definition 1.1. The *total open neighborhood* of a vertex $u \in V$, denoted by $NT(u)$, is the collection of all vertices (except u) adjacent to u together with all edges incident to u.

By definition, $NT(u) = N(u) \cup \{uv : uv \in E\}$. The total closed neighborhood of a vertex u, denoted by $NT[u]$, is obtained by adding u to $NT(u)$.

Definition 1.2. Given a graph $G = (V, E)$. A bijective map

$$f\colon V \cup E \to \{1, 2, \ldots, |V| + |E|\}$$

is called the total local antimagic labeling if, for each edge $uv \in E, w(u) \neq w(v)$, where $w(v)$ is the weight of v given by $w(v) = \sum_{x \in NT(v)} f(x)$. Such a function f is called a *super vertex total local antimagic labeling* if

$$f(V) = \{1, 2, \ldots, |V|\},$$

and it is called a *super edge total local antimagic labeling* if

$$f(E) = \{1, 2, \ldots, |E|\}.$$

A graph which induces super vertex (edge) total local antimagic labeling is called a *super vertex (edge) total local antimagic* graph.

A super vertex (edge) total local antimagic labeling induces a proper vertex coloring of G by considering the vertex weights as colors. Hence, we define the super vertex (edge) total local antimagic chromatic number as follows:

The *super vertex total local antimagic chromatic number* of a graph G, denoted by $\chi_{svtla}(G)$, is the minimum number of colors used over all colorings of G induced by super vertex total local antimagic labelings of G.

The *super edge total local antimagic chromatic number* of a graph G, denoted by $\chi_{setla}(G)$, is the minimum number of colors used over all colorings of G induced by super edge total local antimagic labelings of G. By definition, we have $\chi(G) \leq \chi_{svtla}(G)$ and $\chi(G) \leq \chi_{setla}(G)$. Since $\chi(G) \geq \omega(G)$, $\omega(G) \leq \chi(G) \leq \chi_{svtla}(G)$ and $\omega(G) \leq \chi(G) \leq \chi_{setla}(G)$, where $\omega(G)$ is the clique number of G.

We abbreviate "super vertex total local antimagic" as "svtla" and "super edge total local antimagic" as "setla." In this paper, we discuss the super vertex (edge) total local antimagic chromatic number for a graph and study it for some graph families.

Definition 1.3. A magic rectangle $MR(m, n)$ of size $m \times n$ is a rectangular arrangement of the first mn natural numbers, such that the sum of all entries in each row is the same, and the sum of all entries in each column is the same.

Harmuth gave the following theorem [7], which gives the necessary and sufficient conditions for the existence of a magic rectangle of a given order.

Theorem 1.4 (Harmuth [7]). *The magic rectangle $MR(a, b)$ exists if and only if $a, b > 1$, $ab > 4$, and $a \equiv b \pmod 2$.*

If G admits an svtla labeling f, then the sum $\sum_{x \in V} w(x)$ counts the label of each vertex v exactly $\deg(v)$ times, where $\deg(v)$ is the number of edges in G incident to v. In addition, the label of an edge $e = uv$ is counted only in $w(u)$ and $w(v)$, exactly twice. Thus, we have the following observations:

Observation 1.5. If G admits an svtla labeling f, then

$$\sum_{x \in V} w(x) = \sum_{v \in V} \deg(v) f(v) + 2 \sum_{e \in E} f(e).$$

Similar observations were made for the setla graph.

Observation 1.6. If G admits a setla labeling f, then

$$\sum_{x \in V} w(x) = \sum_{v \in V} \deg(v) f(v) + 2 \sum_{e \in E} f(e).$$

In Section 2, we study the svtla labeling of some graphs, and in Section 3, we study the setla labeling of some graphs.

2 Super vertex total local antimagic labeling

Proposition 2.1. *For any svtla graph G with a vertex v having the largest number ℓ of pendent vertices, $\chi_{svtla}(G) \geq \ell + 1$.*

Proof. Let G be a graph on n vertices and $v_1, v_2, \ldots, v_\ell$ be the pendent vertices adjacent to v. Let f be any svtla labeling of G. Subsequently, the weights of the pendent vertices $w(v_i) = f(v) + f(vv_i)$ are all distinct and $w(v) \neq w(v_i)$ for each i, $1 \leq i \leq \ell$. Hence, f induces a proper vertex coloring of G that need at least $\ell+1$ colors. This proves the proposition. □

The following corollary is evident from Proposition 2.1.

Corollary 2.2. *For the star $K_{1,n}$, $\chi_{svtla}(K_{1,n}) = n + 1$.*

Proof. By Proposition 2.1, $\chi_{svtla}(K_{1,n}) \geq n + 1$. We define super vertex total local antimgaic labeling of $K_{1,n}$ as $f(c) = n + 1$ and $f(v_i) = i$ and $f(cv_i) = n + 1 + i$ for each i, $1 \leq i \leq n$ as shown in Figure 2.1. Therefore, $\chi_{svtla}(K_{1,n}) \leq n + 1$. This proves that $\chi_{svtla}(K_{1,n}) = n + 1$. □

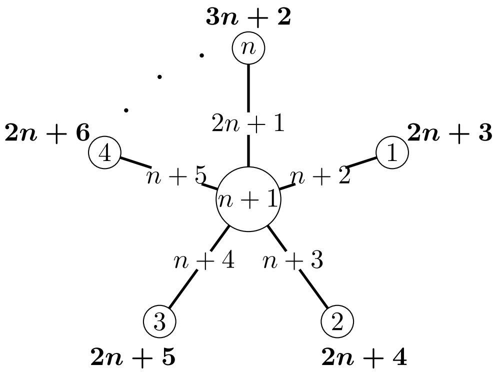


Figure 2.1: svtla labeling of a star.

Corollary 2.3. *If in a tree T the largest number of pendent vertices equals ℓ at a vertex, then $\chi_{svtla}(T) \geq \ell + 1$.*

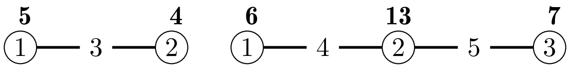


Figure 2.2: svtla labelings of P_2 and P_3.

Theorem 2.4. *For the path P_n with $n \geq 3$, $3 \leq \chi_{svtla}(P_n) \leq 5$.*

Proof. It is easy to see that $\chi_{svtla}(P_2) = 2$ and $\chi_{svtla}(P_3) = 3$ (see Figure 2.2). Let $n \geq 4$ and P_n be the path on n vertices with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and $e_i = v_i v_{i+1}$ be the edges. First, we will prove the lower bound. Let f be a svtla labeling of P_n. Then

$$w(v_1) = f(e_1) + f(v_2)$$

and

$$w(v_3) = f(v_2) + f(v_4) + f(e_2) + f(e_3).$$

If $w(v_1) = w(v_3)$ then we get $f(e_1) = f(e_2) + f(e_3) + f(v_4)$. This is impossible, because $f(e_1) \leq 2n-1$ and $f(e_2)+f(e_3)+f(v_4) \geq 2n+4$. Hence $w(v_1) \neq w(v_3)$. Also, $w(v_1) \neq w(v_2)$ and $w(v_2) \neq w(v_3)$ since $v_1v_2, v_2v_3 \in E(P_n)$ and f is a svtla labeling. Hence, for $n \geq 4$,

$$\chi_{svtla}(P_n) \geq 3. \tag{1}$$

Now, for the upper bound, we have the following two cases:

Case 1: When $n \equiv 3 \pmod 4$, then we define a super vertex total labeling f by

$$f(v_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \text{ and } i \neq n \\ n & \text{if } i = n \end{cases}$$

and

$$f(e_i) = 2n - i.$$

Therefore,

$$\begin{aligned} w(v_1) &= f(v_2) + f(e_1) = 2 + (2n-1) = 2n+1, \\ w(v_{n-1}) &= f(v_{n-2}) + f(v_n) + f(e_{n-2}) + f(e_{n-1}) \\ &= (n-2) + n + (2n-n+2) + (2n-n+1) = 4n+1, \end{aligned}$$

and

$$w(v_n) = f(v_{n-1}) + f(e_{n-1}) = (n-1) + (2n-n+1) = 2n.$$

Now for each i, $2 \leq i \leq n-1$,

$$\begin{aligned} w(v_i) &= f(v_{i-1}) + f(v_{i+1}) + f(e_{i-1}) + f(e_i) \\ &= f(v_{i-1}) + f(v_{i+1}) + (2n-(i-1)) + (2n-i) \\ &= f(v_{i-1}) + f(v_{i+1}) + 4n - 2i + 1 \\ &= 4n - 2i + 1 + \begin{cases} i + (i+1) & \text{if } i \equiv 0 \pmod 4 \\ (i-2)+(i+1) & \text{if } i \equiv 1 \pmod 4 \\ (i-1)+(i+2) & \text{if } i \equiv 2 \pmod 4 \\ (i-1)+i & \text{if } i \equiv 3 \pmod 4 \end{cases} \\ &= \begin{cases} 4n & \text{if } i \text{ is odd} \\ 4n+2 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Therefore, the weights of adjacent vertices are distinct; that is, f is the svtla labeling of P_n, which induces a proper 5-coloring of P_n.

Case 2: When $n \not\equiv 3 \pmod 4$, then we define a super vertex total labeling f by

$$f(v_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

$$f(e_i) = 2n - i.$$

Therefore, $w(v_1) = f(v_2) + f(e_1) = 2 + (2n-1) = 2n+1$,

$$\begin{aligned} w(v_n) &= f(v_{n-1}) + f(e_{n-1}) \\ &= \begin{cases} n + (2n - (n-1)) & \text{if } n \equiv 0 \pmod 4 \\ (n-2) + (2n-(n-1)) & \text{if } n \equiv 1 \pmod 4 \\ (n-1) + (2n-(n-1)) & \text{if } n \equiv 2 \text{ or } 3 \pmod 4 \end{cases} \\ &= \begin{cases} 2n+1 & \text{if } n \equiv 0 \pmod 4 \\ 2n-1 & \text{if } n \equiv 1 \pmod 4 \\ 2n & \text{if } n \equiv 2 \text{ or } 3 \pmod 4), \end{cases} \end{aligned}$$

and for each i, $2 \leq i \leq n-1$,

$$\begin{aligned} w(v_i) &= f(v_{i-1}) + f(v_{i+1}) + f(e_{i-1}) + f(e_i) \\ &= f(v_{i-1}) + f(v_{i+1}) + (2n-(i-1)) + (2n-i)) \\ &= (4n-2i+1) + f(v_{i-1}) + f(v_{i+1}) \\ &= (4n-2i+1) + \begin{cases} i + (i+1) & \text{if } i \equiv 0 \pmod 4 \\ (i-2) + (i+1) & \text{if } i \equiv 1 \pmod 4 \\ (i-1) + (i+2) & \text{if } i \equiv 2 \pmod 4 \\ (i-1) + i & \text{if } i \equiv 3 \pmod 4 \end{cases} \\ &= \begin{cases} 4n & \text{if } i \text{ is odd} \\ 4n+2 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Consider the following two sub-cases:

Subcase (i): When $n \equiv 0 \pmod 4, w(v_1) = w(v_n) = 2n+1$, and for each i, $2 \leq i \leq n-1, w(v_i) = 4n$ or $4n+2$. Hence, $\chi_{svtla}(P_n) \leq 3$. Also, by Equation (1), $\chi_{svtla}(P_n) \geq 3$. Therefore, $\chi_{svtla}(P_n) = 3$.

Subcase (ii): When $n \equiv 1 \text{ or } 2 \pmod 4$, $w(v_1) = 2n+1, w(v_n) = 2n-1$ and for each i, $2 \leq i \leq n-1, w(v_i) = 4n$ or $4n+2$. Therefore $\chi_{svtla}(P_n) \leq 4$.

Hence, $\chi_{svtla}(P_n) = 3$, when $n \equiv 0 \pmod 4$ and

$$\chi_{svtla}(P_n) \leq \begin{cases} 4 & \text{if } n \equiv 1 \text{ or } 2 \pmod 4 \\ 5 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

This completes the proof. □

As illustrated in Figure 2.3,

$$\chi_{svtla}(P_5) = \chi_{svtla}(P_6) = \chi_{svtla}(P_9) = \chi_{svtla}(P_{10}) = 3.$$

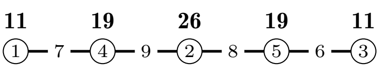
Hence, for $n \equiv 1$ or $2 \pmod 4$, we have examples where χ_{svtla} value of P_n is 3. We pose the following problem.

Problem 2.5. Show that:

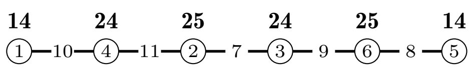
$$\chi_{svtla}(P_n) = 3 \text{ for } n \equiv 1 \text{ or } 2 \pmod 4$$

and

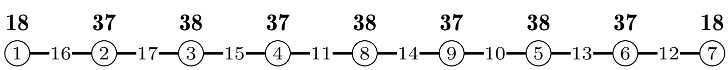
$$\chi_{svtla}(P_n) = 4 \text{ for } n \geq 7 \text{ and } n \equiv 3 \pmod 4.$$



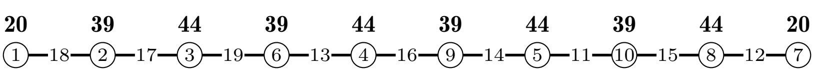
(a) svtla labeling of P_5.



(b) svtla labeling of P_6.



(c) svtla labeling of P_9.



(d) svtla labeling of P_{10}.

Figure 2.3: svtla labelings of some paths.

Theorem 2.6. *For cycle C_n with $n \geq 3$, $\chi_{svtla}(C_n) \leq 4$.*

Proof. Let C_n be a cycle on $n \geq 3$ with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $e_i = v_i v_{i+1}$ be edges, where the subscripts are taken modulo n.

Now we have the following two cases:

Case 1: When $n \equiv 3 \pmod 4$, then we define a super vertex total labeling f by

$$f(v_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \text{ and } i \neq n \\ n & \text{if } i = n \end{cases}$$
$$f(e_i) = 2n - i + 1.$$

Therefore,

$$\begin{aligned} w(v_1) &= f(v_2) + f(v_n) + f(e_1) + f(e_n) \\ &= 2 + n + 2n + (n+1) \\ &= 4n + 3 \end{aligned}$$

and

$$\begin{aligned} w(v_n) &= f(v_1) + f(v_{n-1}) + f(e_{n-1}) + f(e_n) \\ &= 1 + (n-1) + (n+2) + (n+1) \\ &= 3n + 3. \end{aligned}$$

Now for each i, $2 \leq i \leq n-1$,

$$\begin{aligned} w(v_i) &= f(v_{i-1}) + f(v_{i+1}) + f(e_{i-1}) + f(e_i) \\ &= (4n - 2i + 3) + f(v_{i-1}) + f(v_{i+1}) \\ &= \begin{cases} (4n-2i+3) + i + (i+1) & \text{if } i \equiv 0 \pmod 4 \\ (4n-2i+3) + (i-2) + (i+1) & \text{if } i \equiv 1 \pmod 4 \\ (4n-2i+3) + (i-1) + (i+2) & \text{if } i \equiv 2 \pmod 4 \\ (4n-2i+3) + (i-1) + i & \text{if } i \equiv 3 \pmod 4 \end{cases} \\ &= \begin{cases} 4n+4 & \text{if } i \text{ is even} \\ 4n+2 & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

It is easy to see that f is a svtla labeling of C_n and that it induces a 4 proper vertex coloring of C_n.

Case 2: When $n \not\equiv 3 \pmod 4$, then we define a super vertex total labeling f by

$$f(v_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

and

$$f(e_i)\begin{cases}2n-i & \text{if } i \neq n\\ 2n & \text{if } i = n.\end{cases}$$

Therefore,

$$\begin{aligned}
w(v_1) &= f(v_2)+f(v_n)+f(e_1)+f(e_n)\\
&= 2+f(v_n)+(2n-1)+2n\\
&= 4n+1+f(v_n)\\
&= \begin{cases}4n+1+(n-1) & \text{if } n \equiv 0 \pmod 4\\ 4n+1+n & \text{if } n \equiv 1 \text{ or } 2 \pmod 4\end{cases}\\
&= \begin{cases}5n & \text{if } n \equiv 0 \pmod 4\\ 5n+1 & \text{if } n \equiv 1 \text{ or } 2 \pmod 4\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
w(v_n) &= f(v_1)+f(v_{n-1})+f(e_{n-1})+f(e_n)\\
&= 1+f(v_{n-1})+(n+1)+2n\\
&= (3n+2)+f(v_{n-1})\\
&= \begin{cases}(3n+2)+n & \text{if } n \equiv 0 \pmod 4\\ (3n+2)+(n-2) & \text{if } n \equiv 1 \pmod 4\\ (3n+2)+(n-1) & \text{if } n \equiv 2 \pmod 4\end{cases}\\
&= \begin{cases}4n+2 & \text{if } n \equiv 0 \pmod 4\\ 4n & \text{if } n \equiv 1 \pmod 4\\ 4n+1 & \text{if } n \equiv 2 \pmod 4).\end{cases}
\end{aligned}$$

Now for each i, $2 \leq i \leq n-1$,

$$\begin{aligned}
w(v_i) &= f(v_{i-1})+f(v_{i+1})+f(e_{i-1})+f(e_i)\\
&= f(v_{i-1})+f(v_{i+1})+(2n-i+1)+(2n-i)\\
&= (4n-2i+1)+f(v_{i-1})+f(v_{i+1})\\
&= \begin{cases}(4n-2i+1)+i+(i+1) & \text{if } i \equiv 0 \pmod 4\\ (4n-2i+1)+(i-2)+(i+1) & \text{if } i \equiv 1 \pmod 4\\ (4n-2i+1)+(i-1)+(i+2) & \text{if } i \equiv 2 \pmod 4\\ (4n-2i+1)+(i-1)+i & \text{if } i \equiv 3 \pmod 4\end{cases}\\
&= \begin{cases}4n+2 & \text{if } i \text{ is even}\\ 4n & \text{if } i \text{ is odd.}\end{cases}
\end{aligned}$$

This proves that f is a svtla labeling. Now, we have the following subcases:

Subcase (i): When $n \equiv 0 \pmod 4$, $w(v_1) = 5n, w(v_n) = 4n+2$ and for each $i,\ 2 \leq i \leq n-1, w(v_i) = 4n$ or $4n+2$. Hence, $\chi_{svtla}(C_n) \leq 3$.

Subcase (ii): When $n \equiv 1 \pmod 4$, $w(v_1) = 5n+1, w(v_n) = 4n$ and for each $i,\ 2 \leq i \leq n-1, w(v_i) = 4n$ or $4n+2$. Hence, $\chi_{svtla}(C_n) \leq 3$. Also, $3 = \chi(C_n) \leq \chi_{svtla}(C_n) \leq 3$. This proves $\chi_{svtla}(C_n) = 3$.

Subcase (iii): When $n \equiv 2 \pmod 4$, $w(v_1) = 5n+1, w(v_n) = 4n+1$ and for each $i,\ 2 \leq i \leq n-1, w(v_i) = 4n$ or $4n+2$. Hence, $\chi_{svtla}(C_n) \leq 4$.

This completes the proof. □

A svtla labeling of a cycle C_5 is shown in Figure 2.4.

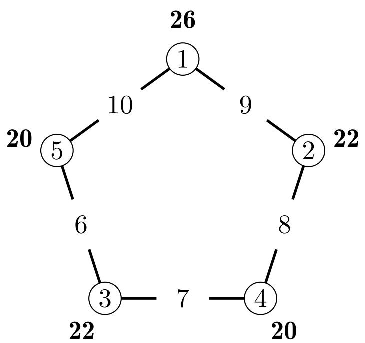


Figure 2.4: svtla labeling of C_5.

Theorem 2.7. *For the complete graph K_n, $n \geq 2$, $\chi_{svtla}(K_n) = n$.*

Proof. For $p \geq 2$, we construct the graph K_p by adding one vertex to K_{p-1} and joining it to the all vertices of K_p by an edge, that is $K_p = K_{p-1} + K_1$. We will prove that $\chi_{svtla}(K_p) = p$ by induction on p. By Theorem 2.4, we know that $\chi_{svtla}(K_2) = 2$ and by Theorem 2.6, $\chi_{svtla}(K_3) = 3$. Therefore, the result is true for $p = 2$ and $p = 3$.

We assume that $p \geq 4$ and the result is true for a given $p = n$ that is $\chi_{svtla}(K_n) = n$ with svtla labeling f. We will prove the result for $p = n+1$

that is $\chi_{svtla}(K_{n+1}) = n+1$. Without loss of generality, we may assume that $v_1, v_2, \ldots, v_n$ are the vertices of K_n such that $w_f(v_1) < w_f(v_2) < \cdots < w_f(v_n)$. Let $e_1, e_2, \ldots, e_m$ be the edges of K_n, where $m = \frac{n(n-1)}{2}$. Let $V(K_1) = v_0$. We define a super vertex total labeling g of $K_{n+1} = K_n + K_1$ by

$$g(x) = \begin{cases} 1 & \text{if } x = v_0 \\ f(v_i) + 1 & \text{if } x = v_i, 1 \le i \le n \\ f(e_i) + 1 & \text{if } x = e_i, 1 \le i \le n \\ m + n + 1 + i & \text{if } x = v_0 v_i, 1 \le i \le n. \end{cases}$$

Therefore,

$$\begin{aligned} w_g(v_0) &= \sum_{v \in V(G)} f(v) + \sum_{i=1}^{n} f(v_0 v_i) \\ &= (2 + 3 + \cdots + (n+1)) + \sum_{i=1}^{n} (m + n + 1 + i) \\ &= \frac{n^3 + 4n^2 + 7n}{2} \end{aligned}$$

and for $1 \le i \le n$,

$$\begin{aligned} w_g(v_i) &= \sum_{y \in NT_{K_n}(v_i)} f(y) + m + n + 1 + i \\ &= w_f(v_i) + 2(n-1) + (m + n + 2 + i). \end{aligned}$$

From the expression for $w_g(v_i)$, it is clear that

$$w_g(v_1) < w_g(v_2) < \cdots < w_g(v_n).$$

Now we show that $w_g(v_0) > w(v_i)$ for each i, $1 \le i \le n$. We consider the vertex v_i. If we assign largest labels from the set $\{1, 2, \ldots, \frac{n(n+1)}{2}\}$ to the elements in $NT(v_i)$, then we obtain

$$\begin{aligned} w_g(v_i) \le \left(\tfrac{n^2+n}{2} - (2n-3)\right) + \left(\tfrac{n^2+n}{2} - (2n-2)\right) + \cdots + \tfrac{n^2+n}{2} \\ = n^3 - 2n^2 + 4n - 3. \end{aligned}$$

Now for $n \ge 3$, $w_g(v_0) = \frac{n^3+4n^2+7n}{2} > n^3 - 2n^2 + 4n - 3 \ge w_g(v_i)$ where $1 \le i \le n$. Hence, all vertex weights in $K_n + K_1$ are distinct. This proves the theorem. □

Now, we calculate the super vertex total local antimagic chromatic number of a complete bipartite graph $K_{m,n}$.

Theorem 2.8. *$\chi_{svtla}(K_{m,n}) = 2$ if and only if $m, n > 1, mn > 4$ and $m \equiv n \pmod 2$.*

Proof. Suppose $m, n > 1, mn > 4$ and $m \equiv n \pmod 2$. Then by Theorem 1.4, there exists a magic rectangle $MR(m,n)$ with row sum $\frac{n(mn+1)}{2}$, column sum $\frac{m(mn+1)}{2}$. Let $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_n\}$ be the bipartition of the vertex set of $K_{m,n}$ with $m \leq n$. Adding $(m+n)$ to the each entry of $MR(m,n)$ we obtain a new rectangle say $R = [r_{i,j}]_{m\times n}$ in which row sum is $\rho = n(m+n) + \frac{n(mn+1)}{2}$ and column sum is $\sigma = m(m+n) + \frac{m(mn+1)}{2}$. Define the super vertex total labeling f by

$$f(a) = \begin{cases} i & \text{if } a = x_i \\ m+i & \text{if } a = y_i \\ r_{i,j} & \text{if } a = x_i y_j. \end{cases}$$

Then for any i, $1 \leq i \leq m$ and for any j, $1 \leq j \leq n$,

$$\begin{aligned} w(x_i) &= \sum_{j=1}^{n} f(y_j) + \rho \\ &= \left(mn + \frac{n(n+1)}{2}\right) + \left(mn + n^2 + \frac{mn^2+n}{2}\right) \\ &= \frac{(2+m)n^2 + (2+4m)n}{2} \end{aligned}$$

and

$$\begin{aligned} w(y_j) &= \sum_{i=1}^{m} f(x_i) + \sigma \\ &= \frac{m(m+1)}{2} + \left(m^2 + mn + \frac{m^2n+m}{2}\right) \\ &= \frac{(n+3)m^2 + (2n+2)m}{2}. \end{aligned}$$

It is easy to observe that weights of vertices in independent sets are the same and $w(x_i) < w(y_j)$ for any values of i, j. Therefore, f is a svtla labeling, which induces 2 colors. Hence, $\chi_{svtla}(K_{m,n}) \leq 2$. We know, $2 = \chi(K_{m,n}) \leq \chi_{svtla}(K_{m,n})$. Therefore, $\chi_{svtla}(K_{m,n}) = 2$.

Conversely, suppose that $\chi_{svtla}(K_{m,n}) = 2$ with svtla labeling f. We must have $w(x_i) \neq w(y_j)$ for any i, $1 \leq i \leq m$ and for any j, $1 \leq j \leq n$ and the weights of vertices in the independent sets must be same. We can form a magic rectangle $MR(m,n)$ with (i,j)-th entry as $f(x_iy_j)-(m+n)$ so that the row sum is equal to $w(y_j)-n(m+n)$, where $w(y_j)$ is same for each $j, 1 \le j \le n$, and the column sum is equal to $w(x_i)-m(m+n)$, where $w(x_i)$ is same for each $i, 1 \le i \le m$. Hence, by Theorem 1.4, m and n satisfy the required conditions. This completes the proof. □

Proposition 2.9. *For any $n \ge 1$,*

$$\chi_{svtla}(K_{2,n}) = \begin{cases} 2 & \text{if } n \ge 4 \text{ and } n \text{ is even} \\ 3 & \text{if } n \text{ is odd or } n = 2. \end{cases}$$

Proof. We know that $\chi_{svtla}(K_{2,1}) = \chi_{svtla}(K_{2,2}) = 3$. Let $n \ge 3$. If n is even by Theorem 2.8, $\chi_{svtla}(K_{2,n}) = 2$. Let n be odd. In this case $\chi_{svtla}(K_{2,n}) = 2$ implies the existence of $MR(2,n)$, which is impossible since n is odd. Hence, $\chi_{svtla}(K_{2,n}) \ge 3$. We will show that $\chi_{svtla}(K_{2,n}) \le 3$. Let $\{x,y\}$ and $\{u_1,u_2,\ldots,u_n\}$ be a bipartition of $K_{2,n}$. From Corollary 2.2, $\chi_{svtla}(K_{2,1}) = \chi_{svtla}(P_3) = 3$ and from Theorem 2.6, $\chi_{svtla}(K_{2,2}) = \chi_{svtla}(C_4) = 3$. For $n \ge 3$, define a super vertex total labeling of $K_{2,n}$ by $f(x)=1$, $f(y)=2$, for each i, $1 \le i \le n$, $f(u_i) = 2+i$, and

$$f(au_i) = \begin{cases} n+2+i & \text{if } a = xu_i \\ 3n+3-i & \text{if } a = yu_i. \end{cases}$$

Now, we calculate the vertex weights: For each i, $1 \le i \le n$, $w(u_i) = 4n+8$, $w(x) = 2n^2+5n$, $w(y) = 3n^2+5n$. We observe that $w(x) < w(u_i) < w(y)$. Hence, f is a svtla labeling that induces 3 colors. Therefore, $\chi_{svtla}(K_{2,n}) \le 3$. Hence, $\chi_{svtla}(K_{2,n}) = 3$. □

From Corollary 2.3, for a tree T other than a star, we have $\chi_{svtla}(T) > \chi(T)$. Whereas, $\chi_{svtla}(C_{4n+1}) = \chi(C_{4n+1})$. Also for complete graphs, some complete bipartite graphs and star graphs their χ_{svtla} values agree with their chromatic number. Hence, we raise the following problem:

Problem 2.10. Characterize graphs G for which $\chi_{svtla}(G) = \chi(G)$.

Let $S_{n,t}$ be a graph obtained by replacing each edge of a star S_n by a path of length $t+1$ (see Figure 2.5). We calculate the svtla chromatic number of $S_{n,t}$ for $t = 1,2$. Let vv_i be replaced by path $vv_{i,1}, v_{i,2}, \ldots, v_{i,t+2}$.

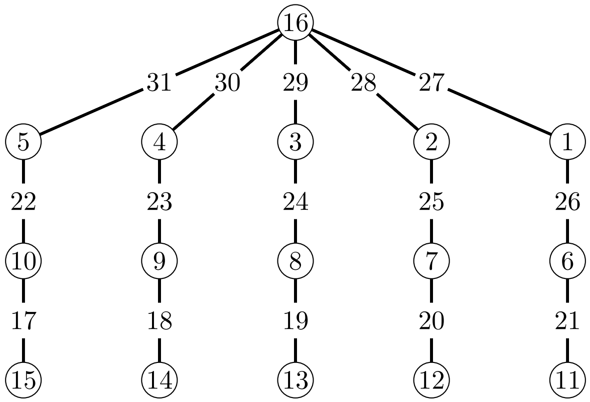


Figure 2.5: svtla labeling of $S_{5,2}$.

Theorem 2.11. *$\chi_{svtla}(S_{n,t}) \le n+t+1$, for $t=1,2$.*

Proof. We define the super vertex total labeling of $S_{n,t}$ by

$$\begin{aligned} f(v) &= nt+n+1, && \\ f(v_{i,j}) &= i+n(j-1), && \text{if } 1 \le i \le n,\ 1 \le j \le t+1, \\ f(vv_{i,1}) &= 2nt+n+1+i, && \text{if } 1 \le i \le n, \\ f(v_{i,j-1}v_{i,j}) &= n(2t+3-j)+2-i, && \text{if } 1 \le i \le n,\ 2 \le j \le t+1. \end{aligned}$$

The weights of the vertices induced by f are

$$\begin{aligned} w(x) = \sum_{i=1}^{n} f(v_{i,1}) + \sum_{i=1}^{n} f(v_{i,1}v_{i,2}) &= \sum_{i=1}^{n} i + \sum_{i=1}^{n}(2nt+n+1+i) \\ &= 2n^2(1+t)+2n \end{aligned}$$

and for each i, $1 \le i \le n$,

$$\begin{aligned} w(v_{i,1}) &= f(v)+f(v_{i,2})+f(vv_{i,1})+f(v_{i,1}v_{i,2}) \\ &= (nt+n+1)+(n+i)+(2nt+n+1+i)+(2nt+n+2-i) \\ &= n(5t+4)+4+i, \\ w(v_{i,2}) &= f(v_{i,1})+f(v_{i,3})+f(v_{i,1}v_{i,2})+f(v_{i,2}v_{i,3}) \\ &= (i)+(i+2n)+(2nt+n+2-i)+(2nt+2-i) \\ &= 4nt+3n+4, \\ w(v_{i,3}) &= f(v_{i,2})+f(v_{i,2}v_{i,3}) \\ &= (n+i)+(2nt+2-i) \\ &= 2nt+n+2. \end{aligned}$$

Clearly, f is an svtla labeling of $S_{n,t}$, and f induces $n+t+1$ distinct colors. Hence, $\chi_{svtla}(S_{n,t}) \leq n+t+1$. □

Let $B_{m,n}$ be the bi-star with vertex set $\{x, y, x_i, y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ with centers x and y and edge set $\{xy\} \cup \{xx_i : 1 \leq i\} \cup \{yy_j :\leq m, 1 \leq j \leq n\}$, where $m \leq n$. We calculate the super vertex total local antimagic chromatic number $\chi_{svtla}(B_{m,n})$.

Theorem 2.12. *For the bi-star $B_{m,n}$, $\chi_{svtla}(B_{m,n}) = n+2$, where $m \leq n$.*

Proof. Let $m \leq n$. By Theorem 2.1, $n+1 \leq \chi_{setla}(B_{m,n})$. First we show that $\chi_{svtla}(B_{m,n}) > n+1$. Let f be any svtla labeling of $B_{m,n}$. We know that the weights of all pendent vertices y_i are distinct and $w(y) \neq w(y_i)$ for any i, $1 \leq i \leq n$. If $w(x) = w(y_i)$ for some i then we obtain $f(x)+f(xx_i) = f(x) + f(xy) + \sum_{i=1}^{n}(f(y_i) + f(yy_i))$. This implies, $f(xx_i) = f(xy) + \sum_{i=1}^{n}(f(y_i)+f(yy_i))$ which is not possible. Therefore, $\chi_{svtla}(B_{m,n}) \geq n+2$. For the upper bound, we define a super vertex total labeling f of $B_{m,n}$ as follows:

$$\begin{aligned}
f(x) &= m+n+2 & \\
f(y) &= n+2 & \\
f(xy) &= 2m+2n+3 & \\
f(x_i) &= i & \text{for } 1 \leq i \leq m\\
f(xx_i) &= m+n+2+i & \text{for } 1 \leq i \leq m\\
f(yy_i) &= 2m+n+2+i & \text{for } 1 \leq i \leq n
\end{aligned}$$

and we label the vertices y_i by $\{m+1, m+2, \ldots, m+n, m+n+1\} - \{n+2\}$ in any manner. The sum of these y_i labels is $\frac{n^2+2mn+2m+n-2}{2}$. Now we calculate the weights:

$$\begin{aligned}
w(x) &= f(y) + f(xy) + \sum_{i=1}^{m}(f(x_i) + f(xx_i))\\
&= (n+2) + (2m+2n+3) + \sum_{i=1}^{m}(m+n+2+2i)\\
&= 2m^2 + mn + 5m + 3n + 5,\\
w(y) &= f(x) + f(xy) + \sum_{i=1}^{n} f(y_i) + \sum_{i=1}^{n} f(yy_i)\\
&= (m+n+2) + (2m+2n+3)\\
&\quad + \frac{n^2+2mn+2m+n-2}{2} + \sum_{i=1}^{n}(2m+n+2+i)\\
&= 2n^2 + 3mn + 4m + 6n + 4.
\end{aligned}$$

The weights of pendent vertices $w(x_i) = f(x) + f(xx_i) = m + n + 4 + i$ for each i, $1 \le i \le m$ and $w(y_j) = f(y) + f(yy_j) = 2m + 2n + 4 + j$ for each j, $1 \le j \le n$. Thus, f is a required svtla labeling that induces $n+2$ colors. Hence, $\chi_{svtla}(B_{m,n}) = n + 2$. □

A svtla labeling of bi-star $B(4,5)$ is illustrated in Figure 2.6.

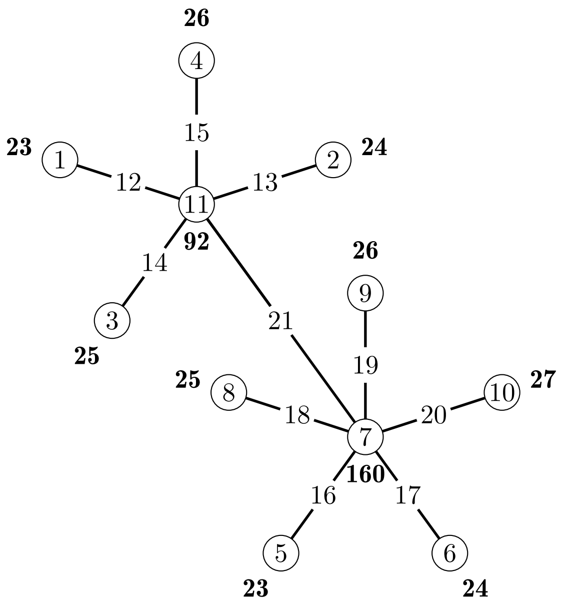


Figure 2.6: svtla labeling of bi-star $B(4,5)$.

Theorem 2.13. *If G is an r-regular graph, then*

$$\chi_{svtla}(G \circ K_1) \le \chi_{svtla}(G) + 1.$$

Proof. Let G be an r-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and m edges. Let f be a svtla labeling of G. Without loss of generality, we may assume that $f(v_i) = i$, for $1 \le i \le n$. Let $\{x_1, x_2, \ldots, x_n\}$ be newly added vertices to obtain $H = G \circ K_1$ such that there is an edge $e_i = v_i x_i$ for each i, $1 \le i \le n$. Define a super vertex total labeling g of $G \circ K_1$ by

$$\begin{aligned} g(v_i) &= f(v_i) = i && \text{for } 1 \le i \le n \\ g(x_i) &= n + i && \text{for } 1 \le i \le n \\ g(e_i) &= 2n + m - i + 1 && \text{for } 1 \le i \le m \\ g(e) &= 2n + f(e) && \text{for } e \in E(G). \end{aligned}$$

Now we calculate the weight of each vertex in $G \circ K_1$ induced by super vertex total labeling g. For each i, $1 \leq i \leq n$, $w_g(x_i) = g(v_i) + g(e_i) = 2n + m + 1$. Now for any vertex $v_i \in V(G)$,

$$\begin{aligned}
w_g(v_i) &= \sum_{x \in NT_H(v_i)} g(x) \\
&= g(x_i) + g(e_i) + \sum_{x \in NT_G(v_i)} g(x) \\
&= (3n + m + 1) + \sum_{x \in NT_G(v_i)} g(x) \\
&= (3n + m + 1) + \sum_{u \in N_G(v_i)} g(u) + \sum_{uv_i \in E(G)} g(uv_i) \\
&= (3n + m + 1) + \sum_{u \in N_G(v_i)} f(u) + \sum_{uv_i \in E(G)} (2n + f(uv_i)) \\
&= (3n + m + 1) + 2nr + \sum_{u \in N_G(v_i)} f(u) + \sum_{uv_i \in E(G)} f(uv_i) \\
&= (3n + m + 1) + 2nr + w_f(v_i),
\end{aligned}$$

which is independent of i. Hence, g is a svtla labeling of $G \circ K_1$ and $\chi_{svtla}(G \circ K_1) \leq \chi_{svtla}(G) + 1$. □

Also, in addition to regular graphs, some non-regular graphs follow the inequality obtained in Theorem 2.13. For example: $\chi_{svtla}(K_4 - e) = 3$ and $\chi_{svtla}((K_4 - e) \circ K_1) = 4 \leq \chi_{svtla}(K_4 - e) + 1$ (see Figure 2.7). The following questions arise naturally.

Problem 2.14. Characterize the graphs G for which

$$\chi_{svtla}(G \circ K_1) = \chi_{svtla}(G) + 1.$$

Problem 2.15. Let G and H be super vertex total local antimagic graphs. Determine the $\chi_{svtla}(G \circ H)$ in terms of $\chi_{svtla}(G)$ and $\chi_{svtla}(H)$.

Remark 2.16. The outcomes presented in Section 3 demonstrate notable parallels to the findings expounded upon in Section 2. While these parallels exist, they are nuanced by slight differences in the bounds, leading to intriguing and distinct results. This distinctive outcome variation underscores the importance of maintaining a separate treatment for Section 3. This approach acknowledges the subtleties and their significance in comprehensively understanding the χ_{setla} values.

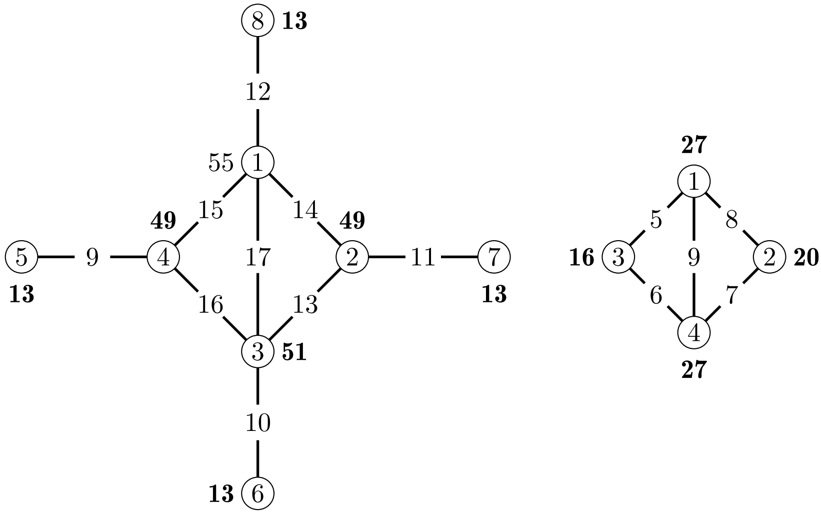


Figure 2.7: svtla labelings of $(K_4 - e) \circ K_1$ and $K_4 - e$.

3 Super edge total local antimagic labeling

We follow the same notations and definitions for the various families of graphs as defined in Section 2.

Proposition 3.1. *For any graph G with a vertex v having the largest number ℓ of pendent vertices, $\chi_{setla}(G) \geq \ell + 1$.*

Proof. Let G be a graph on n vertices and $v_1, v_2, \ldots, v_\ell$ be pendent vertices at v. Let f be a setla labeling. Then the ℓ weights $w(v_i) = f(v) + f(vv_i)$ are all distinct and $w(v) > w(v_i)$ where $1 \leq i \leq \ell$. Hence, f induces a proper vertex coloring of G that needs at least $\ell + 1$ colors. This proves the theorem. □

The proof of the following two corollaries is evident from Proposition 3.1.

Corollary 3.2. *For the star $K_{1,n}$, $\chi_{setla}(K_{1,n}) = n + 1$.*

A setla labeling of star $K_{1,n}$ is illustrated in Figure 3.1.

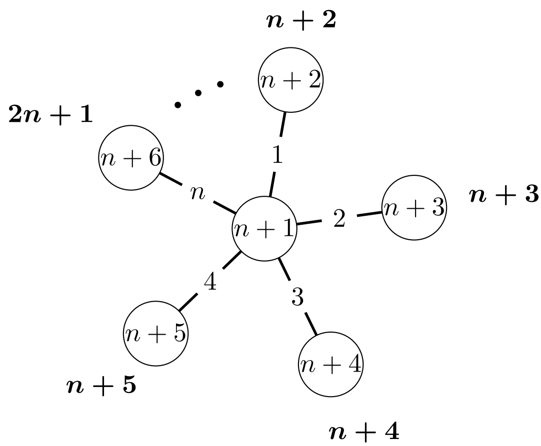


Figure 3.1: setla labeling of a star.

Corollary 3.3. *If a tree T with the largest number of pendent vertices equals ℓ at a vertex, then $\chi_{setla}(T) \geq \ell + 1$.*

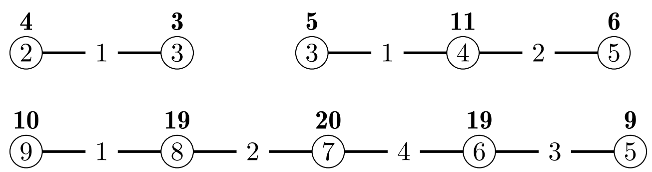


Figure 3.2: setla labeling of P_2, P_3 and P_5.

Theorem 3.4. *For the path P_n, $3 \leq \chi_{setla}(P_n) \leq 5$.*

Proof. It is easy to see that $\chi_{setla}(P_2) = 2$ and $\chi_{setla}(P_3) = 3$ (see Figure 3.2). Let P_n be a path with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $e_i = v_i v_{i+1}$ be edges, where $1 \leq i \leq n-1$, where $n \geq 4$. First, we establish the lower bound. Let f be a setla labeling of P_n. Then, $w(v_1) = f(v_2) + f(e_1)$ and $w(v_3) = f(v_2) + f(v_4) + f(e_2) + f(e_3)$. If $w(v_1) = w(v_3)$ then we obtain $f(e_1) = f(v_4) + f(e_2) + f(e_3)$. Which is impossible since $f(e_1) \leq n-1$ and $f(v_4) + f(e_2) + f(e_3) \geq n$. Therefore, $w(v_1) \neq w(v_3)$. Also $w(v_1) \neq w(v_2)$ and $w(v_2) \neq w(v_3)$ since f is a setla labeling and $v_1v_2, v_2v_3 \in E(P_n)$. This proves that

$$\chi_{setla}(P_n) \geq 3. \tag{2}$$

To prove the upper bound, we consider the following two cases:

Case 1: When $n \equiv 0 \pmod 4$, define a super edge total labeling f by

$$f(e_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \text{ and } i \neq n-2 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \text{ and } i \neq n-1 \\ n-1 & \text{if } i = n-2 \\ n-2 & \text{if } i = n-1 \end{cases}$$

and $f(v_i) = 2n - i$. Therefore,

$$\begin{aligned} w(v_1) &= f(v_2) + f(e_1) = 1 + (2n-2) = 2n-1 \\ w(v_n) &= f(v_{n-1}) + f(e_{n-1}) = (n+1) + (n-2) = 2n-1 \\ w(v_{n-1}) &= f(v_{n-2}) + f(v_n) + f(e_{n-2}) + f(e_{n-1}) \\ &= (n+2) + n + (n-1) + (n-2) = 4n-1 \\ w(v_{n-2}) &= f(v_{n-3}) + f(v_{n-1}) + f(e_{n-3}) + f(e_{n-2}) \\ &= (n+3) + (n+1) + (n-3) + (n-1) = 4n. \end{aligned}$$

Now for each i, $2 \leq i \leq n-3$,

$$\begin{aligned} w(v_i) &= f(v_{i-1}) + f(v_{i+1}) + f(e_{i-1}) + f(e_i) \\ &= 2n - (i-1) + 2n - (i+1) + f(e_{i-1}) + f(e_i) \\ &= 4n - 2i + f(e_{i-1}) + f(e_i) \\ &= 4n - 2i + \begin{cases} i + (i-1) & \text{if } i \equiv 0 \pmod 4 \\ (i-2) + i & \text{if } i \equiv 1 \pmod 4 \\ (i-1) + i & \text{if } i \equiv 2 \pmod 4 \\ (i-1) + (i+1) & \text{if } i \equiv 3 \pmod 4 \text{ and } i \neq n \end{cases} \\ &= \begin{cases} 4n-1 & \text{if } i \text{ is even} \\ 4n-2 & \text{if } i \equiv 1 \pmod 4 \\ 4n & \text{if } i \equiv 3 \pmod 4). \end{cases} \end{aligned}$$

Observe that f is a setla labeling and induces 4 colors.

Case 2: When $n \not\equiv 0 \pmod 4$, define a super edge total labeling f by

$$f(e_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

and $f(v_i) = 2n - i$.

Therefore,

$$\begin{aligned} w(v_1) &= f(v_2) + f(e_1) = (2n-2)+1 = 2n-1, \\ w(v_n) &= f(v_{n-1}) + f(e_{n-1}) \\ &= n+1+\begin{cases} n-2 & \text{if } n \equiv 1 \pmod 4 \\ n-1 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4 \end{cases} \\ &= \begin{cases} 2n-1 & \text{if } n \equiv 1 \pmod 4 \\ 2n & \text{if } n \equiv 2 \text{ or } 3 \pmod 4 \end{cases} \end{aligned}$$

and for $2 \le i \le n-1$,

$$\begin{aligned} w(v_i) &= f(v_{i-1}) + f(v_{i+1}) + f(e_{i-1}) + f(e_i) \\ &= [2n-(i-1)] + [2n-(i+1)] + f(e_{i-1}) + f(e_i) \\ &= (4n-2i) + f(e_{i-1}) + f(e_i) \\ &= (4n-2i) + \begin{cases} i+(i-1) & \text{if } i \equiv 0 \pmod 4 \\ (i-2)+i & \text{if } i \equiv 1 \pmod 4 \\ (i-1)+i & \text{if } i \equiv 2 \pmod 4 \\ (i-1)+(i+1) & \text{if } i \equiv 3 \pmod 4 \end{cases} \\ &= \begin{cases} 4n-1 & \text{if } i \text{ is even} \\ 4n-2 & \text{if } i \equiv 1 \pmod 4 \\ 4n & \text{if } i \equiv 3 \pmod 4). \end{cases} \end{aligned}$$

Subcase (i): When $n \equiv 1 \pmod 4$, $w(v_1) = w(v_n) = 2n-1$ and for each i, $2 \le i \le n-1$, $w(v_i)$ is $4n-1$ or $4n-2$ or $4n$. Therefore, $\chi_{setla}(P_n) \le 4$.

Subcase (ii): When $n \equiv 2$ or $3 \pmod 4$, $w(v_1) = 2n-1$, $w(v_n) = 2n$, and for each i, $2 \le i \le n-1$, $w(v_i)$ is $4n-1$ or $4n-2$ or $4n$. Therefore, $\chi_{setla}(P_n) \le 5$.

Hence, $3 \le \chi_{setla}(P_n) \le 5$.
This completes the proof. □

Proposition 3.5. *For the cycle C_4, $\chi_{setla}(C_4) = 3$.*

Proof. Consider a setla cycle $C_4\colon v_1v_2v_3v_4$ with setla labeling f. Since, f is setla, we have $w(v_1) \neq w(v_2)$ and $w(v_2) \neq w(v_3)$. We will show that $\chi_{setla}(C_4) = 3$. On the contrary, suppose $\chi_{setla}(C_4) = 2$. We must have, $w(v_1) = w(v_3) \implies f(e_1) + f(e_4) = f(e_2) + f(e_3)$ and $w(v_2) = w(v_4) \implies f(e_1) + f(e_2) = f(e_3) + f(e_4)$. From the above two equalities, we have

$f(e_2) = f(e_4)$, which is a contradiction. Hence, $\chi_{setla}(C_4) \geq 3$. Also, from the setla labeling of C_4 as shown in Figure 3.3, $\chi_{setla}(C_4) \leq 3$. Therefore, $\chi_{setla}(C_4) = 3$. □

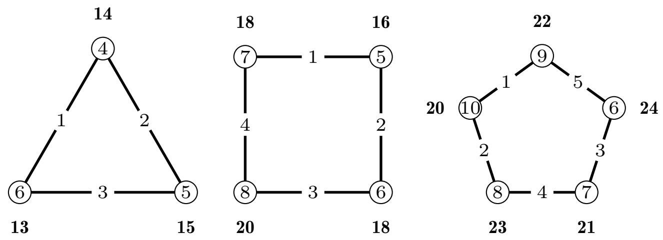


Figure 3.3: setla labelings of C_3, C_4 and C_5.

Theorem 3.6. *For the cycle C_n, $n \geq 3$, $3 \leq \chi_{setla}(C_n) \leq 5$.*

Proof. It is easily observed that $\chi_{setla}(C_3) = 3$. And by Proposition 3.5, $\chi_{setla}(C_4) = 3$. Now consider the cycle C_n with vertex set $\{v_1, v_2, \ldots, v_n\}$ and $e_i = v_i v_{i+1}$ be edges for $n \geq 5$, where subscripts are taken modulo n. We consider the following two cases:

Case 1: When $n \equiv 3 \pmod 4$, define a super edge total labeling f by

$$f(e_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \text{ and } i \neq n \\ n & \text{if } i = n \end{cases}$$

$$f(v_i) = \begin{cases} 2n-i+1 & \text{if } i \neq 1,2 \\ 2n-1 & \text{if } i = 1 \\ 2n & \text{if } i = 2. \end{cases}$$

Therefore,

$$\begin{aligned} w(v_1) &= f(v_2) + f(v_n) + f(e_1) + f(e_n) \\ &= 2n + (n+1) + 1 + n \\ &= 4n + 2, \\ w(v_2) &= f(v_1) + f(v_3) + f(e_1) + f(e_2) \\ &= (2n-1) + (2n-2) + 1 + 2 \\ &= 4n, \\ w(v_3) &= f(v_2) + f(v_4) + f(e_2) + f(e_3) \end{aligned}$$

$$\begin{aligned}
&= 2n + (2n-3) + 2 + 4 \\
&= 4n + 3, \\
w(v_n) &= f(v_1) + f(v_{n-1}) + f(e_{n-1}) + f(e_n) \\
&= (2n-1) + (n+2) + (n-1) + n \\
&= 5n.
\end{aligned}$$

Now for each $i,\ 4 \le i \le n-1$,

$$\begin{aligned}
w(v_i) &= f(v_{i-1}) + f(v_{i+1}) + f(e_{i-1}) + f(e_i) \\
&= [2n - (i-1) + 1] + [2n - (i+1) + 1] + f(e_{i-1}) + f(e_i) \\
&= 4n + 2 - 2i + f(e_{i-1}) + f(e_i) \\
&= 4n + 2 - 2i + \begin{cases} i + (i-1) & \text{if } i \equiv 0 \pmod 4 \\ (i-2) + i & \text{if } i \equiv 1 \pmod 4 \\ (i-1) + i & \text{if } i \equiv 2 \pmod 4 \\ (i-1) + (i+1) & \text{if } i \equiv 3 \pmod 4 \end{cases}
\end{aligned}$$

$$w(v_i) = \begin{cases} 4n+1 & \text{if } i \text{ is even} \\ 4n & \text{if } i \equiv 1 \pmod 4 \\ 4n+2 & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

This proves that f is a setla labeling and induces 5 colors. Hence, $\chi_{setla}(C_n) \le 5$.

Case 2: When $n \not\equiv 3 \pmod 4$, define a super edge total labeling f by

$$f(e_i) = \begin{cases} i-1 & \text{if } i \equiv 0 \pmod 4 \\ i & \text{if } i \equiv 1 \text{ or } 2 \pmod 4 \\ i+1 & \text{if } i \equiv 3 \pmod 4 \end{cases}$$

and

$$f(v_i) = \begin{cases} 2n - i + 1 & \text{if } i \ne 1, 2 \\ 2n-1 & \text{if } i = 1 \\ 2n & \text{if } i = 2. \end{cases}$$

Therefore,

$$\begin{aligned}
w(v_1) &= f(v_2) + f(v_n) + f(e_1) + f(e_n) \\
&= 2n + (n+1) + 1 + \begin{cases} n & \text{if } n \equiv 1 \text{ or } 2 \pmod 4 \text{ and } n \ne 1 \\ n-1 & \text{if } n \equiv 0 \pmod 4 \end{cases} \\
&= \begin{cases} 4n+2 & \text{if } n \equiv 1 \text{ or } 2 \pmod 4 \\ 4n+1 & \text{if } n \equiv 0 \pmod 4, \end{cases}
\end{aligned}$$

$$\begin{aligned}
w(v_2) &= f(v_1) + f(v_3) + f(e_1) + f(e_2) \\
&= (2n-1) + (2n-2) + 1 + 2 = 4n, \\
w(v_3) &= f(v_2) + f(v_4) + f(e_2) + f(e_3) \\
&= 2n + (2n-3) + 2 + 4 = 4n+3, \\
w(v_n) &= f(v_1) + f(v_{n-1}) + f(e_{n-1}) + f(e_n) \\
&= (2n-1) + (n+2) + f(e_{n-1}) + f(e_n) \\
&= (3n+1) + \begin{cases} 2n-1 & \text{if } n \equiv 0 \pmod 4 \\ 2n-2 & \text{if } n \equiv 1 \pmod 4 \\ 2n-1 & \text{if } n \equiv 2 \pmod 4 \end{cases} \\
&= \begin{cases} 5n & \text{if } n \equiv 0 \text{ or } 2 \pmod 4 \\ 5n-1 & \text{if } n \equiv 1 \pmod 4. \end{cases}
\end{aligned}$$

Now for each i, $2 \leq i \leq n-1$,

$$\begin{aligned}
w(v_i) &= (4n-2i+2) + \begin{cases} 2i-1 & \text{if } i \equiv 0 \pmod 4 \\ 2i-2 & \text{if } i \equiv 1 \pmod 4 \\ 2i-1 & \text{if } i \equiv 2 \pmod 4 \\ 2i & \text{if } i \equiv 3 \pmod 4 \end{cases} \\
&= \begin{cases} 4n & \text{if } i \equiv 1 \pmod 4 \\ 4n+1 & \text{if } i \equiv 0 \text{ or } 2 \pmod 4 \\ 4n+2 & \text{if } i \equiv 3 \pmod 4. \end{cases}
\end{aligned}$$

It is easy to verify that the weights of vertices are $4n$, $4n+1$ or $4n+2$, $4n+3$, $5n-1$ or $5n$. Therefore, f is a setla labeling that induces 4 colors. Hence, $\chi_{setla}(C_n) \leq 4$.

This completes the proof. □

Now we calculate the setla labeling for a complete bipartite graph $K_{m,n}$. When $m = n = 2$, then $K_{2,2} \cong C_4$ and by Proposition 3.5, $\chi_{setla}(C_4) = 3$ and for $m = 2$ and $n \geq 3$ we have the following result.

Theorem 3.7. *$\chi_{setla}(K_{m,n}) = 2$ if and only if $m, n > 1$, $mn > 4$ and $m \equiv n \pmod 2$.*

Proof. Suppose $m, n > 1, mn > 4$ and $m \equiv n \pmod 2$. Then by Theorem 1.4, there exists a magic rectangle $MR(m,n) = [r_{i,j}]_{m\times n}$ with row sum $\rho = \frac{n(mn+1)}{2}$, column sum $\sigma = \frac{m(mn+1)}{2}$. Let $\{x_1, x_2, \ldots, x_m\}$ and

$\{y_1, y_2, \ldots, y_n\}$ be a bipartition of the vertex set of $K_{m,n}$ with $m \leq n$. Define the super edge total labeling f by

$$f(a) = \begin{cases} mn + i & \text{if } a = x_i \\ mn + m + i & \text{if } a = y_i \\ r_{i,j} & \text{if } a = x_i y_j. \end{cases}$$

Then for any $1 \leq i \leq m$ and for any $1 \leq j \leq n$,

$$\begin{aligned} w(x_i) &= \sum_{j=1}^{n} f(y_j) + \rho \\ &= n(mn + m) + \frac{n(n+1)}{2} + \frac{n(mn+1)}{2} \\ &= (3m+1)n^2 + (2m+2)n, \end{aligned}$$

$$\begin{aligned} w(y_j) &= \sum_{i=1}^{n} f(x_i) + \sigma \\ &= m(mn) + \frac{m(m+1)}{2} + \frac{m(mn+1)}{2} \\ &= (3n+1)m^2 + 2m. \end{aligned}$$

It is easy to verify that $w(x_i) > w(y_j)$ for any i, $1 \leq i \leq m$ and for any j, $1 \leq j \leq n$. Therefore, f is a setla labeling and induces 2 colors. Hence, $\chi_{setla}(K_{m,n}) \leq 2$. We know, $\chi(K_{m,n}) = 2$. Therefore, $\chi_{setla}(K_{m,n}) \geq \chi(K_{m,n}) = 2$.

Conversely, suppose that $\chi_{setla}(K_{m,n}) = 2$ with setla labeling f. We must have $w(x_i) \neq w(y_j)$ for any i, $1 \leq i \leq m$ and for any j, $1 \leq j \leq n$, and weights of vertices in independent sets must be same. We can form a magic rectangle $MR(m, n)$ with (i, j)-th entry as $f(x_i y_j)$ so that row sum is equal to $w(y_j)$ and the column sum is equal to $w(x_i)$ for any i and j. Then again, by Theorem 1.4, m and n satisfy the required conditions. This completes the proof. □

Proposition 3.8. *For any $n \geq 1$,*

$$\chi_{setla}(K_{2,n}) = \begin{cases} 2 & \text{if } n \geq 4 \text{ and } m \text{ is even} \\ 3 & \text{if } n \text{ is odd or } n = 2. \end{cases}$$

Proof. We know that $\chi_{setla}(K_{2,1}) = \chi_{setla}(K_{2,2}) = 3$. Let $n \geq 3$. If n is even by Theorem 3.7, $\chi_{setla}(K_{2,n}) = 2$. Let n be odd. Again by Theorem 3.7, $\chi_{setla}(K_{2,n}) \geq 3$. We will show that $\chi_{setla}(K_{2,n}) \leq 3$. Let $\{x, y\}$ and $\{u_1, u_2, \ldots, u_n\}$ be a bipartition of $K_{2,n}$, where $n \geq 3$. Define a super edge total labeling of $K_{2,n}$ by $f(x) = 3n+1$, $f(y) = 3n+2$, for each i, $1 \leq i \leq n$, $f(u_i) = 2n+i$, and

$$f(au_i) = \begin{cases} i & \text{if } a = x \\ 2n+1-i & \text{if } a = y. \end{cases}$$

Now, we calculate the weights. For each i, $1 \leq i \leq n$, $w(u_i) = 8n+4$, $w(x) = 3n^2+n$, $w(y) = 4n^2+n$. It is easy verify that $w(u_i) < w(x) < w(y)$ for all i, $1 \leq i \leq n$. Hence, f is a setla labeling and induces 3 colors. Therefore, $\chi_{setla}(K_{2,n}) \leq 3$. Hence, $\chi_{setla}(K_{2,n}) = 3$. □

Then we have the following question:

Problem 3.9. Determine the super vertex (edge) total local antimagic chromatic number for the complete bipartite graph $K_{m,n}$, where $m \not\equiv n \pmod 2$.

Theorem 3.10. *For a bi-star $\chi_{setla}(B_{m,n}) = n+2$, where $m \leq n$.*

Proof. Suppose $m \leq n$. By Theorem 3.1, $n+1 \leq \chi_{setla}(B_{m,n})$. First we show that $\chi_{setla}(B_{m,n}) > n+1$. Let f be any setla labeling of $B_{m,n}$. We know that weights of all pendent vertices y_i are distinct and $w(y) \neq w(y_i)$ for any i, $1 \leq i \leq n$. If $w(x) = w(y_i)$ for any i, then we obtain $f(x) + f(xx_i) = f(x) + f(xy) + \sum_{i=1}^{n}(f(y_i) + f(yy_i))$. This implies, $f(xx_i) = f(xy) + \sum_{i=1}^{n}(f(y_i) + f(yy_i))$, which is not possible. Therefore, $\chi_{setla}(B_{m,n}) \geq n+2$. This proves the lower bound. For an upper bound, we define a super edge total labeling f of $B_{m,n}$ as follows:

$$\begin{aligned} f(x) &= 2m+2n+3, & \\ f(y) &= m+2n+3, & \\ f(xy) &= m+n+1, & \\ f(x_i) &= m+n+1+i, & \text{if } 1 \leq i \leq m \\ f(xx_i) &= i, & \text{if } 1 \leq i \leq m \\ f(yy_i) &= m+i, & \text{if } 1 \leq i \leq n \end{aligned}$$

and we label the vertices y_i by $\{2m+n+2, 2m+n+3, \ldots, 2m+2n+2\} - \{m+2n+3\}$ in any manner. The sum of these y_i labels is

$$\frac{3n^2+4mn+2m+3n-2}{2}.$$

Now we calculate the weights:

$$
\begin{aligned}
w(x) &= f(y) + f(xy) + \sum_{i=1}^{m}(f(x_i) + f(xx_i)) \\
&= (m + 2n + 3) + (m + n + 1) + \sum_{i=1}^{m}(m + n + 1 + 2i) \\
&= 2m^2 + mn + 4m + 3n + 4, \\
w(y) &= f(x) + f(xy) + \sum_{i=1}^{n} f(y_i) + \sum_{i=1}^{n} f(yy_i) \\
&= (2m + 2n + 3) + (m + n + 1) + \tfrac{3n^2+4mn+2m+3n-2}{2} + \sum_{i=1}^{n}(m + i) \\
&= 2n^2 + 3mn + 4m + 5n + 3.
\end{aligned}
$$

The weights of pendent vertices: $w(x_i) = f(x) + f(xx_i) = 2m + 2n + 3 + i$ and $w(y_i) = f(y) + f(yy_i) = 2m + 2n + 3 + i$. Thus f is a required setla that induces $n + 2$ colors. Hence, $\chi_{setla}(B_{m,n}) = n + 2$. □

A setla labeling of bi-star $B(4, 5)$ is illustrated in Figure 3.4.

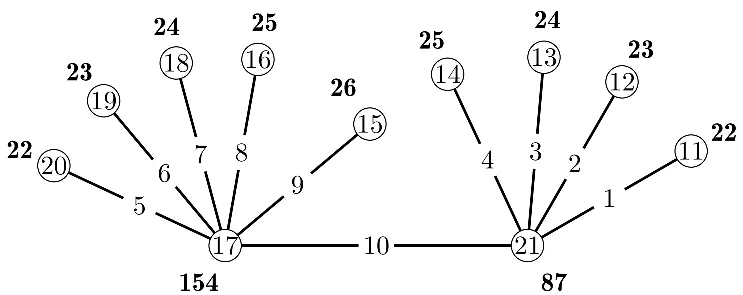


Figure 3.4: setla labeling of bi-star $B(4, 5)$.

Theorem 3.11. *If G is an r-regular graph, then*

$$\chi_{setla}(G \circ K_1) \leq \chi_{setla}(G){+}1.$$

Proof. Let G be an r-regular graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and m edges. Let f be a setla labeling of G. Without loss of generality, we assume that $f(v_i) = m + i$, where $1 \leq i \leq n$. Let $\{x_1, x_2, \ldots, x_n\}$ be newly added vertices to obtain $H = G \circ K_1$ such that there is an edge $e_i = v_i x_i$ for each i, $1 \le i \le n$. We define a super edge total labeling g of H as follows:

$$\begin{array}{ll} g(v_i) = f(v_i) + n = m + n + i & \text{if } 1 \le i \le n \\ g(x_i) = m + 2n + i & \text{if } 1 \le i \le n \\ g(e_i) = m + n + 1 - i & \text{if } 1 \le i \le m \\ g(e) = f(e) & \text{if } e \in E(G). \end{array}$$

Now, we calculate the weight of each vertex in H induced by g. For each i, $1 \le i \le n$, $w_g(x_i) = g(v_i) + g(e_i) = 2m + 2n + 1$ and $g(x_i) + g(e_i) = 2m + 3n + 1$. For any vertex $v_i \in V(G)$,

$$\begin{aligned} w_g(v_i) &= \sum_{x \in NT_H(v_i)} g(x) \\ &= g(x_i) + g(e_i) + \sum_{x \in NT_G(v_i)} g(x) \\ &= 2m + 3n + 1 + \sum_{x \in NT_G(v_i)} g(x) \\ &= 2m + 3n + 1 + \sum_{u \in N_G(v_i)} g(u) + \sum_{uv_i \in E(G)} g(uv_i) \\ &= 2m + 3n + 1 + \sum_{u \in N_G(v_i)} (n + f(u)) + \sum_{uv_i \in E(G)} f(uv_i) \\ &= 2m + 3n + 1 + nr + \sum_{u \in N_G(v_i)} f(u) + \sum_{uv_i \in E(G)} f(uv_i) \\ &= 2m + 3n + 1 + nr + w_f(v_i) \end{aligned}$$

is independent of i. This proves that, for any $i \ne j$, $w_g(v_i) \ne w_g(v_j)$ if and only if $w_f(v_i) \ne w_f(v_j)$. Hence, g is a setla labeling of $G \circ K_1$ and $\chi_{setla}(G \circ K_1) \le \chi_{setla}(G) + 1$. □

We pose the following problems:

Problem 3.12. Characterize graphs whose $\chi_{setla}(G) = \chi(G)$.

Problem 3.13. Characterize graphs whose $\chi_{setla}(G \circ K_1) = \chi_{setla}(G) + 1$.

Problem 3.14. Let G and H be super edge total local antimagic graphs. Determine the $\chi_{setla}(G \circ H)$ in terms of $\chi_{setla}(G)$ and $\chi_{setla}(H)$.

Conjecture 3.15. Every graph without isolated vertices admits a super vertex total local antimagic labeling.

Conjecture 3.16. Every graph without isolated vertices admits a super edge total local antimagic labeling.

4 Conclusion and future directions

We studied and calculated the super vertex (edge) total local antimagic chromatic numbers for some families of graphs, such as paths, cycles, and bipartite graphs. We conclude with some directions for further investigation, which will appear in subsequent papers:

1. Characterize graphs G on n vertices for which $\chi_{svtla}(G) = k$ (or $\chi_{setla}(G) = k$) for a given $2 \leq k \leq n$.

2. Study extensions to closed neighborhoods by defining the weights $w(u) = \sum_{x \in NT[u]} f(x)$, and studying induced coloring schemes under super vertex (edge) total local antimagic labelings.

3. Another possible extension would be to remove the condition of super vertex (edge) by revising the definition of the weights as $w(u) = \sum_{x \in NT(u)} f(x)$.

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