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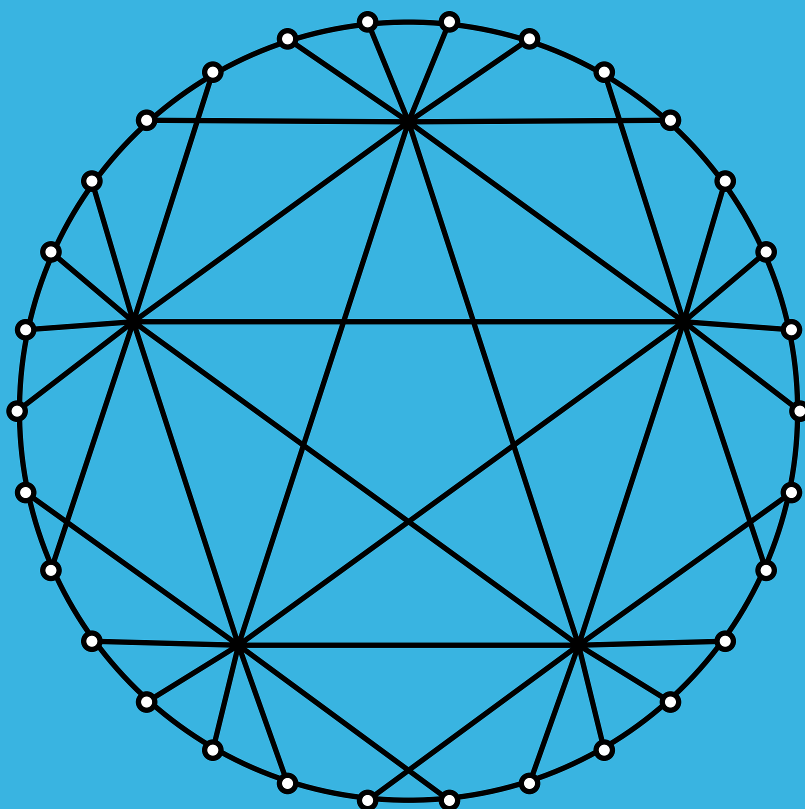

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Mixed hexagon systems

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Dedication. In memory of the late, great Dean Hoffman of Auburn University. He was the Master's thesis advisor and a mentor for the first author. He once commented to the first author that mixed triple systems were "a cute idea!"

Abstract. A decomposition of the complete mixed graph on v vertices into a partial orientation of the 6-cycle with two edges and four arcs is a *mixed hexagon system* of order v. Necessary and sufficient conditions for the existence of a mixed hexagon system of order v are given for each of the 25 such partial orientations of the 6-cycle.

1 Introduction

Graph and digraph decompositions are very widely studied. In particular, isomorphic decompositions of complete graphs and digraphs have received substantial attention. Formally, for K_v (respectively, D_v) the complete graph (digraph) on v vertices and g a subgraph (subdigraph) of K_v (D_v), a g-decomposition of K_v (D_v) is a set $\gamma = \{g_1, g_2, \ldots, g_k\}$ of edge (arc) disjoint isomorphic copies of g such that the union of the edge sets (arc sets) of the g_i is the edge set (arc set) of the complete graph (digraph):

$$\cup_{i=1}^k E(g_i) = E(K_v) \text{ or } \cup_{i=1}^k A(g_i) = A(D_v).$$

Steiner triple systems were the first graph decompositions to be studied; a *Steiner triple system* of order v is a 3-cycle decompositions of K_v. For a detailed history, see [6]. A C_m-decomposition of the complete graph on v vertices, where C_m is a cycle of length m, was a topic of intense

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interest until necessary and sufficient conditions for the existence of such decompositions were given for all m and all v in the early 2000s [1, 12]. An easier and self-contained construction for odd cycle systems is given in [2]. Of particular interest to us is the well-known fact that a 6-cycle decomposition of K_v (called a *hexagon system* of order v) exists if and only if $v \equiv 1$ or 9 (mod 12). A survey of all that has been written about hexagon systems up through 2021, including a list of open problems, can be found in [9].

Automorphisms of hexagon systems have received some attention. A hexagon system which admits a cyclic permutation as an automorphism is a *cyclic hexagon system* and such systems exist of order v if and only if $v \equiv 1$ or 9 (mod 12), $v \geq 13$ [10, 11]. A *k-rotational* hexagon system of order v admits a permutation consisting of a single fixed point and k cycles each of length $(v-1)/k$ as an automorphism. A *reverse* hexagon system admits an involution with a single fixed point as an automorphism. A 2-rotational hexagon system of order v exists if and only if $v \equiv 1$ or 9 (mod 12), a 4-rotational hexagon system of order v exists if and only if $v \equiv 1$ or 9 (mod 12), and a reverse hexagon system of order v exists if and only if $v \equiv 1$ or 9 (mod 12) [8]. All constructions in this paper are based on cyclic permutations. So, the existence conditions for mixed hexagon systems given below are the same as the conditions for the existence of cyclic mixed hexagon systems.

Some digraph decompositions of note are g-decompositions of D_v when g is one of the two orientations of the 3-cycle. These correspond to a *Mendelsohn triple system* of order v (when g is the 3-circuit) and a *directed triple system* of order v when g is the other orientation of the 3-cycle. Each exists if and only if $v \equiv 0$ or 1 (mod 3), except that a Mendelsohn triple system of order 6 does not exist [5, 7].

Harary and Palmer define a mixed graph as containing "both ordinary and oriented lines" [4]. They follow the fairly standard convention of combining two anti-parallel arcs into a single edge. We deviate from this practice, and use the following definition. A *mixed graph* on v vertices is an ordered pair (V, C) where V is a set of vertices with $|V| = v$, and C is a set of ordered pairs and unordered pairs of elements of V. The unordered pairs are called *edges* and the ordered pairs are called *arcs*. The *complete mixed graph* on v vertices, denoted M_v, is the mixed graph (V, C) where, for every pair of distinct vertices $v_1, v_2 \in V$, we have that C contains an edge between v_1 and v_2, an arc from v_1 to v_2, and an arc from v_2 to v_1. Notice that M_v has twice as many arcs as edges. For g a mixed graph, a g-decomposition of M_v is a set $\gamma = \{g_1, g_2, \ldots, g_k\}$ of edge disjoint and arc disjoint isomorphic

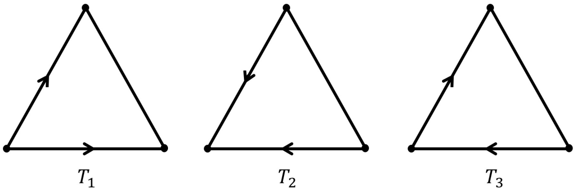

Figure 1. The three mixed triples with one edge and two arcs

copies of g such that the union of the edge sets and arc sets of the g_i are the edge set and the arc set, respectively, of the complete mixed graph:

$$\cup_{i=1}^{k} E(g_i) = E(M_v) \text{ and } \cup_{i=1}^{k} A(g_i) = A(M_v).$$

In [3], decompositions of M_v into each of the partial orientations of the 3-cycle are considered. Since M_v has twice as many arcs as edges, the partial orientation of the 3-cycle must have two arcs and one edge. The three such mixed graphs are given in Figure 1. For $i \in \{1, 2, 3\}$, a T_i-decomposition of M_v is called a *T_i-triple system*, and collectively these are *mixed triple systems*. A T_i-triple system of order v exists if and only if $v \equiv 1 \pmod 2$, except for $v \in \{3, 5\}$ for T_3.

Since M_v has twice as many arcs as edges, then the next size cycle that can be considered in connection with decompositions of M_v is the 6-cycle. We list below the 25 partial orientations of the 6-cycle that have two edges and four arcs; we call these *mixed hexagons*. A decomposition of M_v into one of these partially oriented 6-cycles is a *mixed hexagon system* of order v. The purpose of this paper is give necessary and sufficient conditions for the existence of a mixed hexagon system of order v for each of the 25 mixed hexagons.

2 The mixed hexagons

We consider three categories of mixed hexagons with two edges and four arcs: (1) Category A in which the two edges share a vertex, (2) Category B in which there is a single arc which shares a vertex with each edge, and (3) Category C of all others. Notice that in Category C, the arcs appear as two orientations of paths of length two, where the oriented paths of length two are vertex disjoint. An example of each category is given in Figure 2. Up to isomorphism, there are 10 different Category A mixed hexagons.

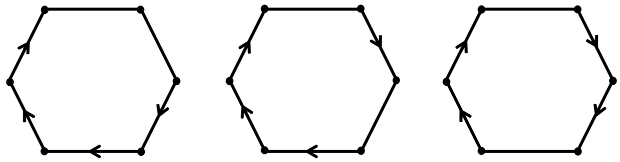

Figure 2. Examples of each category of mixed hexagon.

We introduce a notation and a symbol to represent each of the Category A mixed hexagons, as given in Figure 3. Notice that the Category A mixed hexagons come in the following converse pairs: $A2$ and $A8$, $A3$ and $A5$, $A4$ and $A10$, $A6$ and $A9$ ($A1$ and $A7$ are each self converse).

Up to isomorphism, there are 8 different Category B mixed hexagons. We introduce a notation and a symbol to represent each of the Category B mixed hexagons, as given in Figure 4. Notice that the Category B mixed hexagons come in the converse pairs $B2$ and $B5$, and $B4$ and $B7$ ($B1$, $B3$, $B6$, and $B8$ are each self converse).

Up to isomorphism, there are 7 different Category C mixed hexagons. We introduce a notation and a symbol to represent each of the Category C mixed hexagons, as given in Figure 5. Notice that the Category C mixed hexagons come in the converse pairs $C2$ and $C3$, and $C4$ and $C6$ ($C1$, $C5$, and $C7$ are self converse).

3 Classification of mixed hexagon systems

In this section, we give necessary and sufficient conditions for the existence of a mixed hexagon system of order v, for each Category A, B, and C mixed hexagon.

Lemma 3.1. *If a mixed hexagon system of order v exists for any of the 25 mixed orientations given above, then $v \equiv 1 \pmod 4$, $v \geq 9$.*

Proof. Notice that the complete mixed graph M_v has $v(v-1)$ arcs, and each mixed hexagon has four arcs. So, in a mixed hexagon system of order v it is necessary that $4 \mid v(v-1)$, or that $v \equiv 0$ or $1 \pmod 4$. The total degree (that is, the in-degree plus the out-degree plus the edge degree) of each vertex of M_v is $3(v-1)$. The total degree of each vertex of a mixed

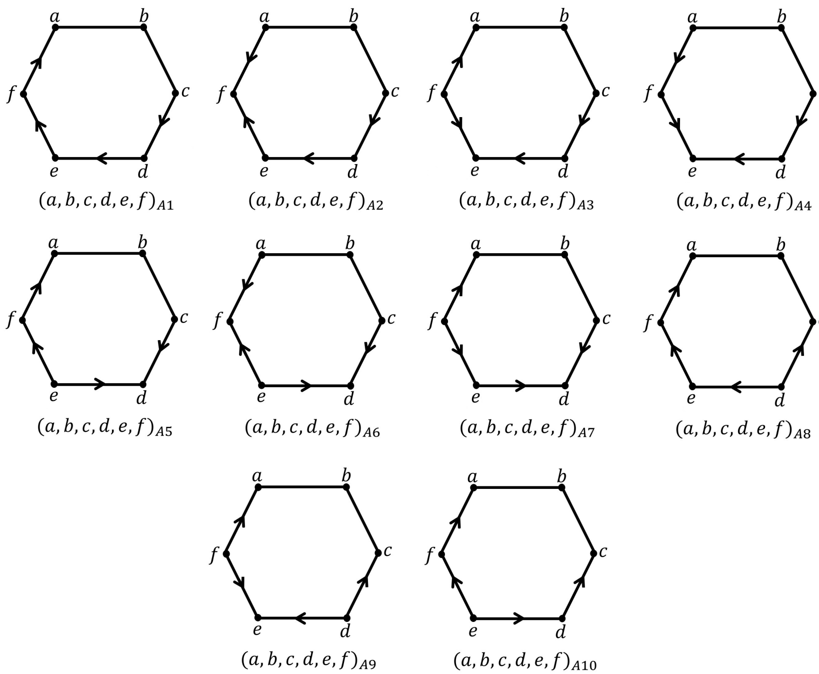

Figure 3. The ten Category A mixed hexagons and our notation for each.

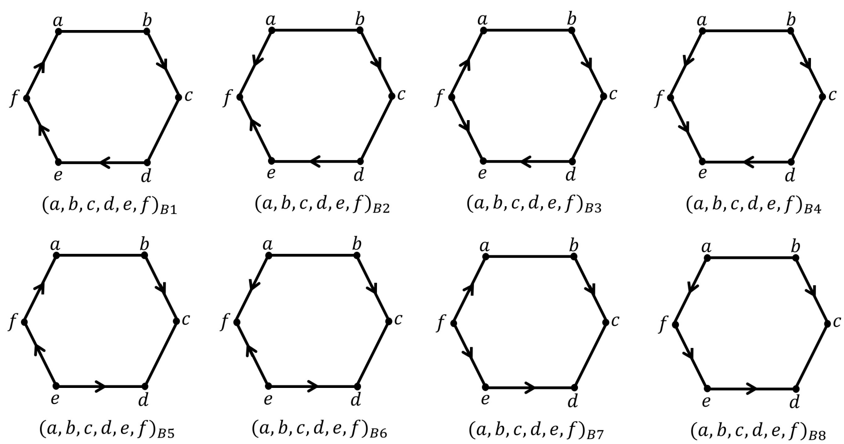

Figure 4. The eight Category B mixed hexagons and our notation for each.

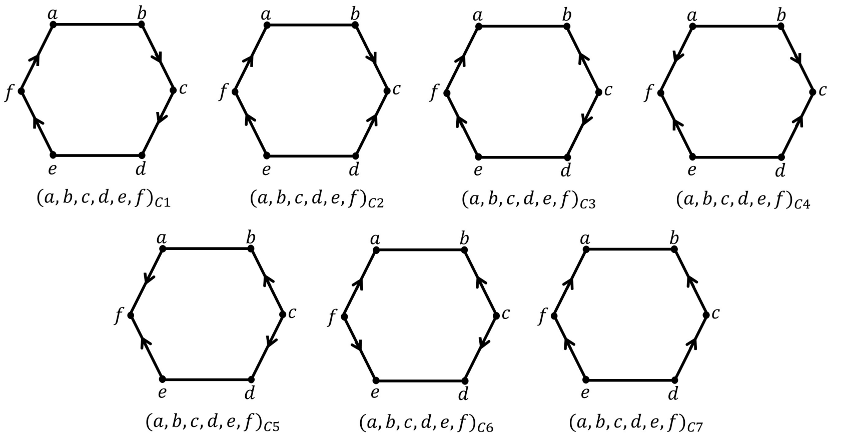

Figure 5. The seven Category C mixed hexagons and our notation for each.

hexagon is two, so in a mixed hexagon system of order v it is also necessary that $2 \mid 3(v-1)$. That is, v must be odd. Therefore, a necessary condition for the existence of a mixed hexagon system of order v is $v \equiv 1 \pmod 4$. $\square$

In order to establish the sufficient conditions for the existence of a mixed hexagon system, we construct such systems for all $v \equiv 1 \pmod 4$, where $v \geq 9$. We take the vertex set of M_v as $\{0, 1, \ldots, v-1\}$. A set $\mathcal{B}$ of mixed hexagons is a set of *base blocks* for a mixed hexagon system of order v, if the images of the elements of $\mathcal{B}$ under the cyclic permutation $\pi = (0, 1, 2, \ldots, v-1)$ form a mixed hexagon system of order v. We present a set of base blocks in each case.

Lemma 3.2. *If $v \equiv 1 \pmod 8$, $v \geq 9$, then an Aj-decomposition of M_v exists for each $j \in \{1, 2, \ldots, 10\}$.*

Proof. Let $v = 8k+1$ where $k \geq 1$. Define the following sets of blocks (here, and throughout, it is understood that vertex labels are reduced modulo v):

$$\begin{aligned}
\mathcal{A}_1 &= \{(0, 4k+1+2i, 8k, 4k-2-2i, 4k-1, 8k-1-2i)_{A1},\\
&\quad (0, 2k-2i, 1, 8k-2i, 4k, 4k-1-2i)_{A1} \mid i = 0, 1, \ldots, k-1\},\\
\mathcal{A}_2 &= \{(0, 3, 8, 1, 5, 6)_{A2}, (0, 2, 1, 6, 5, 3)_{A2}\}, \text{if } k = 1,\\
\mathcal{A}_2 &= \{(0, 2k+1+2i, 8k, 2k-1-2i, 4k+1, 6k-2i)_{A2}, (0, 2+2i, 1,\\
&\quad 6k+3+2i, 4k+1, 2k+1+2i)_{A2} \mid i = 0, 1, \ldots, k-1\}, \text{if } k \geq 2,\\
\mathcal{A}_3 &= \{(0, 2k+1+2i, 8k, 2k-2-2i, 4k, 6k+1+2i)_{A3},\\
&\quad (0, 8k-2i, 1, 6k+3+2i, 4k+1, 2k-2i)_{A3} \mid i = 0, 1, \ldots, k-2\}\\
&\quad \cup \{(0, 4k+1, 8k, 4k-1, 4k, 8k-1)_{A3},
\end{aligned}$$

$$
\begin{aligned}
& (0, 6k+2, 1, 4k+2, 4k+1, 2)_{A3}\},\\
\mathcal{A}_4 &= \{(0, 2k-2i, 1, 8k-2i, 4k, 4k-1-2i)_{A4}, (0, 2k+1+2i, 8k,\\
& \quad 1+2i, 4k+1, 4k+2+2i)_{A4} \mid i = 0, 1, \ldots, k-1\},\\
\mathcal{A}_6 &= \{(0, 6k-1-2i, 8k, 4k-1-2i, 4k, 8k-1-2i)_{A6}, (0, 6k+2+2i,\\
& \quad 1, 4k+2+2i, 4k+1, 2+2i)_{A6} \mid i = 0, 1, \ldots, k-1\},\\
\mathcal{A}_7 &= \{(0, 2k+1+2i, 8k, 1+2i, 4k+1, 4k+2+2i)_{A7},\\
& \quad (0, 2k-2i, 1, 8k-2i, 4k, 4k-1-2i)_{A7} \mid i = 0, 1, \ldots, k-1\}.
\end{aligned}
$$

The set $\mathcal{A}_j$ forms a set of base blocks for a mixed hexagon system of type Aj of order $v \equiv 1 \pmod 8$, where $j \in \{1, 2, 3, 4, 6, 7\}$. Since the Category A mixed hexagons include the converse pairs $A2$ and $A8$, $A3$ and $A5$, $A4$ and $A10$, and $A6$ and $A9$ and M_v is self converse then these imply Aj decompositions of M_v for $j \in \{5, 8, 9, 10\}$ where $v \equiv 1 \pmod 8$. □

Lemma 3.3. *If $v \equiv 5 \pmod 8$, $v \geq 13$, then an Aj-decomposition of M_v exists for each $j \in \{1, 2, \ldots, 10\}$.*

Proof. Let $v = 8k+5$ where $k \geq 1$. Define the following sets of blocks:

$$
\begin{aligned}
\mathcal{A}_1 &= \{(0, 2, 1, 3, 4, 9)_{A1}, (0, 5, 11, 1, 7, 3)_{A1}, (0, 4, 1, 12, 6, 2)_{A1}\} \text{ if } k = 1,\\
\mathcal{A}_1 &= \{(0, 2+2i, 1, 3+2i, 4k+4, 4k+5+2i)_{A1} \mid i = 0, 1, \ldots, k-1\}\\
& \quad \cup \{(0, 6k+1-2i, 1, 8k+4-2i, 4k+2, 4k+1-2i)_{A1} \mid\\
& \quad i = 0, 1, \ldots, k-2\} \cup \{(0, 2k+3, 6k+5, 1, 4k+3, 2k+1)_{A1},\\
& \quad (0, 2k+2, 1, 6k+2, 4k-1, 2k-1)_{A1}\} \text{ if } k \geq 2,\\
\mathcal{A}_2 &= \{(0, 2+2i, 8k+4, 4k+3-i, 4k+5+i, 1+2i)_{A2},\\
& \quad (0, 6k+2-2i, 8k+4, 5k+3+i, 3k+4+3i, 6k+5+2i)_{A2} \mid\\
& \quad i = 0, 1, \ldots, k-1\} \cup \{(0, 4k+3, 4k+2, 8k+4, 6k+3, 2k+1)_{A2}\},\\
\mathcal{A}_3 &= \{(0, 2+2i, 8k+4, 1+2i, 4k+2+i, 4k+1-i)_{A3},\\
& \quad (0, 6k+2-2i, 8k+4, 3k-i, k+1+i, 3k+1-i)_{A3} \mid\\
& \quad i = 0, 1, \ldots, k-1\} \cup \{(0, 8k+4, 4k+2, 2k+1, 6k+4, 4k+3)_{A3}\}.
\end{aligned}
$$

The set $\mathcal{A}_j$ forms a set of base blocks for a mixed hexagon system of type Aj of order $v \equiv 1 \pmod 8$, where $j \in \{1, 2, 3\}$.

Let $v = 16k + 5$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{A}_4 &= \{(0, 1+4i, 16k+3, 2+4i, 4+8i, 3+4i)_{A4}, (0, 2+4i, 16k+3, \\
&\quad 4k+2+4i, 8k+4+8i, 4k+3+4i)_{A4}, (0, 4k+1+4i, 16k+3, \\
&\quad 8k+2+4i, 16k+4+8i, 8k+3+4i)_{A4}, (0, 4k+2+4i, 16k+3, \\
&\quad 12k+2+4i, 8k-1+8i, 12k+1+4i)_{A4} \mid i = 0, 1, \ldots, k-1\} \\
&\quad \cup \{(0, 8k+1, 16k+3, 16k+2, 16k-1, 16k+1)_{A1}\}, \\
\mathcal{A}_6 &= \{(0, 16k+2-4i, 16k+3, 2+4i, 1, 3+4i)_{A6}, (0, 16k+1-4i, \\
&\quad 16k+3, 4k+2+4i, 1, 4k+3+4i)_{A6}, (0, 12k+1-4i, \\
&\quad 16k+3, 12k+2+4i, 1, 12k+3+4i)_{A6} \mid i = 0, 1, \ldots, k-1\} \\
&\quad \cup \{(0, 12k+2-4i, 16k+3, 8k+2+4i, 1, 8k+3+4i)_{A6} \mid \\
&\quad i = 0, 1, \ldots, k-1, i \neq k/2\} \\
&\quad \cup \{(0, 10k+2, 16k+3, 10k, 16k+4, 10k+3)_{A6} \mid k/2 \in \mathbb{N}\} \\
&\quad \cup \{(0, 8k+2, 16k+3, 16k+2, 16k+4, 16k+1)_{A6}\}, \\
\mathcal{A}_7 &= \{(0, 5, 6, 10, 3, 19)_{A7}, (0, 14, 11, 19, 16, 15)_{A7}, (0, 2, 19, 10, 1, 11)_{A7}, \\
&\quad (0, 6, 19, 14, 1, 7)_{A7}, (0, 9, 19, 18, 1, 3)_{A7}\} \text{ if } k = 1, \\
\mathcal{A}_7 &= \{(0, 16k-6-4i, 16k+3, 10+4i, 1, 16k-5-4i)_{A7} \mid \\
&\quad i = 0, 1, \ldots, k-3\} \cup \{(0, 5, 6, 10, 3, 16k+3)_{A7}, \\
&\quad (0, 16k-2, 16k-5, 16k+3, 16k, 16k-1)_{A7}\} \\
&\quad \cup \{(0, 2+4i, 16k+3, 4k+2+4i, 1, 12k+3-4i)_{A7}, \\
&\quad (0, 4k+1+4i, 16k+3, 8k+2+4i, 1, 8k+3-4i)_{A7}, \\
&\quad (0, 4k+2+4i, 16k+3, 12k+2+4i, 1, 4k+3-4i)_{A7} \mid \\
&\quad i = 0, 1, \ldots, k-1\} \cup \{(0, 8k+1, 16k+3, 16k+2, 1, 3)_{A7}\} \text{ if } k \geq 2.
\end{aligned}$$

The set $\mathcal{A}_j$ forms a set of base blocks for a mixed hexagon system of type Aj of order $v \equiv 5 \pmod{16}$, where $j \in \{4, 6, 7\}$.

Let $v = 16k + 13$ where $k \geq 0$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{A}_4 &= \{(0, 1+4i, 16k+11, 2+4i, 4+8i, 3+4i)_{A4}, (0, 4k+2+4i, \\
&\quad 16k+11, 8k+6+4i, 16k+12+8i, 8k+5+4i)_{A4} \mid \\
&\quad i = 0, 1, \ldots, k\} \\
&\quad \cup \{(0, 2+4i, 16k+11, 4k+6+4i, 8k+12+8i, 4k+5+4i)_{A4}, \\
&\quad (0, 4k+5+4i, 16k+11, 12k+10+4i, 8k+7+8i, \\
&\quad 12k+9+4i)_{A4} \mid i = 0, 1, \ldots, k-1\} \\
&\quad \cup \{(0, 8k+5, 16k+11, 16k+10, 16k+7, 16k+9)_{A4}\}, \\
\mathcal{A}_6 &= \{(0, 4k+2+4i, 16k+11, 8k+6+4i, 1, 8k+7+4i)_{A6} \mid \\
&\quad i = 0, 1, \ldots, k\} \cup \{(0, 5+4i, 16k+11, 6+4i, 1, 7+4i)_{A6}, \\
&\quad (0, 2+4i, 16k+11, 4k+6+4i, 1, 4k+7+4i)_{A6}, \\
&\quad (0, 4k+5+4i, 16k+11, 12k+10+4i, 1, 12k+11+4i)_{A6} \mid \\
&\quad i = 0, 1, \ldots, k-1\} \cup \{(0, 16k+12, 2, 5, 3, 4)_{A6}, \\
&\quad (0, 8k+5, 16k+11, 16k+10, 16k+12, 16k+9)_{A6}\},
\end{aligned}$$

$$\begin{aligned}\mathcal{A}_7 = & \{(0, 5+4i, 16k+11, 6+4i, 1, 16k+7-4i)_{A7}, (0, 2+4i,\\ & 16k+11, 4k+6+4i, 1, 12k+7-4i)_{A7}, (0, 4k+5+4i, 16k+11,\\ & 12k+10+4i, 1, 4k+3-4i)_{A7} \mid i = 0, 1, \ldots, k-1\}\\ & \cup\{(0, 4k+2+4i, 16k+11, 8k+6+4i, 1, 8k+7-4i)_{A7} \mid\\ & i = 0, 1, \ldots, k\} \cup \{(0, 3, 2, 5, 1, 16k+12)_{A7},\\ & (0, 8k+5, 16k+11, 16k+10, 1, 3)_{A7}\}.\end{aligned}$$

The set $\mathcal{A}_j$ forms a set of base blocks for a mixed hexagon system of type Aj of order $v \equiv 13 \pmod{16}$, where $j \in \{4, 6, 7\}$.

Since the Category A mixed hexagons include the converse pairs $A2$ and $A8$, $A3$ and $A5$, $A4$ and $A10$, and $A6$ and $A9$, and M_v is self converse then these imply Aj decompositions of M_v for $j \in \{5, 8, 9, 10\}$ where $v \equiv 5$ (mod 8), $v \geq 13$. □

Lemma 3.4. *If $v \equiv 1$ (mod 8) then a Bj-decomposition of M_v exists for each $j \in \{1, 2, \ldots, 8\}$.*

Proof. Let $v = 8k+1$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}\mathcal{B}_1 = & \{(0, 6k-1-2i, 6k+1, 1+2i, 4k, 8k-2i)_{B1}, (0, 2+2i, 6k+1,\\ & 6k-2i, 4k-1-4i, 2k-1-2i)_{B1} \mid i = 0, 1, \ldots, k-1\},\\ \mathcal{B}_2 = & \{(0, 2k-2i, 6k+2, 4k+3+2i, 4k+2, 4k-2i)_{B2}, (0, 2k+1+2i,\\ & 6k, 4k-2-2i, 4k, 4k+1+2i)_{B2} \mid i = 0, 1, \ldots, k-1\},\\ \mathcal{B}_3 = & \{(0, 2k+1+2i, 6k, 4k-2-2i, 4k, 4k+1+2i)_{B3}, (0, 2k-2i,\\ & 6k+1, 4k+2+2i, 4k, 4k-1-2i)_{B3} \mid i = 0, 1, \ldots, k-1\},\\ \mathcal{B}_4 = & \{(0, 6k-1-2i, 6k+1, 1+2i, 4k, 8k-2i)_{B4}, (0, 2k-2i,\\ & 6k+2, 4k+3+2i, 4k+1, 1+2i)_{B4} \mid i = 0, 1, \ldots, k-1\},\\ \mathcal{B}_6 = & \{(0, 1, 4, 2, 6, 8)_{B6}, (0, 3, 7, 2, 4, 1)_{B6}\} \text{ if } k = 1,\\ \mathcal{B}_6 = & \{(0, 2k-2i, 6k+2, 4k+3+2i, 4k+2, 4k-2i)_{B6} \mid\\ & i = 0, 1, \ldots, k-1\} \cup \{(0, 2k+1+2i, 6k, 4k-2-2i, 4k-1,\\ & 4k+1+2i)_{B6} \mid i = 0, 1, \ldots, k-2\}\\ & \cup\{(0, 4k-1, 6k-1, 2k-1, 4k+1, 6k+2)_{B6}\} \text{ if } k \geq 2,\\ \mathcal{B}_8 = & \{(0, 2, 8, 7, 6, 4)_{B8}, (0, 5, 8, 2, 3, 7)_{B8}\} \text{ if } k = 1,\\ \mathcal{B}_8 = & \{(0, 2k-2i, 6k+2, 4k+3+2i, 4k+2, 4k-2i)_{B8} \mid\\ & i = 0, 1, \ldots, k-1\} \cup \{(0, 2k+1+2i, 6k,\\ & 4k-2-2i, 4k-1, 4k+1+2i)_{B8} \mid i = 0, 1, \ldots, k-2\}\\ & \cup\{(0, 4k+1, 6k+2, 2k, 4k-1, 6k-1)_{B8} \text{ if } k \geq 2.\end{aligned}$$

The set $\mathcal{B}_j$ forms a set of base blocks for a mixed hexagon system of type Bj of order $v \equiv 1 \pmod 8$, where $j \in \{1, 2, 3, 4, 6, 8\}$. Since the Category B

mixed hexagons include the converse pairs $B2$ and $B5$, and $B4$ and $B7$, and M_v is self converse then these imply Bj decompositions of M_v for $j \in \{5,7\}$ where $v \equiv 1 \pmod 8$. $\square$

Lemma 3.5. *If $v \equiv 5 \pmod 8$, $v \geq 13$, then a Bj-decomposition of M_v exists for each $j \in \{1,2,\ldots,8\}$.*

Proof. Let $v = 8k+5$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{B}_1 \;=\;& \{(0, 6k+1-2i, 2k-1, 4k+4+2i, 4k+3, 2+2i)_{B1} \mid \\
& i = 0,1,\ldots,k-2\} \cup \{(0, 2+2i, 4k+3, \\
& 4k+2-2i, 4k+4, 4k+5+2i)_{B1} \mid i = 0,1,\ldots,k-1\} \\
& \cup\{(0, 2k+3, 4k+4, 1, 4k+3, 2k+1)_{B1}, \\
& (0, 2k+2, 8k+3, 6k+2, 4k-1, 2k-1)_{B1}\},
\end{aligned}$$

$$\mathcal{B}_2 \;=\; \{(0,5,1,10,8,12)_{B2}, (0,12,4,11,8,3)_{B2}, (0,2,8,5,7,1)_{B2}\} \text{ if } k=1,$$

$$\begin{aligned}
\mathcal{B}_2 \;=\;& \{(0, 4+2i, 4k+9+3i, 4k+4+i, 8k+4, 3+2i)_{B2} \mid \\
& i = 0,1,\ldots,k-2\} \cup \{(0, 2k+2+2i, \\
& 7k+6+3i, 5k+3+i, 8k+4, 6k+5+2i)_{B2} \mid i = 0,1,\ldots,k-1, \\
& i \neq (k-2)/3\} \cup \{(0, (16k+10)/3, (8k+5)/3, (16k+7)/3, \\
& 8k+4, (20k+11)/3))_{B2} \mid (k-2)/3 \in \mathbb{N}\} \cup \{(0, 8k+4, 4k, \\
& 8k+3, 6k+2, 2k+1)_{B2}, (0, 2, 4k+4, 4k+1, 4k+3, 1)_{B2}\} \text{ if } k \geq 2,
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_3 \;=\;& \{(0, 2+2i, 4k+6+3i, 4k+3+i, 4k+5+3i, 4k+4+i)_{B3}, \\
& (0, 6k+2-2i, 4k+3, 6k+5+2i, k+1+i, 3k+1-i)_{B3} \mid \\
& i = 0,1,\ldots,k-1\} \cup \{(0, 4k+3, 2k+2, 2k+1, 6k+3, 4k+2)_{B3}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_8 \;=\;& \{(0, 2k+2+2i, 4k+2, 2k-1-2i, 7k+3-i, 5k+4+i)_{B8}, \\
& (0, 2+2i, 1, 8k+3-2i, 4k+2-i, 4k+4+i)_{B8} \mid \\
& i = 0,1,\ldots,k-1\} \cup \{(0, 8k+4, 6k+3, 2k+1, 6k+4, 4k+3)_{B8}\}.
\end{aligned}$$

The set $\mathcal{B}_j$ forms a set of base blocks for a mixed hexagon system of type Bj of order $v \equiv 5 \pmod 8$, where $j \in \{1,2,3,8\}$.

Let $v = 16k+5$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{B}_4 \;=\;& \{(0, 16k+2-4i, 1, 2+4i, 4+8i, 3+4i)_{B4}, \\
& (0, 16k+1-4i, 4k, 4k+2+4i, 8k+4+8i, 4k+3+4i)_{B4}, \\
& (0, 12k+1-4i, 8k, 12k+2+4i, 8k-1+8i, 12k+3+4i)_{B4} \mid \\
& i = 0,1,\ldots,k-1\} \cup \{(0, 12k+2-4i, 4k+1, \\
& 8k+2+4i, 16k+4+8i, 8k+3+4i)_{B4} \mid i = 0,1,\ldots,k-1, \\
& i \neq k/2\} \cup \{(0, 6k+1, 16k+3, 10k, 4k-1, 10k+3)_{B4} \mid k/2 \in \mathbb{N}\} \\
& \cup\{(0, 8k+2, 8k+1, 16k+2, 16k-1, 16k+1)_{B4}\},
\end{aligned}$$

$$\begin{aligned}\mathcal{B}_6 &= \{(0,14,18,2,20,1)_{B6},(0,18,9,10,1,11)_{B6},(0,17,4,6,1,7)_{B6},\\ &\quad (0,13,8,14,1,15)_{B6},(0,10,9,18,20,17)_{B6}\} \text{ if } k=1,\\ \mathcal{B}_6 &= \{(0,16k-6-4i,1,10+4i,16k+4,9+4i)_{B6} \mid i=0,1,\dots,k-3\}\\ &\quad \cup\{(0,16k-2,16k+2,2,16k+4,1)_{B6},(0,1,7,4,16k+2,5)_{B6}\}\\ &\quad \cup\{(0,16k+1-4i,4k,4k+2+4i,1,4k+3+4i)_{B6},\\ &\quad (0,12k+1-4i,8k,12k+2+4i,1,12k+3+4i)_{B6} \mid\\ &\quad i=0,1,\dots,k-1\}\cup\{(0,12k+2-4i,4k+1,\\ &\quad 8k+2+4i,1,8k+3+4i)_{B6} \mid i=0,1,\dots,k-1, i\neq k/2\}\\ &\quad \cup\{(0,6k+1,16k+3,10k,16k+4,10k+3)_{B6} \mid k/2\in\mathbb{N}\}\\ &\quad \cup\{(0,8k+2,8k+1,16k+2,16k+4,16k+1)_{B6}\} \text{ if } k\geq 2.\end{aligned}$$

The set $\mathcal{B}_j$ forms a set of base blocks for a mixed hexagon system of type Bj of order $v \equiv 5 \pmod{16}$, where $j \in \{4,6\}$.

Let $v = 16k+13$ where $k \geq 0$. Define the following sets of blocks:

$$\begin{aligned}\mathcal{B}_4 &= \{(0,1+4i,5+8i,2+4i,4+8i,3+4i)_{B4},(0,4k+2+4i,\\ &\quad 12k+10+8i,8k+6+4i,16k+12+8i,8k+7+4i)_{B4} \mid\\ &\quad i=0,1,\dots,k\}\cup\{(0,2+4i,4k+10+8i,4k+6+4i,\\ &\quad 8k+12+8i,4k+5+4i)_{B4},(0,4k+5+4i,4+8i,\\ &\quad 12k+10+4i,8k+7+8i,12k+9+4i)_{B4} \mid i=0,1,\dots,k-1\}\\ &\quad \cup\{(0,8k+5,8k+4,16k+10,16k+7,16k+9)_{B4}\},\\ \mathcal{B}_6 &= \{(0,2,10,6,1,7)_{B6},(0,5,9,2,1,3)_{B6},(0,1,11,8,10,9)_{B6}\} \text{ if } k=0,\\ \mathcal{B}_6 &= \{(0,2,14,10,1,11)_{B6},(0,9,4,22,1,23)_{B6},(0,6,22,14,1,15)_{B6},\\ &\quad (0,10,28,16,26,17)_{B6},(0,5,9,2,1,3)_{B6},(0,1,7,4,26,5)_{B6},\\ &\quad (0,13,12,26,1,27)_{B6}\} \text{ if } k=1,\\ \mathcal{B}_6 &= \{(0,9+4i,21+8i,10+4i,1,11+4i)_{B6} \mid i=0,1,\dots,k-2\}\\ &\quad \cup\{(0,2+4i,4k+10+8i,4k+6+4i,1,4k+7+4i)_{B6},\\ &\quad (0,4k+5+4i,4+8i,12k+10+4i,1,12k+11+4i)_{B6} \mid\\ &\quad i=0,1,\dots,k-1\}\cup\{(0,4k+2+4i,12k+10+8i,\\ &\quad 8k+6+4i,1,8k+7+4i)_{B6} \mid i=0,1,\dots,k\}\\ &\quad \cup\{(0,5,9,2,1,3)_{B6},(0,1,7,4,16k+10,5)_{B6},\\ &\quad (0,8k+5,8k+4,16k+10,1,16k+11)_{B6}\} \text{ if } k\geq 2.\end{aligned}$$

The set $\mathcal{B}_j$ forms a set of base blocks for a mixed hexagon system of type Bj of order $v \equiv 13 \pmod{16}$, where $j \in \{4,6\}$.

Since the Category B mixed hexagons include the converse pairs $B2$ and $B5$, and $B4$ and $B7$, and M_v is self converse then these imply Bj decompositions of M_v for $j \in \{5,7\}$ where $v \equiv 5 \pmod 8$, $v \geq 13$. □

Lemma 3.6. *If $v \equiv 1 \pmod 8$ then a Cj-decomposition of M_v exists for each $j \in \{1,2,\dots,7\}$.*

Proof. Let $v = 8k + 1$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{C}_1 &= \{(0, 2k-2i, 6k+1, 6k-2i, 4k+1, 4k-1-2i)_{C1}, (0, 2k+1+2i,\\
&\quad 6k+1, 6k+2+2i, 4k, 4k+2+2i)_{C1} \mid i = 0, 1, \ldots, k-1\},\\
\mathcal{C}_2 &= \{(0, 2k-2i, 6k+2, 6k+1-2i, 4k+2, 4k-2i)_{C2}, (0, 2k+1+2i,\\
&\quad 6k+1, 6k+2+2i, 4k, 4k+2+2i)_{C2} \mid i = 0, 1, \ldots, k-1\},\\
\mathcal{C}_4 &= \{(0, 2k-2i, 6k+2, 6k-2i, 4k+1, 4k-2i)_{C4}, (0, 2k+1+2i,\\
&\quad 6k, 6k+2+2i, 4k, 4k+1+2i)_{C4} \mid i = 0, 1, \ldots, k-1\},\\
\mathcal{C}_5 &= \{(0, 4, 8, 2, 3, 1)_{C5}, (0, 3, 1, 7, 5, 4)_{C5}\} \text{ if } k = 1,\\
\mathcal{C}_5 &= \{(0, 2k+1+2i, 6k+1, 6k+2+2i, 4k, 8k-1-2i)_{C5} \mid\\
&\quad i = 0, 1, \ldots, k-2\} \cup \{(0, 2k-2i, 6k+1, 2k+2+2i,\\
&\quad 3+4i, 2+2i)_{C4} \mid i = 0, 1, \ldots, k-1, i \neq (k-1)/2\}\\
&\quad \cup\{(0, 4k+1, 6k+3, 1, 4k, 6k+1)_{C4}\}\\
&\quad \cup\{(0, 7k+1, 4k, k, 2k+1, k+1)_{C4} \mid (k-1)/2 \in \mathbb{N}\} \text{ if } k \geq 2,\\
\mathcal{C}_7 &= \{(0, 2k-1-2i, 2k+1, 6k+1-2i, 4k+1, 8k-2i)_{C7},\\
&\quad (0, 2k+2+2i, 2k, 6k+1+2i, 4k, 1+2i)_{C7} \mid i = 0, 1, \ldots, k-2\}\\
&\quad \cup\{(0, 4k-1, 6k-1, 8k, 4k, 6k)_{C7},\\
&\quad (0, 2, 6k+1, 4k+2, 4k+1, 2k-1)_{C7}\}.
\end{aligned}$$

The set $\mathcal{C}_j$ forms a set of base blocks for a mixed hexagon system of type Cj of order $v \equiv 1 \pmod 8$, where $j \in \{1, 2, 4, 5, 7\}$. Since the Category C mixed hexagons include the converse pairs $C2$ and $C3$, $C4$ and $C6$, and M_v is self converse then these imply Cj decompositions of M_v for $j \in \{3, 6\}$ where $v \equiv 1 \pmod 8$. □

Lemma 3.7. *If $v \equiv 5 \pmod 8$, $v \geq 13$, then a Cj-decomposition of M_v exists for each $j \in \{1, 2, \ldots, 7\}$.*

Proof. Let $v = 8k + 5$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{C}_1 &= \{(0, 6k+1-2i, 2k-1, 2k-2-2i, 4k+3, 4k+1-2i)_{C1} \mid\\
&\quad i = 0, 1, \ldots, k-2\} \cup \{(0, 8k+4-2i, 1, 4k+2-2i,\\
&\quad 4k+4, 4k+5+2i)_{C1} \mid i = 0, 1, \ldots, k-1\}\\
&\quad \cup\{(0, 2k+3, 4k+4, 1, 4k+3, 2k+1)_{C1},\\
&\quad (0, 2k+2, 8k+3, 6k, 4k-1, 2k-1)_{C1}\},\\
\mathcal{C}_2 &= \{(0, 4, 8, 10, 5, 1)_{C2}, (0, 2, 10, 9, 6, 8)_{C3}, (0, 12, 6, 3, 10, 7)_{C2}\} \text{ if } k = 1,\\
\mathcal{C}_2 &= \{(0, 2k+2+2i, 7k+6+3i, k+1+i, 7k+3-i, 5k+4+i)_{C2},\\
&\quad (0, 2+2i, 4k+6+3i, 4k+5+i, 4k+2-i, 4k+4+i)_{C2} \mid\\
&\quad i = 0, 1, \ldots, k-1\}\\
&\quad \cup\{(0, 8k+4, 4k+2, 2k+1, 6k+4, 4k+3)_{C2}\} \text{ if } k \geq 2,
\end{aligned}$$

The set $\mathcal{C}_j$ forms a set of base blocks for a mixed hexagon system of type Cj of order $v \equiv 5 \pmod 8$, where $j \in \{1, 2\}$.

Let $v = 16k + 5$ where $k \geq 1$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{C}_4 &= \{(0,5,7,6,20,3)_{C4}, (0,1,13,2,20,9)_{C4}, (0,2,8,3,20,7)_{C4},\\
&\quad (0,6,20,5,18,13)_{C4}, (0,9,6,10,20,19)_{C4}\} \text{ if } k=1,\\
\mathcal{C}_4 &= \{(0,9+4i,19+8i,10+4i,16k+4,11+4i)_{C4} \mid i=0,1,\dots,k-3\}\\
&\quad \cup\{(0,5,7,6,16k+4,3)_{C4}, (0,1,9,2,16k+4,5)_{C4}\}\\
&\quad \{(0,2+4i,4k+4+8i,3+4i,16k+4,4k+3+4i)_{C4},\\
&\quad (0,4k+2+4i,16k+4+8i,4k+1+4i,16k+2,12k+1+4i)_{C4} \mid\\
&\quad i=0,1,\dots,k-1\} \cup \{(0,4k+1+4i,12k+5+8i,\\
&\quad 4k+4+4i,1,8k+3+4i)_{C4} \mid i=0,1,\dots,k-1, i\neq k/2\}\\
&\quad \cup\{(0,10k+2,4k+1,10k+5,1,10k+3)_{C5} \mid k/2\in\mathbb{N}\}\\
&\quad \cup\{(0,8k+1,8k-2,8k+2,16k+4,16k+3)_{C4}\} \text{ if } k\geq 2,\\
\mathcal{C}_5 &= \{(0,4k+1+4i,4k-2,4k+2+4i,16k+4,1+4i)_{C5},\\
&\quad (0,1+4i,8k+3,2+4i,16k+4,8k+1+4i)_{C5},\\
&\quad (0,2+4i,12k+4,3+4i,16k+4,4k+1+4i)_{C5},\\
&\quad (0,4k+2+4i,8k+6,4k+5+4i,1,12k+3+4i)_{C5} \mid\\
&\quad i=0,1,\dots,k-1\}\\
&\quad \cup\{(0,8k+1,8k+3,8k+2,16k+4,16k+1)_{C5}\}.\\
\mathcal{C}_7 &= \{(0,1+4i,16k+3,16k+2-4i,16k-1-8i,16k+1-4i)_{C7},\\
&\quad (0,2+4i,12k+4,8k+3-4i,8k-1-8i,12k+1-4i)_{C7},\\
&\quad (0,4k+1+4i,12k+3,4k+2-4i,16k+4-8i,8k+3-4i)_{C7},\\
&\quad (0,4k+2+4i,8k+6,12k+8-4i,8k+4-8i,4k+1-4i)_{C7} \mid\\
&\quad i=0,1,\dots,k-1\} \cup \{(0,8k+1,8k+3,8k+7,4,1)_{C7}\}.
\end{aligned}$$

The set $\mathcal{C}_j$ forms a set of base blocks for a mixed hexagon system of type Cj of order $v \equiv 5 \pmod{16}$, where $j \in \{4,5,7\}$.

Let $v = 16k + 13$ where $k \geq 0$. Define the following sets of blocks:

$$\begin{aligned}
\mathcal{C}_4 &= \{(0,2,1,3,12,9)_{C4}, (0,10,3,11,12,7)_{C4}, (0,5,9,8,1,3)_{C4}\} \text{ if } k=0,\\
\mathcal{C}_4 &= \{(0,9+4i,21+8i,12+4i,1,11+4i)_{C4} \mid i=0,1,\dots,k-2\}\\
&\quad \cup\{(0,2+4i,4k+10+8i,5+4i,1,4k+7+4i)_{C4},\\
&\quad (0,4k+5+4i,4+8i,4k+8+4i,1,12k+11+4i)_{C4} \mid\\
&\quad i=0,1,\dots,k-1\} \cup \{(0,4k+2+4i,12k+10+8i,\\
&\quad 4k+5+4i,1,8k+7+4i)_{C4} \mid i=0,1,\dots,k, i\neq (k+1)/2\}\\
&\quad \cup\{(0,6k+4,16k+12,6k+3,16k+10,10k+7)_{C4} \mid\\
&\quad (k+1)/2\in\mathbb{N}\} \cup \{(0,1,9,2,16k+12,5)_{C4}, (0,5,9,8,1,3)_{C4},\\
&\quad (0,8k+5,8k+4,8k+8,1,16k+11))_{C4}\} \text{ if } k\geq 1,\\
\mathcal{C}_5 &= \{(0,16k+10-4i,16k+7-8i,16k+11-4i,16k+12,\\
&\quad 1+4i)_{C5}, (0,4k+2+4i,12k+8,4k+3+4i,16k+12,\\
&\quad 8k+5+4i)_{C5} \mid i=0,1,\dots,k\} \cup \{(0,2+4i,12k+8,3+4i,
\end{aligned}$$

$$\begin{aligned}
& 16k+12, 4k+5+4i)_{C5}, (0, 4k+5+4i, 8k+7, 4k+6+4i, \\
& 16k+12, 12k+9+4i)_{C5} \mid i = 0, 1, \ldots, k-1\} \\
& \cup\{(0, 8k+5, 8k+7, 8k+6, 16k+12, 16k+9)_{C5}\}, \\
\mathcal{C}_7 \ = \ & \{(0, 1+4i, 16k+11, 16k+10-4i, 16k+7-8i, 16k+9-4i)_{C7} \mid \\
& i = 0, 1, \ldots, k\} \cup \{(0, 8k+5, 8k+7, 8k+11, 4, 1)_{C7}\} \\
& \cup\{(0, 2+4i, 12k+8, 8k+3-4i, 8k-1-8i, 12k+5-4i)_{C7}, \\
& (0, 4k+5+4i, 8k+7, 12k+11-4i, 8k+4-8i, 4k+1-4i)_{C7} \mid \\
& i = 0, 1, \ldots, k-1\} \cup \{(0, 4k+2+4i, 12k+8, 4k+3-4i, \\
& 16k+12-8i, 8k+5-4i)_{C7} \mid i = 0, 1, \ldots, k, i \neq (k+1)/2\} \\
& \cup\{(0, 6k+4, 12k+10, 2k+1, 12k+8, 6k+3)_{C7} \mid (k+1)/2 \in \mathbb{N}\}.
\end{aligned}$$

The set $\mathcal{C}_j$ forms a set of base blocks for a mixed hexagon system of type Cj of order $v \equiv 13 \pmod{16}$, where $j \in \{4, 5, 7\}$.

Since the Category C mixed hexagons include the converse pairs $C2$ and $C3$, $C4$ and $C6$, and M_v is self converse then these imply Cj decompositions of M_v for $j \in \{3, 6\}$ where $v \equiv 5 \pmod 8$, $v \geq 13$. □

We now have the following necessary and sufficient conditions for the existence of a mixed hexagon system for each of the Category A, B, and C mixed hexagons.

Theorem 3.8. *For each of the 25 partial orientations of a hexagon with four arcs and two edges, there exists a mixed hexagon system of order v if and only if $v \equiv 1 \pmod 4$, $v \geq 9$.*

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