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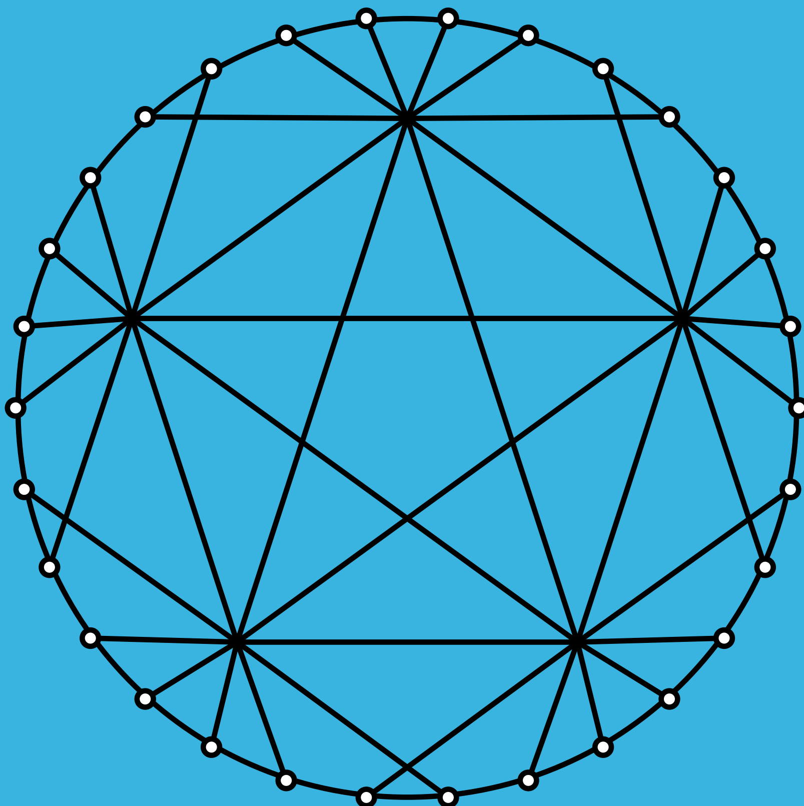

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Peg duotaire on graphs: jump versus merge

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Abstract. Numerous papers have explored the one-player game of peg solitaire on graphs. In most papers, moves are performed by jumping over adjacent pegs. In others, moves are performed by merging two pegs into a mutually adjacent hole. This paper introduces a two-player game in which players remove pegs from the graph on alternating turns. One player removes pegs with jumps and the other removes pegs with merges. When a player cannot make a valid move on their turn, the game ends.

In this paper, two options for play are explored. In the first, the last player to remove a peg wins. Which player has a winning strategy depends not only on the specific graph, but also which player goes first and whether they are using jump moves or merge moves. In the second, one player seeks to maximize the number of pegs in the final configuration, while the other tries to minimize this number. We consider both variations of this game on several infinite families of graphs such as paths, cycles, double stars, and complete bipartite graphs. In all cases, we present explicit strategies. Several open problems related to this study are also given.

1 Introduction

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (in other words, a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z. For more information on traditional peg solitaire, refer to [3, 14]. In [9], peg solitaire is expanded for play on graphs. A graph, $G = (V, E)$, is a set of vertices,

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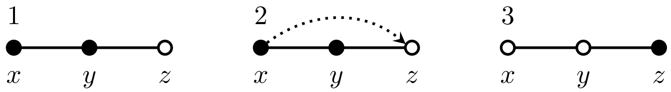

Figure 1: A typical peg solitaire jump, $x \cdot \overrightarrow{y} \cdot z$

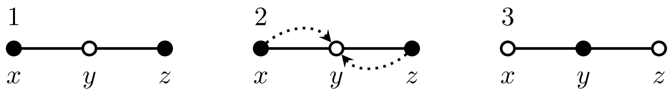

Figure 2: An example of a merge, $(x, z) \to y$

V, together with a set of edges, E. If there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z, provided that $xy, yz \in E$. Such a jump will be denoted $x \cdot \overrightarrow{y} \cdot z$. For all undefined graph theory terminology, refer to West [27].

The game begins with a hole in a single vertex. When no moves are available, the set of pegs remaining on the graph is called the *terminal set*, denoted T. A number of papers seek to minimize the cardinality of T over all starting configurations that consist of a single hole (see for example [1, 5, 8, 9, 10, 13, 17, 21, 22]).

In [19], Engbers and Weber introduce a variation in which the principal move is a *merge*. Let $x, y, z \in V(G)$ and $xy, yz \in E(G)$. If there are pegs in x and z and a hole in y, then we can *merge* the pegs in x and z into a single peg in y (see Figure 2). This move is denoted $(x, z) \to y$. For this "merge only" variant, the solvability of several families of graphs has been determined [19]. Subsequently, a variation which allowed the player to utilize both the jump move and the merge move was introduced in [11]. Other variations of peg solitaire on graphs are studied in [7, 12, 15, 16, 18].

In this paper, we consider the two-player variant, *duotaire*. For more information on traditional duotaire, refer to [20, 23]. In [6], the authors introduce peg duotaire on graphs. The first player (Player One) selects the initial hole. The players (beginning with Player Two) then alternate removing pegs from the graph using peg solitaire jumps. When no moves are available, the game ends, and the last player to complete a turn is the winner.

In this paper, we introduce and explore a new version of duotaire. In this version, we follow the same rules; however, one player (Jumper) only makes moves by jumping pegs. The other player (Merger) only makes moves by merging pegs. When one player is unable to make their respective move, the game ends. Inspired by the convention used in [2], we will assume that Jumper is male and that Merger is female. In Section 2, the last player to make a move on the graph wins. In Section 3, the players are trying to maximize or minimize the number of pegs in the terminal set. For both options of play, we give explicit strategies on the complete graph, the path, the cycle, the complete bipartite graph, and the double star. These are the same graph families studied in [6]. As we will see in Section 3, these graph families will provide examples for interesting phenomenon. In Section 4, we present open problems related to this study.

2 Winning strategies

Typically, the goal in combinatorial game theory is to determine which player has a winning strategy. By the Fundamental Theorem of Combinatorial Game Theory (see for example [4, 26]), either both players have a drawing strategy or one player has a winning strategy. Since draws are impossible in duotaire, one player has a winning strategy for each graph. In this section, we consider peg duotaire on several families of graphs. For each family, we characterize those graphs in which each player has a winning strategy. The strategies presented often depend on whether Jumper or Merger plays first. If Jumper (Merger) is Player One, then we refer to the game as a Jumper-game (Merger-game). For both the Jumper-game and the Merger-game, we present simple explicit strategies for each graph.

The following observations will be useful for numerous results.

Observation 2.1. *Because one peg is removed during each turn, Player One wins if and only if $|V| - |T|$ is odd. Player Two wins if and only if $|V| - |T|$ is even.*

Observation 2.2. *Merger can never move into a vertex of degree one. For this reason, if G has a vertex of degree one, then Jumper has a winning strategy in the Jumper-game.*

We now consider the game on multiple well-known families of graphs.

Theorem 2.3. *For the complete graph, K_n, Player One has a winning strategy if and only if $n = 1$ or n is even. Player Two has a winning strategy if and only if $n \neq 1$ and n is odd.*

Proof. If $n \leq 2$, then no further moves are possible after Player One selects the initial hole. Assume that $n \geq 3$. Up to automorphism on the vertices, all moves are forced until one peg remains. Thus $|T| = 1$. The result follows from Observation 2.1. □

The *path* on n vertices is the graph with vertex set $V = \{v_0, v_1, ..., v_{n-1}\}$ and edge set $E = \{v_i v_{i+1} : i = 0, 1, ..., n-2\}$. This graph is denoted P_n.

Theorem 2.4. *For the path, P_n, Player One has a winning strategy if and only if $n \leq 3$. Jumper has a winning strategy if and only if $n \geq 4$.*

Proof. Per Observation 2.2, Jumper always wins in the Jumper-game. For this reason, we consider the Merger-game for the remainder of the proof.

If $n \leq 3$, then Merger selects v_1 for the initial hole. In this case, no jumps are possible and Merger wins.

Assume that $n \geq 4$. Suppose Merger places the initial hole in vertex v_i. In this case, Jumper jumps $v_{i-2} \cdot \overrightarrow{v_{i-1}} \cdot v_i$ or $v_{i+2} \cdot \overrightarrow{v_{i+1}} \cdot v_i$ (whichever is possible) to end the game with no merges possible. □

The *cycle* on n vertices, where $n \geq 3$ is the graph with vertex set $V = \{v_0, v_1, ..., v_{n-1}\}$ and edge set $E = \{v_i v_{i+1} : i = 0, 1, ..., n-1\}$ (all computations on the indices are done modulo n). This graph is denoted C_n. While the cycle graph is obtained from the path by adding a single edge, we will see that its strategy is more complex than the strategy presented in the proof of Theorem 2.4.

Theorem 2.5. *For the cycle, C_n, Merger has a winning strategy if and only if $n \leq 11$ and $n \neq 5$ in the Jumper-game. Otherwise, Jumper has a winning strategy.*

Proof. We first consider the Jumper-game where $3 \leq n \leq 7$. Without loss of generality assume that Jumper selects v_0 for the initial hole and that Merger merges $(v_{n-1}, v_1) \to v_0$. If $n \in \{3, 4\}$, then this ends the game. If $n \in \{5, 6, 7\}$, then we may assume that Jumper jumps $v_3 \cdot \overrightarrow{v_2} \cdot v_1$ because of symmetry. When $n = 5$, this ends the game. Otherwise, Merger merges $(v_0, v_{n-2}) \to v_{n-1}$. If $n \in \{6, 7\}$, then the game is over.

Suppose that $n \geq 8$. Without loss of generality, we can assume that the initial hole is in v_2. The first merge $(v_1, v_3) \to v_2$ is forced. Up to automorphism on the vertices, we can assume that the first jump is $v_5 \cdot \overrightarrow{v_4} \cdot v_3$

which forces the merge $(v_0, v_2) \to v_1$. Hence for $n \geq 8$, we will begin on Jumper's first meaningful move, namely on the configuration with pegs in $\{v_1, v_3, v_6, ..., v_{n-1}\}$. Note that Jumper's first meaningful move is either $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ or $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. When $n \in \{8, 9\}$, the merge $(v_1, v_3) \to v_2$ ends the game.

Suppose $n \in \{10, 11\}$. If Jumper's first meaningful jump is $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, then Merger responds with $(v_1, v_3) \to v_2$. At this point, Jumper may either jump $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ or $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. If he jumps $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ ($v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$), then Merger responds with $(v_5, v_7) \to v_6$ ($(v_0, v_2) \to v_1$) which ends the game. Suppose instead that Jumper's first meaningful jump is $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. This forces $(v_1, v_3) \to v_2$. At this point, Jumper may either jump $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ or $v_{n-4} \cdot \overrightarrow{v_{n-3}} \cdot v_{n-2}$. In either case, the merge $(v_0, v_2) \to v_1$ ends the game.

Suppose $n \geq 12$. On Jumper's first meaningful move, he jumps $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$ which forces $(v_1, v_3) \to v_2$. Jumper then jumps $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ which forces $(v_0, v_2) \to v_1$. Finally, Jumper jumps $v_{n-4} \cdot \overrightarrow{v_{n-3}} \cdot v_{n-2}$. This ends the game with pegs in v_1, v_5, v_8,...,v_{n-5}, and v_{n-2} and holes elsewhere.

We conclude by showing that Jumper has a winning strategy in the Merger-game. We assume, without loss of generality that Merger selects v_0 for the initial hole. Jumper jumps $v_2 \cdot \overrightarrow{v_1} \cdot v_0$ to end the game with no merges possible. $\square$

We now consider the complete bipartite graph, $K_{n,m}$, with partition sets of cardinality n and m, where $n \geq 1$ and $m \geq 1$. This is the graph with vertex set $V = \{x_1, ..., x_n, y_1, ..., y_m\}$ and edge set $E = \{x_i y_j : i = 1, ..., n, j = 1, ..., m\}$. Note that when $n = 1$ or $m = 1$, the complete bipartite graph is often called a *star*. Let $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$. For purposes of exposition, we will also utilize notation introduced in [5]. For any configuration of pegs on $K_{n,m}$, let $\rho(X)$ and $\rho(Y)$ denote the number of pegs currently in the sets X and Y, respectively.

Theorem 2.6. *For the complete bipartite graph, $K_{n,m}$, if $n = 1$ or $m = 1$, then Player One has a winning strategy. If $n \geq 2$ and $m \geq 2$, then:*

(i) *In the Jumper-game, Jumper has a winning strategy if and only if $n+m$ is odd.*

(ii) *In the Merger-game, Jumper has a winning strategy if and only if n and m are both even.*

Proof. Suppose that $n = 1$. Per Observation 2.2, Jumper wins in the Jumper-game by selecting y_1 for the initial hole. Likewise, Merger wins in the Merger-game by selecting x_1 for the initial hole. In either case, no further moves are possible. An analogous argument holds for the case where $m = 1$. For the remainder of the proof, we will assume $n \geq 2$ and $m \geq 2$.

(i) Suppose that Jumper selects the initial hole.

Suppose that $n + m$ is odd. Without loss of generality, this means that n is even and m is odd. Jumper selects x_n for the initial hole. Merger is forced to merge $(y_m, y_{m-2}) \to x_n$ and Jumper is forced to jump $y_{m-1} \cdot \overrightarrow{x_n} \cdot y_{m-2}$. If $n = 2$ and $m = 3$, then the game is over and Jumper wins. Otherwise, $\rho(X)$ and $\rho(Y)$ are both odd, there are holes in both X and Y, and it is Merger's turn. Observe that any merge will change the parity of either $\rho(X)$ or $\rho(Y)$, but not both and that after the merge $\rho(X) \geq 1$ and $\rho(Y) \geq 1$. After any merge move, if $\rho(X)$ $(\rho(Y))$ is odd, then Jumper responds with $x_i \cdot \overrightarrow{y_j} \cdot x_k$ $(y_i \cdot \overrightarrow{x_j} \cdot y_k)$. Jumper's response guarantees that $\rho(X)$ and $\rho(Y)$ are both odd. Hence, Merger can never end the game. It follows that this gives Jumper's winning strategy.

Suppose that $n + m$ is even. Thus either n and m are both even or n and m are both odd. After Jumper selects the initial hole, the first merge and first jump are forced. If $n = m = 2$, then this ends the game and Merger wins. Otherwise, without loss of generality, $\rho(X)$ is even and $\rho(Y)$ is odd after the first jump. We now give Merger's strategy. After any of Jumper's turns, if $\rho(X)$ $(\rho(Y))$ is odd, then Merger responds with $(y_i, y_j) \to x_k$ $((x_i, x_j) \to y_k)$. Merger's response guarantees that $\rho(X)$ and $\rho(Y)$ are both even on Jumper's turn. Hence, Jumper can never end the game and Merger will win.

(ii) Suppose that Merger selects the initial hole.

Suppose that n and m are both even. Observe that after the placement of the initial hole and the initial jump that $\rho(X)$ and $\rho(Y)$ are both odd. This is analogous to the case in (i) where $n+m$ is odd after the first jump, hence the result follows.

Conversely, suppose that at least one of n and m is odd. Without loss of generality, assume that n is odd. Note that regardless of the initial hole and the first jump that $\rho(X)$ is even. On each of her moves, Merger merges $(x_i, x_j) \to y_k$. If $n = 3$, then this ends the game after the first merge. Otherwise, this forces Jumper to always respond with $x_p \cdot \overrightarrow{y_q} \cdot x_r$ (if available). This strategy will result in Merger removing all pegs from X, thus leaving no jumps available. Ergo, Merger has a winning strategy. □

The *double star* is the graph with vertex set $\{x, y, x_1, ..., x_n, y_1, ..., y_m\}$ and edge set $\{xy, xx_1, ..., xx_n, yy_1, ..., yy_m\}$, where $n \geq 1$ and $m \geq 1$. The double star with parameters n and m is denoted $S_{n,m}$. We denote $X =$

$\{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$. As in the proof of Theorem 2.6, for any configuration of pegs and holes we will let $\rho(X)$ and $\rho(Y)$ denote the number of pegs in X and Y, respectively.

Theorem 2.7. *For the double star, $S_{n,m}$, Jumper has a winning strategy in the Jumper-game. In the Merger-game, Jumper has a winning strategy if and only if n and m are both odd.*

Proof. Per Observation 2.2, Jumper wins in the Jumper-game by selecting any pendant for the initial hole. For the remainder of the proof, we consider the Merger-game.

Assume that n and m are both odd. If $n = m = 1$, then this graph is isomorphic to P_4 and the result is given by Theorem 2.4. For the remainder of the proof, we assume that $n \geq 3$.

If the initial hole is in a pendant, then Jumper jumps into the pendant, which results in an odd number of pegs in both X and Y and holes in x and y after his turn. If the initial hole is in x, then Jumper is forced to jump $y_m \cdot \overrightarrow{y} \cdot x$. If $m = 1$, then this ends the game. Otherwise, if $m \geq 3$, then Merger must then make one of two merges, $(x, y_{m-1}) \to y$ or $(y_{m-1}, y_{m-2}) \to y$. If Merger chooses $(x, y_{m-1}) \to y$ and $m = 3$, then Jumper responds with $y_{m-2} \cdot \overrightarrow{y} \cdot x$ to end the game. If Merger chooses $(x, y_{m-1}) \to y$ and $m \geq 5$, then Jumper responds with $y_{m-2} \cdot \overrightarrow{y} \cdot y_m$. If Merger chooses $(y_{m-1}, y_{m-2}) \to y$, then Jumper responds with $x \cdot \overrightarrow{y} \cdot y_{m-2}$. The case of when the initial hole is in y is analogous. In all cases, Jumper's response results in a configuration where $\rho(X)$ and $\rho(Y)$ are both odd and there are holes in x and y after his turn.

Once this is achieved, Jumper responds to $(x_i, x_j) \to x$ ($(y_i, y_j) \to y$) with $x_k \cdot \overrightarrow{x} \cdot x_i$ ($y_k \cdot \overrightarrow{y} \cdot y_i$). This strategy ensures that $\rho(X)$ is odd, $\rho(Y)$ is odd, and there are holes in x and y after each of Jumper's turns. Further, either $\rho(X)$ or $\rho(Y)$ will decrease by two between Jumper's turns. Thus Merger will eventually have a configuration with $\rho(X) = \rho(Y) = 1$ and holes in x and y. As there are no merges available, Jumper wins.

Suppose that n or m is even. Without loss of generality, assume that n is even. If $n = 2$, then Merger selects y for the initial hole. This forces Jumper to jump $x_1 \cdot \overrightarrow{x} \cdot y$, and Merger merges $(x_2, y) \to x$ to end the game.

If $n \geq 4$, then Merger selects x_1 for the initial hole. This forces $y \cdot \overrightarrow{x} \cdot x_1$ and Merger responds with $(x_1, x_2) \to x$. From this point on, Merger responds to $x_i \cdot \overrightarrow{x} \cdot x_j$ ($x_i \cdot \overrightarrow{x} \cdot y$) with $(x_j, x_k) \to x$ ($(x_k, y) \to x$). This strategy guarantees that $\rho(X)$ is even, x has a peg, and y has a hole after each of Merger's turns. Further, $\rho(X)$ will decrease by two between Merger's turns. Thus Jumper will eventually have a configuration with $\rho(X) = 0$, $\rho(Y) = m$, a peg in x, and a hole in y. As there are no jumps available, Merger wins. $\square$

3 Competitive graph parameters

In the game considered in Section 2, there is a clear way of determining the winner. However, when we consider competitive graph parameters, there is no clearly defined win condition. *Competitive graph parameters* were introduced by Phillips and Slater in [24]. These games are played between the *maximizer* and the *minimizer*. Typically, players alternate adding elements to a common set S so long as the resulting set has some predetermined property. When both players make optimal moves, the cardinality of the resulting set is fixed. This cardinality gives a competitive graph parameter.

Motivated by the above comment, we consider a variation of Jumper versus Merger duotaire in which the players are only concerned with the cardinality of the terminal set. This is in contrast to the variation in Section 2, where the players are only concerned with the parity of the terminal set. In this variation explored in this section, the *maximizer* seeks to make the terminal set as large as possible while the *minimizer* tries to make it as small as possible.

When Jumper plays first on graph G and adopts the role of the maximizer (minimizer), we denote the resulting competitive graph parameter $J^+(G)$ ($J^-(G)$). Likewise, when Merger plays first and adopts the role of the maximizer (minimizer), we denote the resulting graph parameter $M^+(G)$ ($M^-(G)$). We determine these competitive graph parameters for several classes of graphs.

We begin this section with a few simple propositions.

Proposition 3.1. *If G has a vertex of degree one, then $J^+(G) = |V(G)|-1$.*

Proof. Suppose that Jumper plays first and adopts the role of the maximizer. By selecting a vertex of degree one for the initial hole, there are no merges available. Hence $J^+(G) = |V(G)| - 1$. $\square$

Since any move by either player on the complete graph will result in a move by the other player, the following proposition follows, regardless of strategy.

Proposition 3.2. *For K_n, the complete graph on n vertices, where $n \geq 2$:*

$$J^+(K_n) = J^-(K_n) = M^+(K_n) = M^-(K_n) = 1.$$

n	2	3	4	5	6	7	8	9	10	11	12	13
$J^-(P_n)$	1	1	2	2	2	3	2	3	2	3	2	3

Table 1: Values of $J^-(P_n)$ for small n

We now consider the game on multiple families of graphs. We begin with the path and the cycle.

Theorem 3.3. *For P_n, the path on n vertices:*

(i) $J^+(P_n) = n - 1$*;*

(ii) *For $n \leq 13$, $J^-(P_n)$ is given in Table 1. For $n \geq 14$, $J^-(P_n) = n - 11$.*

(iii) *For $n \leq 3$, $M^+(P_n) = n - 1$. For $n \geq 4$, $M^+(P_n) = n - 2$.*

(iv) $M^-(P_2) = 1$ *and* $M^-(P_n) = n - 2$ *for* $n \geq 3$.

Proof.

(i) This follows from Proposition 3.1.

(ii) Clearly, $J^-(P_2) = 1$. For $n \in \{3, 4, 5\}$, Jumper selects v_1 as the initial hole, which forces $(v_0, v_2) \to v_1$. If $n = 3$ or $n = 4$, then there are no jumps available and the result follows. If $n = 5$, then Jumper's additional move $v_4 \cdot \overrightarrow{v_3} \cdot v_2$ ends the game with pegs in v_1 and v_2.

For $n \in \{6, 7\}$, Jumper begins with the initial hole in v_2, which forces $(v_1, v_3) \to v_2$. Jumper responds with $v_5 \cdot \overrightarrow{v_4} \cdot v_3$, which forces $(v_0, v_2) \to v_1$. This ends the game with two pegs if $n = 6$ or three pegs if $n = 7$.

Let $n \in \{8, 9, 10, 11, 12, 13\}$. Jumper begins with the initial hole in v_4, which forces $(v_3, v_5) \to v_4$. Jumper responds with $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, which forces $(v_2, v_4) \to v_3$. Jumper then jumps $v_0 \cdot \overrightarrow{v_1} \cdot v_2$, which forces $(v_3, v_5) \to v_4$. This ends the game with two pegs if $n = 8$ and three pegs if $n = 9$.

For $n \in \{10, 11\}$, Jumper follows the above strategy until there are pegs in v_2, v_4, v_8, v_9, and (in the case of $n = 11$) v_{10} and holes elsewhere. At this point, he jumps $v_9 \cdot \overrightarrow{v_8} \cdot v_7$, which forces $(v_2, v_4) \to v_3$. For $n = 10$ ($n = 11$), this ends the game with two (three) pegs.

For $n \in \{12, 13\}$, Jumper follows the strategy for $n \in \{10, 11\}$ until there are pegs in v_3, v_7, v_{10}, v_{11}, and (in the case of $n = 13$) v_{12} and holes elsewhere. At this point, he jumps $v_{11} \cdot \overrightarrow{v_{10}} \cdot v_9$, which forces a final merge

$(v_7, v_9) \to v_8$. When $n = 12$ ($n = 13$), this ends the game with two (three) pegs.

For $n \leq 13$, Merger's moves were forced. Hence, she cannot do better. It is also straightforward to check that the Jumper can do no better.

For $n \geq 14$, we instead consider the infinite path P_∞ with vertex set $V(P_\infty) = \{u_k : k \in \mathbb{Z}\}$ and edge set $E(P_\infty) = \{u_k u_{k+1} : k \in \mathbb{Z}\}$. Suppose that Jumper plays first on this graph. Without loss of generality, the initial hole is in u_0, which forces the first merge $(u_{-1}, u_1) \to u_0$. Without loss of generality, the first jump is $u_{-3} \cdot \overrightarrow{u_{-2}} \cdot u_{-1}$, which forces the second merge $(u_0, u_2) \to u_1$. Note that at this point, up to automorphism on the vertices, neither player has had a meaningful choice. Thus, Jumper's first meaningful move will occur on a board in which there are holes in u_{-3}, u_{-2}, u_0, and u_2 and pegs elsewhere.

Case 1: Suppose that the first meaningful jump is $u_4 \cdot \overrightarrow{u_3} \cdot u_2$. This forces

$$(u_{-1}, u_1) \to u_0.$$

Jumper may either jump $u_{-5} \cdot \overrightarrow{u_{-4}} \cdot u_{-3}$ or $u_6 \cdot \overrightarrow{u_5} \cdot u_4$.

Case 1a: Suppose that Jumper jumps $u_{-5} \cdot \overrightarrow{u_{-4}} \cdot u_{-3}$. This forces

$$(u_0, u_2) \to u_1.$$

If Jumper jumps $u_6 \cdot \overrightarrow{u_5} \cdot u_4$, then the game ends with nine pegs removed. Suppose instead that Jumper jumps $u_{-7} \cdot \overrightarrow{u_{-6}} \cdot u_{-5}$. This forces

$$(u_{-5}, u_{-3}) \to u_{-4}.$$

Now either $u_6 \cdot \overrightarrow{u_5} \cdot u_4$ or $u_{-9} \cdot \overrightarrow{u_{-8}} \cdot u_{-7}$ ends the game with eleven pegs removed. Since all merges were forced, this shows that $J^-(P_\infty) \leq n - 11$. To show equality, we need only show that the minimizer, Jumper, cannot do better by making different choices.

Case 1b: Suppose instead that Jumper jumps $u_6 \cdot \overrightarrow{u_5} \cdot u_4$. Merger then makes the merge $(u_2, u_4) \to u_3$. Jumper can either jump $u_{-5} \cdot \overrightarrow{u_{-4}} \cdot u_{-3}$ or $u_8 \cdot \overrightarrow{u_7} \cdot u_6$. Either of these moves ends the game with only nine pegs removed. Hence the strategy given in *Case 1a* is better for Jumper.

Case 2: Suppose that Jumper's first meaningful jump is $u_{-5} \cdot \overrightarrow{u_{-4}} \cdot u_{-3}$. Merger responds with $(u_{-1}, u_1) \to u_0$. Without loss of generality, Jumper jumps $u_{-7} \cdot \overrightarrow{u_{-6}} \cdot u_{-5}$, which forces $(u_{-5}, u_{-3}) \to u_{-4}$. If Jumper responds with $u_{-9} \cdot \overrightarrow{u_{-8}} \cdot u_{-7}$, then the game ends with nine pegs removed. If instead Jumper responds with $u_4 \cdot \overrightarrow{u_3} \cdot u_2$, then $(u_0, u_2) \to u_1$ is forced. Jumper may either jump $u_{-9} \cdot \overrightarrow{u_{-8}} \cdot u_{-7}$ or $u_6 \cdot \overrightarrow{u_5} \cdot u_4$. Either of these moves ends the game with eleven pegs removed. Since Merger has a strategy that

n	3	4	5	6	7	8	9	10	11	12	13
$J^+(C_n)$	1	2	2	2	3	2	3	2	3	3	4
$J^-(C_n)$	1	2	2	2	3	2	3	2	3	2	3

Table 2: Values of $J^+(C_n)$ and $J^-(C_n)$ for small n

ensures that Jumper can do no better when his first meaningful move is $u_{-5}\cdot \overrightarrow{u_{-4}} \cdot u_{-3}$, there is no advantage for him to choose this jump.

To see how the strategy on the infinite path P_∞ translates to the finite path P_n, where $n \geq 14$, simply apply the mapping $u_k \to v_{k+7}$ for $k \geq -7$ and ignore any vertices on P_∞ with index less than -7 or greater than $n-8$. Hence, it follows that $J^-(P_n) = n-11$ for $n \geq 14$.

(iii) For $n \leq 3$, Merger selects v_1 for the initial hole. As this leaves no jumps available, the result follows. For $n \geq 4$, Merger selects v_0 as the initial hole. Jumper is forced to jump $v_2\cdot \overrightarrow{v_1} \cdot v_0$. This leaves holes in v_1 and v_2 and pegs elsewhere. Note that the only jump was forced and any initial hole will allow for a jump. Hence, neither player can do better, and the result follows.

(iv) It is obvious that $M^-(P_2) = 1$. If $n = 3$, then Merger selects v_0 for the initial hole and the jump $v_2\cdot \overrightarrow{v_1} \cdot v_0$ is forced. As only one peg remains and the only jump was forced, neither player can do better. Assume that $n \geq 4$ and that Merger selects v_i for the initial hole. At least one of the jumps $v_{i-2}\cdot \overrightarrow{v_{i-1}} \cdot v_i$ or $v_{i+2}\cdot \overrightarrow{v_{i+1}} \cdot v_i$ is available and therefore, forced. In either case, there are no merges available and the result follows. □

Theorem 3.4. *For C_n, the cycle on n vertices:*

(i) *For $n \leq 11$, $J^+(C_n)$ is given in Table 2. For $n \geq 12$, $J^+(C_n) = n-9$.*

(ii) *For $n \leq 12$, $J^-(C_n)$ is given in Table 2. For $n \geq 13$, $J^-(C_n) = n-11$.*

(iii) *$M^+(C_n) = M^-(C_n) = n-2$.*

Proof. Suppose that Jumper plays first. Let $n \leq 5$. Without loss of generality, Jumper places the initial hole in v_1 and the first merge is $(v_0, v_2) \to v_1$. When $n = 3$ or $n = 4$, this results in no further jumps. Thus $J^+(C_3) = J^-(C_3) = 1$ and $J^+(C_4) = J^-(C_4) = 2$. When $n = 5$, there is an additional jump, say $v_4\cdot \overrightarrow{v_3} \cdot v_2$, which ends the game with $J^+(C_5) = J^-(C_5) = 2$.

For $n \geq 6$, up to automorphisms on the vertices, the first four turns are forced. So without loss of generality, we assume that Jumper begins with the initial hole in v_2, Merger merges $(v_1, v_3) \to v_2$, Jumper jumps $v_5\cdot \overrightarrow{v_4}$

$\cdot v_3$, and Merger merges $(v_0, v_2) \to v_1$. This results in holes in v_0, v_2, v_4, and v_5 and pegs in v_1, v_3, v_6,...,v_{n-1}. Observe that for $n = 6$, this ends the game with pegs in v_1 and v_3. Thus $J^+(C_6) = J^-(C_6) = 2$. Likewise, for $n = 7$, this ends the game with pegs in v_1, v_3, and v_6. Thus, $J^+(C_7) = J^-(C_7) = 3$.

As in the proof of Theorem 2.5, for $n \geq 8$, we begin on Jumper's first meaningful turn on the configuration where there are holes in v_0, v_2, v_4, and v_5 and pegs in v_1, v_3, v_6,...,v_{n-1}. For $n \in \{8, 9\}$, Jumper has two possible moves on his first meaningful turn. If he jumps $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, then Merger must either merge $(v_1, v_3) \to v_2$, $(v_3, v_5) \to v_4$ or, if $n = 9$, $(v_8, v_1) \to v_0$. If he instead jumps $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$, then Merger must merge $(v_1, v_3) \to v_2$. Any of these possibilities end the game. Thus $J^+(C_8) = J^-(C_8) = 2$ and $J^+(C_9) = J^-(C_9) = 3$.

For $n = 10$, Jumper's first meaningful move is either $v_8 \cdot \overrightarrow{v_9} \cdot v_0$ or $v_7 \cdot \overrightarrow{v_6} \cdot v_5$. If he jumps $v_8 \cdot \overrightarrow{v_9} \cdot v_0$, then $(v_1, v_3) \to v_2$ is forced. At this point, Jumper can either jump $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, which forces $(v_0, v_2) \to v_1$ or $v_6 \cdot \overrightarrow{v_7} \cdot v_8$, which forces either $(v_0, v_2) \to v_1$ or $(v_0, v_8) \to v_9$. In either case, this ends the game with two remaining pegs. We will show that $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ does no better when either player acts as the minimizer. Merger has three possible responses to this move: $(v_1, v_9) \to v_0$, $(v_1, v_3) \to v_2$, or $(v_3, v_5) \to v_4$. The merge $(v_1, v_9) \to v_0$ ends the game with four remaining pegs. This is clearly not optimal for Jumper when he is the minimizer. For this reason, he will not choose $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ for his first meaningful move when he is the minimizer. Thus, $J^-(C_{10}) = 2$. Suppose that Jumper is the maximizer and his first meaningful move is $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, followed by $(v_1, v_3) \to v_2$. This merge can be followed with either $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ or $v_8 \cdot \overrightarrow{v_9} \cdot v_0$. Either of these moves will set up a final merge, yielding a terminal set with two pegs. Suppose instead that Merger responds to Jumper's first meaningful move with $(v_3, v_5) \to v_4$. This can be followed with either $v_8 \cdot \overrightarrow{v_9} \cdot v_0$ or $v_9 \cdot \overrightarrow{v_8} \cdot v_7$, either of which ends the game with three pegs remaining. As either player can make choices which yield a terminal state with two pegs when they act as the minimizer, $J^+(C_{10}) = J^-(C_{10}) = 2$.

Suppose that $n = 11$. On his first meaningful turn, suppose that Jumper jumps $v_9 \cdot \overrightarrow{v_{10}} \cdot v_0$, which forces $(v_1, v_3) \to v_2$. The jump $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ forces $(v_0, v_2) \to v_1$. This ends the game with pegs in v_1, v_5, and v_8. Since all merges were forced, $J^+(C_{11}) \geq 3$ and $J^-(C_{11}) \leq 3$. If instead Jumper's first meaningful jump is $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, then Merger has three possible moves: $(v_1, v_{10}) \to v_0$, $(v_1, v_3) \to v_2$, or $(v_3, v_5) \to v_4$. If Merger merges $(v_1, v_{10}) \to v_0$, then Jumper can either jump $v_8 \cdot \overrightarrow{v_9} \cdot v_{10}$ or $v_9 \cdot \overrightarrow{v_8} \cdot v_7$. The jump $v_8 \cdot \overrightarrow{v_9} \cdot v_{10}$ forces $(v_3, v_5) \to v_4$ and then either $v_{10} \cdot \overrightarrow{v_0} \cdot v_1$ or $v_0 \cdot \overrightarrow{v_{10}} \cdot v_9$. As this results in a terminal set with two pegs, this is not optimal when either players acts as the maximizer. If instead Jumper jumps $v_9 \cdot \overrightarrow{v_8} \cdot v_7$, then there is one additional merge, which ends the game with three pegs. If instead Merger responds to Jumper's first meaningful move with $(v_1, v_3) \to v_2$, then Jumper can either jump $v_9 \cdot \overrightarrow{v_{10}} \cdot v_0$ or $v_9 \cdot \overrightarrow{v_8} \cdot v_7$. Either of these forces another merge, resulting in a terminal set with three pegs. If Merger responds to Jumper's first meaningful move with $(v_3, v_5) \to v_4$, then either $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ forces $(v_1, v_{10}) \to v_0$, which ends the game with three pegs, or $v_9 \cdot \overrightarrow{v_{10}} \cdot v_0$ ends the game with four pegs. As either player can make choices that result in a terminal set with three pegs, $J^+(C_{11}) = J^-(C_{11}) = 3$.

(i) Suppose that $n \geq 12$ and that Jumper acts as the maximizer. Suppose that on his first meaningful turn he jumps $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. This forces $(v_1, v_3) \to v_2$. Jumper jumps $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, which forces $(v_0, v_2) \to v_1$. The jump $v_{n-4} \cdot \overrightarrow{v_{n-3}} \cdot v_{n-2}$ ends the game with $n-9$ pegs remaining. As all merges were forced, Merger can do no better. Thus, $J^+(C_n) \geq n-9$ for $n \geq 12$. Observe that if Jumper replaces $v_{n-4} \cdot \overrightarrow{v_{n-3}} \cdot v_{n-2}$ with $v_9 \cdot \overrightarrow{v_8} \cdot v_7$, then this sets up at least one more move. Hence it is not optimal for Jumper. Note that if Jumper's second meaningful move is $v_{n-4} \cdot \overrightarrow{v_{n-3}} \cdot v_{n-2}$, then Merger can respond with $(v_0, v_2) \to v_1$. The jump $v_{n-6} \cdot \overrightarrow{v_{n-5}} \cdot v_{n-4}$ sets up at least one more move, hence it is not optimal for Jumper. Whereas $v_7 \cdot \overrightarrow{v_6} \cdot v_5$ ends the game with $n-9$ pegs remaining.

If instead Jumper's first meaningful move is $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, then Merger can respond with one of $(v_1, v_{n-1}) \to v_0$, $(v_1, v_3) \to v_2$, or $(v_3, v_5) \to v_4$. If Merger merges $(v_1, v_{n-1}) \to v_0$, then Jumper can either jump $v_{n-3} \cdot \overrightarrow{v_{n-2}} \cdot v_{n-1}$ or $v_9 \cdot \overrightarrow{v_8} \cdot v_7$, either of which allows at least two more moves, resulting in a terminal set with at most $n-9$ pegs. If Merger instead merges $(v_1, v_3) \to v_2$, then Jumper must respond with either $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$ or $v_9 \cdot \overrightarrow{v_8} \cdot v_7$. The jump $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$ forces $(v_0, v_2) \to v_1$ and forces another jump. Note that $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ sets up an additional merge. Whereas $v_{n-4} \cdot \overrightarrow{v_{n-3}} \cdot v_{n-2}$ ends the game with $n-9$ pegs remaining. Finally, suppose that Merger merges $(v_3, v_5) \to v_4$. Jumper can respond with either $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ or $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. The jump $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ will set up at least two additional moves, resulting in a terminal set with at most $n-9$ pegs. Whereas $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$ ends the game with pegs in v_0, v_1, v_4, v_8,...,v_{n-3}. As this results in a terminal set with $n-7$ pegs, Merger's choice of $(v_3, v_5) \to v_4$ was not her optimal move. As all possibilities have been examined, we have that $J^+(C_n) = n-9$ for $n \geq 12$.

(ii) Suppose that $n \geq 12$ and that Jumper acts as the minimizer. Suppose that on his first meaningful turn he jumps $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. This forces $(v_1, v_3) \to v_2$. Jumper jumps $v_7 \cdot \overrightarrow{v_6} \cdot v_5$, which forces $(v_0, v_2) \to v_1$. The jump $v_9 \cdot \overrightarrow{v_8} \cdot v_7$ forces an additional merge $(v_5, v_7) \to v_6$. For $n = 12$

($n = 13$), this ends the game with two (three) pegs remaining. As all merges were forced, $J^-(C_{12}) \leq 2$ and $J^-(C_{13}) \leq 3$. In order for Jumper as the minimizer to do better when $n = 12$, the final merge would have to result in two adjacent pegs. As this is impossible on the cycle, $J^-(C_{12}) = 2$. A similar argument shows that $J^-(C_{13}) = 3$.

For $n \geq 14$, we apply the same argument from the proof of Theorem 3.3 that shows $J^-(P_n) = n - 11$ for $n \geq 14$. Hence, $J^-(C_n) = n - 11$ for $n \geq 14$.

(iii) Without loss of generality, Merger selects v_0 for the initial hole. Jumper either jumps $v_2 \cdot \overrightarrow{v_1} \cdot v_0$ or $v_{n-2} \cdot \overrightarrow{v_{n-1}} \cdot v_0$. In either case, there are no merges available. Hence the result follows. □

We now determine the competitive graph parameters for the complete bipartite graph, $K_{n,m}$. As in the proof of Theorem 2.6, we will use $\rho(X)$ and $\rho(Y)$ denote the number of pegs currently in the partition sets X and Y, respectively.

Theorem 3.5. *For $K_{n,m}$, the complete bipartite graph:*

(i) $J^+(K_{1,m}) = M^+(K_{1,m}) = m$ *and* $J^-(K_{1,m}) = M^-(K_{1,m}) = 1$. $J^+(K_{n,1}) = M^+(K_{n,1}) = n$ *and* $J^-(K_{n,1}) = M^-(K_{n,1}) = 1$.

(ii) $J^+(K_{2,m}) = m$ *and* $J^+(K_{n,2}) = n$. *If* $n \geq 3$, $m \geq 3$, *and* $n + m$ *is even, then* $J^+(K_{n,m}) = 2$. *If* $n \geq 3$, $m \geq 3$, *and* $n + m$ *is odd, then* $J^+(K_{n,m}) = 3$.

(iii) $J^-(K_{2,m}) = J^-(K_{n,2}) = 2$. *If* $n \geq 3$, $m \geq 3$, *and* $n + m$ *is odd, then* $J^-(K_{n,m}) = 1$. *If* $n \geq 3$, $m \geq 3$, *and* n *and* m *are both odd, then* $J^-(K_{n,m}) = 2$. *If* $n \geq 4$, $m \geq 4$, *and* n *and* m *are both even, then* $J^-(K_{n,m}) = \min\{n, m\}$.

(iv) $M^+(K_{2,m}) = M^+(K_{n,2}) = 2$. *If* n *is odd,* m *is even, and* $m \geq 4$, *then* $M^+(K_{n,m}) = m$. *If* n *is even,* m *is odd, and* $n \geq 4$, *then* $M^+(K_{n,m}) = n$. *If* $n \geq 3$, $m \geq 3$, *and* n *and* m *are both odd, then* $M^+(K_{n,m}) = \max\{n, m\}$. *If* $n \geq 4$, $m \geq 4$, *and* n *and* m *are both even, then* $M^+(K_{n,m}) = 1$.

(v) *If* $n+m$ *is odd, then* $M^-(K_{n,m}) = 2$. *If* n *is even and* $m \in \{2, 4\}$, *then* $M^-(K_{n,m}) = 2$. *If* m *is even and* $n \in \{2, 4\}$, *then* $M^-(K_{n,m}) = 2$. *If* $n+m$ *is even,* $n \geq 5$, *and* $m \geq 5$ *or* $n = m = 3$, *then* $M^-(K_{n,m}) = 3$.

Proof.

(i) Per Proposition 3.1, $J^+(K_{1,m}) = 1$. Similarly, if Merger plays first, then she selects the center vertex x_1 as the initial hole. As this leaves no jumps available, $M^+(K_{1,m}) = m$.

The other results are similar to the "star purge" utilized in [11]. Suppose that Jumper plays first. He selects the center vertex x_1 for the initial hole. On her ith move, where $i = 1, ..., \lfloor m/2 \rfloor$, Merger is forced to merge $(y_1, y_{2i}) \to x_1$. Jumper responds with $y_{2i+1} \cdot \overrightarrow{x_1} \cdot y_1$, if this jump is available. If m is even, then this results in a single peg in x_1. If m is odd, then this results in a single peg in y_1. In either case, $J^-(K_{1,m}) = 1$.

Suppose that Merger plays first. She selects y_1 for the initial hole. On his ith move, where $i = 1, ..., \lfloor m/2 \rfloor$, Jumper is forced to jump $y_{2i} \cdot \overrightarrow{x_1} \cdot y_1$. Merger responds with $(y_1, y_{2i+1}) \to x_1$, if this merge is available. If m is even, then this results in a single peg in y_1. If m is odd, then this results in a single peg in x_1. In either case, $M^-(K_{1,m}) = 1$.

Similar arguments hold for $K_{n,1}$. For the remainder of the proof, assume that $n \geq 2$ and $m \geq 2$.

(ii) Assume that Jumper chooses the initial hole and that he is the maximizer.

Suppose that $n = 2$. Jumper selects y_m for the initial hole. This forces $(x_1, x_2) \to y_m$, which leaves no jumps available. The only merge was forced, so Merger cannot do better. Regardless of the placement of the initial hole, there will be at least one merge. So Jumper cannot do better. Thus $J^+(K_{2,m}) = m$. A similar argument holds when $m = 2$.

Suppose that $n \geq 3$, $m \geq 3$, and $n + m$ is even.

We begin by examining a configuration in which $\rho(X) = 2$, $\rho(Y) = 3$, and it is Merger's turn. For purposes of exposition, this will be referred to as a (3,3)-*configuration.* Note that the merge $(y_i, y_j) \to x_k$ forces $y_\ell \cdot \overrightarrow{x_k} \cdot y_i$. The merge $(x_i, x_j) \to y_\ell$ ends the game with two pegs. It is easy to check that neither player can do better once the (3,3)-configuration has been reached. Thus, $J^+(K_{3,3}) = 2$.

We now consider the graph $K_{3,m}$, where m is odd and $m \geq 5$. If the initial hole is in X, then the merge $(y_i, y_j) \to x_i$ is forced. Jumper responds with $y_k \cdot \overrightarrow{x_i} \cdot y_j$. Note that if $\rho(X) = 2$ and Merger merges the two pegs in X, then the game ends with $\rho(Y)$ pegs remaining. As she is the minimizer, this is not optimal for her. So she will merge $(y_i, y_j) \to x_k$ and Jumper will respond with $y_\ell \cdot \overrightarrow{x_k} \cdot y_i$ until the (3,3)-configuration is reached. Since all of Merger's moves were forced, $J^+(K_{3,m}) \geq 2$, where m is odd and $m \geq 5$. Note that when the initial hole is in X, all of Jumper's moves were also forced. Thus we consider the possibility that he places the initial hole in Y. This forces the next three moves, namely $(x_i, x_j) \to y_k$, $x_\ell \cdot \overrightarrow{y_k} \cdot x_i$, and $(y_i, y_j) \to x_\ell$. We will refer to configuration in which $\rho(X) = 2$, $\rho(Y) = p$, and Jumper to play as the $(2, p)$-*configuration.* Once this configuration has been reached, if Jumper continues to jump over pegs in Y, then we will eventually reach the (3,3)-configuration. If Jumper instead jumps over a peg in X, then he forces the merge $(y_i, y_j) \to x_k$, resulting in a $(2, p-2)$-configuration. Therefore, Merger will eventually play on a $(3,3)$-configuration, or Jumper will eventually play on a $(2,2)$-configuration. We have already examined the (3,3)-configuration. In the case of the $(2,2)$-configuration, all moves are forced (up to automorphism on the vertices) and we are again left with two pegs. Hence, neither player can do better. Thus, $J^+(K_{3,m}) = 2$ when m is odd.

We now consider $K_{n,m}$, where n is odd, m is odd, and $m \geq n \geq 5$. Again, Jumper begins with the hole in X, forcing $(y_i, y_j) \to x_k$ and $y_\ell \cdot \overrightarrow{x_k} \cdot y_i$. Notice that $\rho(X)$ is even and $\rho(Y)$ is odd and there are holes in both X and Y. If Merger merges $(x_i, x_j) \to y_k$ on any of her turns, then either we have the $(2,p)$-configuration or Jumper responds with $x_k \cdot \overrightarrow{y_i} \cdot x_j$. If Merger merges $(y_i, y_j) \to x_k$, then Jumper responds with $y_\ell \cdot \overrightarrow{x_k} \cdot y_i$. This again results in a configuration in which $\rho(X)$ is even and $\rho(Y)$ is odd and there are holes in both X and Y. Note that if Jumper instead responds by jumping over a peg in Y, then we get an analogous configuration in which $\rho(Y)$ is even and $\rho(X)$ is odd. Thus eventually the $(2,p)$-configuration will be reached and will result in two pegs remaining. The case where the initial hole is in Y follows similarly. Thus, $J^+(K_{n,m}) = 2$ when n and m are both odd.

Consider $K_{n,m}$, where n and m are both even. After the placement of the initial hole exactly one of $\rho(X)$ and $\rho(Y)$ is even. Thus this reduces to the case where n and m are both odd after the first merge and the first jump. Thus, $J^+(K_{n,m}) = 2$ when n and m are both even.

Suppose that $n \geq 3$, $m \geq 3$, and $n+m$ is odd. Since $n+m$ is odd, we can assume without loss of generality that n is even and m is odd. Suppose that the initial hole is in Y. As long as $\rho(X) \geq 3$ on her turn, Merger merges $(x_i, x_j) \to y_k$. Until $\rho(X) \leq 2$, this forces $x_p \cdot \overrightarrow{y_q} \cdot x_r$. Thus after each player has made $(n-2)/2$ moves, there are pegs in x_1, x_2, y_1,...,y_{m-1} and holes elsewhere. If $\rho(X) \leq 2$ and $\rho(Y) \geq 3$ on her turn, then Merger instead merges $(y_i, y_j) \to x_k$. Since n is even and m is odd, this strategy will eventually result in Merger playing on a graph with (without loss of generality) pegs in x_1, x_2, y_1, and y_2 and holes elsewhere (i.e., a *(2,2)-configuration*) or a board with pegs in x_1, x_2, x_3, and y_1 and holes elsewhere (i.e., a *(3,1)-configuration*). On the (2,2)-configuration, a merge will end the game with three pegs. On the (3,1)-configuration, it is easy to see that the best either player can do is two pegs. We now show that Merger cannot force the (3,1)-configuration. Since n is even and m is odd, then regardless of strategy, on each of Merger's turns, $\rho(X)$ and $\rho(Y)$ are either both even or both odd. Likewise, on each of Jumper's turns exactly one of $\rho(X)$ and $\rho(Y)$ is even. Thus, once each side has a hole, he can choose which side he wants to jump over. Hence, at some point Jumper can jump over the odd side of the partition so that Merger is playing on a board in which $\rho(X)$ and $\rho(Y)$ are both even. Indeed, his selection of the initial hole in the above strategy guarantees this from the beginning. To complete the proof, notice that once the initial hole is selected, all of Jumper's moves were forced until there are pegs in x_1, x_2, y_1,...,y_{m-1} and holes elsewhere. After that, Merger adopts a strategy that will ensure that either $\rho(X) \geq 2$ and $\rho(Y) \geq 2$ or (without loss of generality) $\rho(X) = 1$ and $\rho(Y) \geq 3$ on each of Jumper's turns. Both of these scenarios were examined and resulted in the same endgame. Further, if Jumper places the initial hole in X, then Merger adopts an analogous strategy. However, due to the parity of the sets her final merge occurs on a graph with pegs in x_1, x_2, x_3, and y_1 and holes elsewhere. As we saw above, this is not better for Jumper who is playing as the maximizer. Hence, neither player can do better, and $J^+(K_{n,m}) = 3$ when $n \geq 3$, $m \geq 3$, and $n + m$ is odd.

(iii) Assume that Jumper selects the initial hole and that he is the minimizer.

If $n = 2$, then Jumper places the initial hole in X. On each of Jumper's turns, he jumps $y_i\cdot \overrightarrow{x_j} \cdot y_k$. This strategy forces $(y_p, y_q) \to x_r$ on each of Merger's turns. If m is even, then this will result in pegs in x_1 and x_2 and holes elsewhere after Merger's last turn. If m is odd, then we have an additional peg in y_1, which forces an additional jump. In either case, two pegs remain. Since all merges were forced, $J^-(K_{2,m}) \leq 2$. After the selection of the initial hole, all moves by both players were forced. Hence neither player can do better if the initial hole is in X. If instead the initial hole is in y_m, then Merger ends the game with $(x_1, x_2) \to y_m$. This results in m pegs on the graph at the end of the game. As Jumper is the minimizer, this choice of initial hole is not optimal for him. Ergo, $J^-(K_{2,m}) = 2$. A similar argument holds for $K_{n,2}$.

Suppose that $n \geq 3$, $m \geq 3$, and $n + m$ is odd. Since $n + m$ is odd, we can assume without loss of generality that n is even and m is odd. Jumper begins with the initial hole in X. After the first merge and the first jump, there is a hole in X, a hole in Y, and $\rho(X)$ and $\rho(Y)$ are both odd. If Merger merges $(x_i, x_j) \to y_k$ ($(y_i, y_j) \to x_k$), then Jumper responds with $x_p\cdot \overrightarrow{y_k} \cdot x_i$ ($y_p\cdot \overrightarrow{x_k} \cdot y_i$). Jumper maintains this strategy until he is playing on a graph with pegs in x_1, y_1, and y_2 (without loss of generality) and holes elsewhere. He then jumps $y_1\cdot \overrightarrow{x_1} \cdot y_3$, forcing a final merge. This ends the game with one peg. Clearly, Jumper cannot do better. As Jumper had responses to all of Merger's possible moves, she cannot do better. Thus $J^-(K_{n,m}) = 1$ when $n \geq 3$, $m \geq 3$, and $n + m$ is odd.

Suppose that $n \geq 3$, $m \geq 3$, and n and m are both odd. Jumper adopts an analogous strategy as in the case where $n+m$ is odd until $\rho(X) \leq 2$ or $\rho(Y) \leq 2$. If $\rho(X) = 2$ ($\rho(Y) = 2$), then Jumper instead jumps $y_i \cdot \overrightarrow{x_j} \cdot y_k$ ($x_i \cdot \overrightarrow{y_j} \cdot x_k$), which forces $(y_k, y_p) \to x_j$ ($(x_k, x_p) \to y_i$). Due to the different parity, Jumper's endgame is now played on a graph with pegs in x_1, y_1, and y_2 and holes elsewhere. He jumps $x_1 \cdot \overrightarrow{y_2} \cdot x_3$, which forces a final merge $(x_2, x_3) \to y_2$. This ends the game with two pegs. Since Jumper has responses for each of Merger's possible moves, Merger cannot do better. Note that Merger can force all of Jumper's moves by merging $(y_i, y_j) \to x_i$ until there is a single peg in Y. At which point, Merger is forced to merge $(x_i, x_j) \to y_k$. Note that once $\rho(Y) = 2$ ($\rho(X) = 2$), Jumper must remove pegs from X (Y); otherwise Merger can end the game with $(y_i, y_j) \to x_i$ ($(x_i, x_j) \to y_k$). As Jumper is the minimizer, he will avoid this as long as possible. Thus, the game will continue until Jumper is playing on a board with pegs in x_1, x_2, y_1, and y_2 and holes elsewhere. Ergo, he cannot do better, and $J^-(K_{n,m}) = 2$ when $n \geq 3$, $m \geq 3$, and n and m are both odd.

Suppose that $n \geq 4$, $m \geq 4$, and n and m are both even. Without loss of generality, assume that $n \leq m$. Jumper begins with the initial hole in X. On each of her turns, Merger merges $(y_i, y_j) \to x_k$. If $\rho(Y) = 2$ on Merger's turn, then this ends the game with n pegs remaining in X. If $\rho(Y) \geq 4$ on Merger's turn, then this forces $y_p \cdot \overrightarrow{x_q} \cdot y_r$. Since Merger was able to force all of Jumper's moves, Jumper cannot do better after the selection of the initial hole. If instead Jumper places the initial hole in Y, then Merger adopts an analogous strategy, which results in m pegs at the end of the game. As $n \leq m$, this is not optimal for Jumper as the minimizer. Hence $J^-(K_{n,m}) \geq n$. To see that Merger cannot do better, consider a configuration with at least $n+1$ pegs. If there are pegs in both X and Y, then there is at least one move available for each player. Thus we assume that all pegs are in Y. In this case, there is clearly another merge. If the game were to end after Merger's turn, then her final merge would be $(x_1, x_2) \to y_i$. However, Jumper can avoid the configuration with pegs in x_1, x_2, and Y and a hole in Y on Merger's turn by instead jumping $y_i \cdot \overrightarrow{x_2} \cdot y_j$. Thus neither player can do better, and $J^-(K_{n,m}) = \min\{n, m\}$ when $n \geq 4$, $m \geq 4$, and n and m are both even.

(iv) Suppose that Merger selects the initial hole and that she is the maximizer. Note that after the selection of the initial hole and the first jump, there is a hole in x_n, a hole in y_m, and pegs elsewhere. This being the case, we begin on Merger's first meaningful move, namely on the configuration in which there is a hole in x_n, a hole in y_m, and pegs elsewhere.

Let $n = 2$. For each of Merger's moves, she is forced to merge $(y_i, y_j) \to x_k$, which forces $y_p \cdot \overrightarrow{x_q} \cdot y_r$ (if available). If m is odd, then this continues until there are two pegs in X. If m is even, then this continues until there is a peg in X and a peg in Y. As both players' moves are forced up to automorphism on the vertices, neither can do better, and $M^+(K_{2,m}) = 2$. A similar argument holds when $m = 2$.

Suppose that n is odd, m is even, and $m \geq 4$. On each of Merger's turns, she merges $(x_i, x_j) \to y_k$. Either this merge ends the game with m pegs in Y or it forces $x_p \cdot \overrightarrow{y_q} \cdot x_r$. Since all of Jumper's moves are forced, he cannot do better. Hence $M^+(K_{n,m}) \geq m$. To see that Merger cannot do better, consider a set of k pegs, where $k \geq m+1$. If there are pegs in X and Y, then both players have an available move. Hence we can assume that all k pegs are in X. If it is Merger's turn, then clearly she has another available move. Thus we can assume that it is Jumper's turn. If $k = n$, then Merger's moves would have all been of the form $(y_p, y_q) \to x_r$ and Jumper's moves would have all been of the form $y_i \cdot \overrightarrow{x_r} \cdot y_j$. However, since m is even, this would result in $\rho(X) = n - 1$ and $\rho(Y) = 1$ on Merger's turn. Hence, she would have another available move. Thus we assume that $m + 1 \leq k \leq n - 1$. For a set of k pegs to be in X, Merger would have to merge $(y_1, y_2) \to x_k$ on a graph with pegs in x_1,...,x_{k-1}, y_1, and y_2 and holes elsewhere. For this to happen, Jumper's previous jump would have been $y_3 \cdot \overrightarrow{x_k} \cdot y_2$ on a board with pegs in x_1,...,x_k, y_1, y_3 and holes elsewhere. However, Jumper could have instead jumped $x_k \cdot \overrightarrow{y_3} \cdot x_{k+1}$ on this board. As this forces at least two more moves, at most m pegs can be left on the board. Hence, $M^+(K_{n,m}) = m$ when n is odd, m is even, and $m \geq 4$. By reversing the roles of X and Y, the same argument shows that $M^+(K_{n,m}) = n$ when n is even, m is odd, and $n \geq 4$. Likewise, if n and m are both odd with $n \geq 4$, then the same argument shows that $M^+(K_{n,m}) = \max\{n, m\}$.

Suppose that n and m are both even with $n \geq 4$ and $m \geq 4$. After the selection of the initial hole and the first jump, $\rho(X)$ and $\rho(Y)$ are both odd. As before, any merge will change the parity of exactly one of $\rho(X)$ and $\rho(Y)$. Thus after the first merge, either $\rho(X) = n$ or $\rho(Y) = m$. Without loss of generality, assume that $\rho(X) = n$. This forces $y_i \cdot \overrightarrow{x_j} \cdot y_k$. We now examine two cases.

Case 1: So long as $\rho(Y) \geq 2$, Merger only makes merges of the form $(y_p, y_q) \to x_r$. This forces $y_i \cdot \overrightarrow{x_r} \cdot y_j$. Eventually Jumper plays on a board with pegs in x_1,...,x_n, and y_1 and holes elsewhere. Thus $y_1 \cdot \overrightarrow{x_n} \cdot y_2$ and $(x_{n-2}, x_{n-1}) \to y_1$ are forced. At this point, Jumper does not want to jump over any of the pegs in X, as this would allow Merger to end the game with $(y_p, y_q) \to x_r$. So Jumper is forced to continue to make jumps of the form $x_i \cdot \overrightarrow{y_j} \cdot x_k$, which forces $(x_p, x_q) \to y_r$. Since n and m are both even, eventually Jumper plays on a graph with pegs in x_1, y_1, and y_2 and holes elsewhere. Jumper jumps $y_1 \cdot \overrightarrow{x_1} \cdot y_3$, which forces $(y_2, y_3) \to x_1$. This ends the game with one peg.

Case 2: Suppose that Merger makes a merge of the form $(x_p, x_q) \to y_r$ before she is forced to do so. Without loss of generality, assume that her second merge is of this form. From this point, exactly one of $\rho(X)$ and $\rho(Y)$ will be odd on each of Jumper's turns. If $\rho(X)$ $(\rho(Y))$ is odd on Jumper's turn, then he jumps $x_i\cdot \overrightarrow{y_j} \cdot x_k$ $(y_i\cdot \overrightarrow{x_j} \cdot y_k)$. Notice that this strategy forces Merger to play on a board in which $\rho(X)$ and $\rho(Y)$ are both odd. Hence Jumper can continue this strategy until he plays on a graph with (without loss of generality) pegs in x_1, y_1, and y_2 and holes elsewhere. Jumper jumps $y_1\cdot \overrightarrow{x_1} \cdot y_3$, which forces $(y_2, y_3) \to x_1$. This ends the game with one peg.

In both cases, clearly Jumper cannot do better as the minimizer. As we examined all possibilities for Merger, she cannot do better as the maximizer. Hence $M^+(K_{n,m}) = 1$ when $n \geq 4$, $m \geq 4$, and n and m are both even.

(v) Suppose that Merger selects the initial hole and that she is the minimizer. As in (iv), we begin on Merger's first meaningful move where there is a hole in x_n, a hole in y_m, and pegs elsewhere.

Suppose that n is even. Note that on Merger's first meaningful move, $\rho(X)$ is odd. As long as $\rho(X) \geq 3$, Merger merges $(x_i, x_j) \to y_k$, which forces $x_p\cdot \overrightarrow{y_q} \cdot x_r$. This strategy continues until (up to automorphism on the vertices) there are pegs in x_1 and y_1,...,y_{m-1} and holes elsewhere at the beginning of Merger's turn. If $m = 2$, then the game ends with two pegs. As all of Jumper's moves are forced, he cannot do better. Since $m = 2$, Merger's moves are also forced. Thus $M^-(K_{n,2}) = 2$ when n is even. Note that Merger adopts an analogous strategy when $n = 2$.

Thus we may assume that n is even, $n \geq 4$, and $m \geq 3$. We begin on Merger's turn with pegs in x_1 and y_1,...,y_{m-1} and holes elsewhere. At this point, Merger merges $(y_{m-2}, y_{m-1}) \to x_2$ resulting in pegs in x_1, x_2, y_1,...,y_{m-3} and holes elsewhere. If $m = 3$, then the game ends with two pegs in X. Again, both players' moves were forced, so neither can do better. If $m = 4$, then Jumper can either jump $y_1\cdot \overrightarrow{x_2} \cdot y_2$ or $x_2\cdot \overrightarrow{y_1} \cdot x_3$. In the first case, the game ends with two pegs. However, $x_2\cdot \overrightarrow{y_1} \cdot x_3$ sets up $(x_1, x_3) \to y_1$, which ends the game with one peg. As we have examined all possibilities, $M^-(K_{n,4}) = 2$ when n is even. An analogous argument holds when $n = 4$. We now examine two cases.

Case 1: Assume that n is even, m is odd, $n \geq 6$, and $m \geq 5$. We begin on Jumper's turn with pegs in x_1, x_2, y_1,...,y_{m-3} and holes elsewhere. Note that $\rho(X) = 2$ and $\rho(Y)$ is even. After each of Jumper's next turns, if $\rho(X) \leq 2$, then Merger merges $(y_i, y_j) \to x_k$. We note that this merge is either forced or must be taken to continue to game after Merger's turn. If $\rho(X) \geq 3$, then Merger merges $(x_i, x_j) \to y_k$. This ensures that

$\rho(X) \leq 3$ on Jumper's turns. Merger adopts analogous strategies when $\rho(Y) \leq 3$. This continues until, without loss of generality, Jumper plays on a board with pegs in x_1, y_1, y_2, and y_3 and holes elsewhere (i.e., a *(1,3)-configuration*) or a board with pegs in x_1, x_2, y_1, and y_2 and holes elsewhere (i.e., a *(2,2)-configuration*). If Jumper plays on the (1,3)-configuration and jumps $y_3 \cdot \overrightarrow{x_1} \cdot y_4$, then Merger responds with $(y_1, y_2) \to x_1$, and $y_4 \cdot \overrightarrow{x_1} \cdot y_1$ or $x_1 \cdot \overrightarrow{y_4} \cdot x_2$ ends the game with one peg. As Jumper is the maximizer, this is not optimal. So instead, he jumps $x_1 \cdot \overrightarrow{y_3} \cdot x_2$. Likewise, on the (2,2)-configuration, he jumps $y_1 \cdot \overrightarrow{x_2} \cdot y_3$. In either case, a final merge ends the game with two pegs. As all jumps (except perhaps a non-optimal one) are forced or Merger has a response, Jumper cannot do better. In order for Merger to do better, the game would have to end with one peg after a jump, say $x_1 \cdot \overrightarrow{y_1} \cdot x_2$. Without loss of generality, the previous merge would be $(x_2, x_3) \to y_1$. Therefore, the previous jump would be $x_4 \cdot \overrightarrow{y_1} \cdot x_3$. The configuration at this point is precisely the (1,3)-configuration. However, we have already examined this configuration and have seen that Jumper cannot do better. Thus, $M^-(K_{n,m}) = 2$ when n is even, m is odd, $n \geq 6$, and $m \geq 5$.

Case 2: Assume that n is even, m is even, $n \geq 6$, and $m \geq 6$. We begin on Jumper's turn with pegs in x_1, x_2, y_1,...,y_{m-3} and holes elsewhere. Note that $\rho(X) = 2$ and $\rho(Y)$ is odd. Merger uses the strategy from Case 1 above, Jumper jumps $x_i \cdot \overrightarrow{y_j} \cdot x_k$ on his first unforced move, and $y_i \cdot \overrightarrow{x_j} \cdot y_k$ on each of his remaining moves. Because it is not optimal for Merger as minimizer to merge $(x_i, x_k) \to y_k$ if only two pegs are in X on her turn, this forces Merger to merge $(y_p, y_q) \to x_i$ on her next moves. This eventually forces Jumper to play on a board with, without loss of generality, pegs in x_1, x_2, y_1, y_2, and y_3 and holes elsewhere. The jump $x_1 \cdot \overrightarrow{y_3} \cdot x_3$ forces $(y_1, y_2) \to x_1$, which ends the game with three pegs in X. If Jumper instead jumps $y_3 \cdot \overrightarrow{x_1} \cdot y_4$, then it is easy to see that the best Jumper can do is two pegs at the end of the game. In general, Jumper cannot do better because Merger has responses to all of his jumps, with the exception of the one non-optimal one. The argument that Merger cannot do better is similar to the case above. Thus $M^-(K_{n,m}) = 3$ when n is even, m is even, $n \geq 6$, and $m \geq 6$.

Suppose that n and m are both odd. Merger adopts an analogous strategy to the case where n is even. Since the parity of n has changed, Merger's endgame is played on a board with pegs in x_1, x_2, y_1, and y_2 and holes elsewhere. A final merge ends the game with three pegs remaining. The argument that neither player can do better follows similarly to above. Hence $M^-(K_{n,m}) = 3$ when n and m are both odd. $\square$

The competitive graph parameters on the complete bipartite graph have several interesting properties. Consider the parameters in which each player adopts the same role (i.e., the minimizer versus the maximizer). For instance, for both $J^+(G)$ and $M^-(G)$, Jumper is the maximizer and Merger is the minimizer. The difference is which player moves first. Likewise, for both $J^-(G)$ and $M^+(G)$, Jumper is the minimizer and Merger is the maximizer. Since $J^+(K_{1,m}) = M^+(K_{1,m}) = m$ and $J^-(K_{1,m}) = M^-(K_{1,m}) = 1$, the differences $J^+(K_{1,m}) - M^-(K_{1,m}) = m-1$ and $M^+(K_{1,m}) - J^-(K_{1,m}) = m-1$ can be made arbitrarily large.

Note that the complete bipartite graph $K_{n,m}$, where $n \geq 3$, $m \geq 3$, and $n+m$ is even, is an infinite family of graphs where $M^-(K_{n,m}) > J^+(K_{n,m})$. In other words, both the maximizer and the minimizer do strictly better when they are the second player. Seo and Slater define such graphs as *second player optimal*. Second player optimal graphs are discussed in the context of other competitive graph parameters in [25]. Our last family of graphs, the double star, provide examples in which not only is $J^-(G) > M^+(G)$, but the difference $J^-(G) - M^+(G)$ can be made arbitrarily large. As in the proof of Theorem 2.7, we will let $\rho(X)$ and $\rho(Y)$ denote the number of pegs in X and Y, respectively.

Theorem 3.6. *For $S_{n,m}$, the double star:*

(i) $J^+(S_{n,m}) = n + m + 1$.

(ii) *If n or m is even, then $J^-(S_{n,m}) = 2$. If n and m are both odd, then $J^-(S_{n,m}) = \min\{n+1, m+1\}$.*

(iii) *$M^+(S_{n,1}) = n+1$. If n even and m are both even, then $M^+(S_{n,m}) = \max\{n+1, m+1\}$. If n is even, m is odd, and $m \geq 3$, then $M^+(S_{n,m}) = m+1$. If n is odd, m is even, and $n \geq 3$, then $M^+(S_{n,m}) = n+1$. If n and m are both odd, $n \geq 3$, and $m \geq 3$, then $M^+(S_{n,m}) = 2$.*

(iv) *If n and m are both even, then $M^-(S_{n,m}) = \min\{n+1, m+1\}$. If $n \in \{1,3\}$ or $m \in \{1,3\}$, then $M^-(S_{n,m}) = 2$. If n is odd and $n \geq 5$ or if m is odd and $m \geq 5$, then $M^-(S_{n,m}) = 4$.*

Proof.

(i) This follows from Proposition 3.1.

(ii) Assume that Jumper plays first and that he is the minimizer. If Jumper places the initial hole in a vertex of degree one, then the game ends with $n+m-1$ pegs by Proposition 3.1. As this is not optimal for the minimizer, we assume that Jumper places the initial hole in either x or y throughout this case.

Suppose that n or m is even. Without loss of generality, assume that m is even. Jumper places the initial hole in y, and his strategy is to maintain an even number of pegs in Y and a peg in x until $\rho(Y) = 2$ on Merger's turn. To do so, if Merger merges $(y_i, y_j) \to y$ $((x, y_i) \to y)$, then Jumper responds with $y_k \cdot \overrightarrow{y} \cdot y_j$ $(y_j \cdot \overrightarrow{y} \cdot x)$. Because m is even, this strategy results in an even number of pegs in Y and a peg in x on each of Merger's turns. This continues until $\rho(Y) = 2$ and a peg is in x on Merger's turn. Merger may now make one of two merges. If Merger merges $(y_i, y_j) \to y$ $((y_i, x) \to y)$, then Jumper responds with $x \cdot \overrightarrow{y} \cdot y_i$ $(y_k \cdot \overrightarrow{y} \cdot y_i)$. In either case, Merger is forced to merge two pegs from X into x on her next turn. On Jumper's remaining turns, he jumps $x_i \cdot \overrightarrow{x} \cdot x_j$, which forces $(x_k, x_\ell) \to x$ on all of Merger's remaining turns. If n is even, then the game ends with a peg in x and a peg in y_i. If n is odd, then the game ends with a peg in x_j and a peg in y_i. Since Jumper has a response for all of Merger's moves, she can do no better. Thus, $J^-(S_{n,m}) \leq 2$.

We now show that Jumper cannot do better. Suppose that Jumper allows x to have a hole on any of Merger's turns before $\rho(Y) = 2$. Merger then merges $(x_i, x_j) \to x$. At this point, Merger can force play on X, x, and y by responding to $x_i \cdot \overrightarrow{x} \cdot y$ $(x_i \cdot \overrightarrow{x} \cdot x_j)$ with $(x_j, y) \to x$ $((x_j, x_k) \to x)$. If n is even, then Merger can force the game to end with pegs remaining in x and Y. If n is odd, then Merger can prevent Jumper from removing all but one peg from the graph by forcing play on X, x, and y as described above until $\rho(X) = 1$ on Jumper's turn. At which point, if Jumper jumps $x_i \cdot \overrightarrow{x} \cdot y$, then the game ends with $\rho(Y) + 1$ pegs. Hence Jumper must jump over x and into x_i. Merger can then force at least two pegs remain at the end of the game by merging into y on each of her remaining turns, never leaving a peg in x on Jumper's turn.

Likewise, if Jumper follows the above strategy until $\rho(Y) = 2$, then when $\rho(Y) = 2$ and a peg is in x on Merger's turn, Jumper's responses to Merger's next turn both result in a peg in Y. Any other response to this turn is either not possible or will end the game. Since Jumper is the minimizer, it is not optimal for him to end the game early. Because Merger can always respond to a jump of the form $x_i \cdot \overrightarrow{x} \cdot y$ with $(x_j, y) \to x$, Merger can ensure that a peg is never in y on Jumper's turn. Likewise, because all of Jumper's jumps at this point in the game will eliminate a peg from x, a peg will never be in x on Merger's turn. Thus, at least two pegs will remain at the end of the game.

Finally, consider the case where Jumper selects x for the initial hole. Merger merges $(x_i, y) \to x$ on her first move. If Jumper jumps $x_j \cdot \overrightarrow{x} \cdot y$ $(x_j \cdot \overrightarrow{x} \cdot x_\ell)$, then Merger merges $(x_k, y) \to x$ $((x_k, x_\ell) \to x)$. If n is odd, then this eventually results in a peg in x, m pegs in Y, and holes elsewhere on Jumper's turn. As no jumps are possible, the game is over with $m + 1$ pegs remaining. If n is even, then the argument is similar to when the initial hole is in y. As all possibilities are exhausted, $J^-(S_{n,m}) = 2$ if n or m is even.

Suppose that n and m are both odd. Without loss of generality, assume that $m \geq n$. Jumper places the initial hole in y. Merger's strategy is to force Jumper to only play on x, y, and Y. Merger merges $(y_1, x) \to y$ on her first turn. If Jumper jumps $y_i \cdot \overrightarrow{y} \cdot y_j$ ($y_i \cdot \overrightarrow{y} \cdot x$), then Merger responds with $(y_j, y_k) \to y$ ($(x, y_k) \to y$). Because m is odd, this continues until $\rho(Y) = 0$, a hole is in x, and all other vertices have pegs on Jumper's turn. As no jumps are available, the game is over with $n + 1$ pegs remaining. We note that Merger cannot end the game any sooner, as all other merges result in a peg in y (x) and a peg in some y_i (x_i) on Jumper's turn.

If Jumper selects x for the initial hole, then Merger uses a similar strategy to force the game to end with $m + 1$ pegs. As we assumed that $m \geq n$, this is not optimal for Jumper as the minimizer. As all possibilities are exhausted, $J^-(S_{n,m}) = \min\{n + 1, m + 1\}$ if n and m are both odd.

(iii) Suppose that Merger plays first and that she is the maximizer.

For $S_{n,1}$, Merger places the initial hole in x. This forces $y_1 \cdot \overrightarrow{y} \cdot x$, which ends the game with $n + 1$ pegs. As any placement of the initial hole will result in at least one jump, this is optimal when Merger is the maximizer. Ergo, $M^+(S_{n,1}) = n + 1$.

Suppose that n and m are both even. Without loss of generality, assume that $n \geq m$. Merger selects x for the initial hole. Jumper is forced to jump $y_1 \cdot \overrightarrow{y} \cdot x$. Merger merges $(x, y_2) \to y$. If $m = 2$, then the game is over and the result follows. Assume that $m \geq 4$. Merger's strategy is to restrict Jumper's play to x, y, and Y. If Jumper jumps $y_i \cdot \overrightarrow{y} \cdot y_j$ ($y_i \cdot \overrightarrow{y} \cdot x$) on any of his turns, then Merger responds with $(y_j, y_k) \to y$ ($(y_j, x) \to y$). Because m is even, this will eventually result in a peg in y, n pegs in X, and holes elsewhere. As no jumps are available, the game ends. As Merger had responses to all of Jumper's moves, he can do no better. Hence, $M^+(S_{n,m}) \geq n + 1$. To show that Merger cannot do better, consider a terminal configuration of $n + 2\ell + 1$ pegs, where $\ell \geq 1$. Because of parity, this must occur at the beginning of Merger's turn. Hence both x and y must be filled, otherwise a Merge is available. However, Jumper can always remove a peg from the center on his turn. Likewise, a terminal configuration of $n + 2\ell$ pegs, where $\ell \geq 1$, must occur at the beginning of Jumper's turn. As Merger can never remove both pegs from the center, a jump is available in this scenario. Thus $n + 1$ pegs is the best that Merger can do. Ergo, $M^+(S_{n,m}) = n + 1$ when n and m are both even and $n \geq m$.

An analogous argument holds for the case where n is odd, m is even, and $n \geq 3$ and the case where m is odd, n is even, and $m \geq 3$.

We now consider the case where n and m are both odd and $n \geq m \geq 3$. If on any of Merger's turns $\rho(X)$ and $\rho(Y)$ are both odd and there are holes in both x and y, then we will refer to this as an *(odd,odd)-configuration.* On the (odd,odd)-configuration, Jumper can eventualy force the game to end with two pegs by responding to the merge $(x_i, x_j) \to x$ $((y_i, y_j) \to y)$ with $x_k \cdot \overrightarrow{x} \cdot x_j$ $(y_k \cdot \overrightarrow{y} \cdot y_j)$ until one peg remains in each of X and Y. For this reason, Merger tries to avoid the (odd,odd)-configuration.

If Merger selects a pendant, say x_1, for the initial hole, then Jumper can force the (odd,odd)-configuration with $y \cdot \overrightarrow{x} \cdot x_1$. Assume that Merger selects x for the initial hole. Jumper is forced to jump $y_1 \cdot \overrightarrow{y} \cdot x$. If Merger merges $(y_i, y_j) \to y$ $((x, y_i) \to y)$, then Jumper jumps $x \cdot \overrightarrow{y} \cdot y_i$ $(y_j \cdot \overrightarrow{y} \cdot y_i)$. This results in the (odd,odd)-configuration. Hence, Merger cannot do better than two pegs at the end of the game. The case where Merger selects y for the initial hole is analogous.

We now show that Jumper cannot force the game to end with only one peg. Notice that the players are playing on two star subgraphs, and since n and m are both odd, a peg will remain in one of the pendants, say x_1. For Jumper to remove the peg in x_1, there must be a peg in x on his turn. However, Merger can remove a peg in x with $(y_i, x) \to y$. Likewise, Merger cannot merge into a pendant. Hence the result follows.

(iv) Suppose that Merger plays first and that she is the minimizer.

Suppose that n and m are both even. Without loss of generality, assume that $n \geq m$. If Merger selects x_1 (y_1) for the initial hole, then Jumper responds with $y \cdot \overrightarrow{x} \cdot x_1$ $(x \cdot \overrightarrow{y} \cdot y_1)$. On any of her turns if Merger merges $(x_i, x_j) \to x$ or $(y, x_i) \to x$, then either the game ends or Jumper responds with $x_k \cdot \overrightarrow{x} \cdot y$ (or $x_p \cdot \overrightarrow{x} \cdot x_q$ if x and y both have pegs on his turn). This results in the game ending after a merge with $m+1$ pegs remaining. Further, if Merger's second move is $(y_i, y_j) \to y$, then Jumper can adopt an analogous strategy to eliminate the pegs from Y and leave the pegs in X alone. As this results in $n+1$ pegs remaining, it is not optimal for Merger as the minimizer.

Suppose instead that Merger selects y for the initial hole. Jumper is forced to jump $x_1 \cdot \overrightarrow{x} \cdot y$. If Merger merges $(x_i, x_j) \to x$ $((x_i, y) \to x)$, then Jumper responds with $y \cdot \overrightarrow{x} \cdot x_i$ $(x_j \cdot \overrightarrow{x} \cdot x_i)$. Hence, this reduces to the above argument. Likewise, if Merger selects x as the initial hole, then Jumper's above strategy results in $n+1$ pegs remaining. In any case, Jumper has a response for every possible merge. Thus, Merger cannot do better and $M^-(S_{n,m}) \geq m+1$ when n and m are both even and $n \geq m$.

To see that Jumper cannot do better, consider a set of at least $m+2$ pegs. This either results in a configuration with at least one additional move or one in which $m+2$ pegs are in $X \cup \{y\}$. However, as noted above, Merger can make choices that prevent leaving pegs in X. Thus the result follows.

We now consider the case where n or m is odd. Note that if $n = m = 1$, then $S_{1,1}$ is isomorphic to P_4, and the result follows from Theorem 3.3.

Suppose that $n = 1$ and $m \geq 2$. Merger selects x_1 for the initial hole, which forces $y \cdot \overrightarrow{x} \cdot x_1$. Merger is forced to merge $(y_1, y_2) \to y$. If $m = 2$, then the game is over. If $m = 3$, then the game is over after an additional jump. Assume that $m \geq 4$. If Jumper jumps $y_i \cdot \overrightarrow{y} \cdot y_j$ ($y_i \cdot \overrightarrow{y} \cdot x$) on any of his turns, then Merger responds with $(y_j, y_k) \to y$ ($(x, y_k) \to y$). This continues until two pegs remain. If m is even, then the pegs are in x_1 and y. If m is odd, then the pegs are in x_1 and either x or y_1.

To show that Merger cannot force the game to end with only one peg remaining, we note that if the initial hole is in y, then the game is over after the first jump. Assume that the initial hole is in x. This forces $y_1 \cdot \overrightarrow{y} \cdot x$ and $(y_2, x) \to y$. After which, Jumper can restrict play to y and Y with jumps of the form $y_i \cdot \overrightarrow{y} \cdot y_j$, resulting in two pegs at the end of the game. If instead Merger places the initial hole in y_1 and Jumper responds with $x \cdot \overrightarrow{y} \cdot y_1$, then this reduces to the case where there are pegs in x_1, y_1,...,y_m and holes elsewhere. Hence the result follows. The case where $m = 1$ and $n \geq 2$ is analogous.

Suppose that $n = 3$ and $m \geq 2$. Merger begins with the hole in y, which forces $x_3 \cdot \overrightarrow{x} \cdot y$. Merger responds with $(x_1, x_2) \to x$, which reduces this to the case where $n = 1$ after the placement of the initial hole. Thus Jumper cannot do any better than two pegs at the end. To see that Merger cannot do better, it suffices to look at other placements of the initial hole. If the initial hole is in x_i, then Jumper jumps $x_j \cdot \overrightarrow{x} \cdot x_i$. This reduces to the case where the initial hole is in y after the first jump. If the initial hole is in x, then Jumper is forced to jump $y_i \cdot \overrightarrow{y} \cdot x$. If $m = 2$, then Merger is forced to respond with $(y_j, x) \to y$ which ends the game with four pegs (hence it is not optimal for her). Otherwise, so long as $\rho(Y) \geq 1$, Jumper responds to $(y_i, y_j) \to y$ ($(x, y_i) \to y$) with $y_k \cdot \overrightarrow{y} \cdot y_\ell$ ($y_j \cdot \overrightarrow{y} \cdot y_i$). If $\rho(Y) = 1$, then Jumper jumps $x \cdot \overrightarrow{y} \cdot y_i$. Note that Jumper's strategy ensures that there will be at least two pegs at the end of the game. If the initial hole is in y_i, then Jumper jumps $y_j \cdot \overrightarrow{y} \cdot y_i$. This reduces to the case where the initial hole is in x after the first jump. The case where $m = 3$ and $n \geq 2$ is analogous.

Suppose that n is odd, $n \geq 5$, and $m \notin \{1, 3\}$. Merger begins with the initial hole in y, which forces $x_i \cdot \overrightarrow{x} \cdot y$. So long as $\rho(X) \geq 4$ at the beginning of her turn, Merger responds to $x_i \cdot \overrightarrow{x} \cdot y$ with $(x_j, y) \to x$. Likewise, so long as $\rho(X) \geq 4$ at the beginning of her turn, Merger responds to $x_i \cdot \overrightarrow{x} \cdot x_j$ with $(x_j, x_k) \to x$. This strategy continues until at the start of Merger's turn either $\rho(X) = 2$, $\rho(Y) = m$, a peg is in y, and a hole is in x (which we will refer to as the $(2,m)$-*configuration*) or $\rho(X) = 3$, $\rho(Y) = m$, and holes are in x and y (which we will refer to as the $(3,m)$-*configuration*). In case of the $(2,m)$-configuration, the Merger merges $(x_i, x_j) \to x$, which forces $y\cdot \overrightarrow{x} \cdot x_i$. At this point, Merger merges $(y_i, y_j) \to y$. If on any of his remaining turns Jumper jumps $y_i\cdot \overrightarrow{y} \cdot x$ ($y_i\cdot \overrightarrow{y} \cdot y_j$), then either the game immediately ends with two pegs or Merger responds with $(y_j, x) \to y$ ($(y_j, y_\ell) \to y$). This move also will eventually result in a terminal state with two pegs. However, Jumper can avoid the jump $x_i\cdot \overrightarrow{x} \cdot y$, which leads to the $(2,m)$-configuration by jumping $x_i\cdot \overrightarrow{x} \cdot x_j$ instead. This results in the $(3,m)$-configuration. As we will see, the $(3,m)$-configuration results in a better outcome for Jumper as the maximizer. If Merger merges $(x_i, x_j) \to x$ at any point, then Jumper can end the game with $\rho(Y)+1$ pegs by jumping $x_k\cdot \overrightarrow{x} \cdot y$. As Merger is the minimizer, she will avoid this move as long as possible. Thus Merger merges $(y_i, y_j) \to y$. If on any of his remaining moves, Jumper jumps $y_i\cdot \overrightarrow{y} \cdot x$ ($y_i\cdot \overrightarrow{y} \cdot y_j$), then either the game ends with four pegs or Merger responds with $(y_j, x) \to y$ ($(y_j, y_k) \to y$). Note that if m is even, then on Jumper's turn, he will eventually play on a board with $\rho(X) = 3$, $\rho(Y) = 2$, a peg in y, and holes elsewhere. If he jumps $y_i\cdot \overrightarrow{y} \cdot x$, then the merge $(y_j, x) \to y$ ends the game with four pegs. If he instead jumps $y_i\cdot \overrightarrow{y} \cdot y_j$, then Merger instead responds with $(x_i, x_j) \to x$ which forces at least one more jump. Hence the terminal configuration will have at most three pegs. As Jumper is the maximizer, this choice is not optimal for him. Thus, $M^-(S_{n,m}) \le 4$ when m is even. If m is odd and $m \notin \{1, 3\}$, then an analogous strategy will result in Jumper playing on a board with $\rho(X) = 3$, $\rho(Y) = 1$, a peg in y, and holes elsewhere. Jumper ends the game by jumping $y_i\cdot \overrightarrow{y} \cdot x$. A similar argument to the even case shows that $M^-(S_{n,m}) \le 4$ when m is odd.

We have argued that Merger cannot do better once the $(3,m)$-configuration has been reached. Thus it suffices to examine her moves prior to this configuration. We begin by examining what happens if Merger selects a different initial hole. If Merger begins with the initial hole in y_i, then Jumper jumps $x\cdot \overrightarrow{y} \cdot y_i$. If Merger merges $(x_i, x_j) \to x$ and $n = 5$, then Jumper responds with $x_i\cdot \overrightarrow{x} \cdot x_j$. This results in the $(3,m)$-configuration. If Merger merges $(x_i, x_j) \to x$ and $n \ge 7$, then Jumper responds with $x_k\cdot \overrightarrow{x} \cdot y$. This reduces to the case with the initial hole in y after the first jump has been made. If Merger merges $(y_i, y_j) \to y$, then Jumper responds with $y_k\cdot \overrightarrow{y} \cdot y_i$. This continues until Merger merges $(x_i, x_j) \to x$ or the game ends with $n+1$ pegs. If Merger merges $(x_i, x_j) \to x$, then this results in a case analogous to the one in which n is odd, $n \ge 5$, and the initial hole is in y. If Merger instead begins with the hole in x and m is odd, then we reverse the roles of X and Y so that this is equivalent to starting with the hole in y. So assume that m is even. Jumper is forced to jump $y_i \cdot \overrightarrow{y} \cdot x$. If Merger merges $(y_i, y_j) \to y$ $((y_i, x) \to y)$ on any of her moves, then either the game ends or Jumper responds with $y_k \cdot \overrightarrow{y} \cdot y_i$ $(y_j \cdot \overrightarrow{y} \cdot x)$. This strategy ensures that $\rho(Y)$ is odd at the beginning of each of Merger's turns. This continues until there are n pegs in X, there is a peg in either y or y_i, and holes elsewhere at the beginning of Jumper's turn. As no jumps are available, the game ends with $n+1$ pegs. Hence, starting with the hole in x is worse for Merger when she is the minimizer. If instead Merger begins with the initial hole in x_i, then Jumper can jump $x_j \cdot \overrightarrow{x} \cdot x_i$ to reduce this to the case where the initial hole is y after the first jump.

To continue this argument, we now examine the impact of Merger's moves after the first jump, assuming that the initial hole is in y. Suppose that Merger responds to $x_i \cdot \overrightarrow{x} \cdot y$ with $(x_j, x_k) \to x$. If $\rho(X) \geq 4$, then Jumper responds with $x_\ell \cdot \overrightarrow{x} \cdot x_i$ to reduce to a configuration analogous to the case where the initial hole is in x after the first jump has been made. If $\rho(X) = 2$, then Jumper instead responds with $y \cdot \overrightarrow{x} \cdot x_i$, which results in the $(3, m)$-configuration. In either case, Jumper ensures that the game will end with at least four pegs. $\square$

Observe that if n and m are both odd with $m, n \geq 3$, then $M^+(S_{n,m}) = 2$ and $J^-(S_{n,m}) = \min\{n+1, m+1\}$. Hence these graphs are second player optimal for the game in which Jumper is the minimizer and Merger is the maximizer. Further, the difference $J^-(S_{n,m}) - M^+(S_{n,m})$ can be made arbitrarily large.

4 Open problems

In this section, we present open problems related to this study as possible avenues of future research. It may be interesting to consider a *misère* version of the game. In the misère version, the last player to remove a peg loses. Again, the goal would be to characterize those graphs for which each player has a winning strategy in the misère version.

The *pie rule* (also known as the swap rule or Nash's rule from Hex) is a common method for mitigating the advantage of going first. If the pie rule is implemented, then after the first move is made, the second player has one of two options. If they let the move stand, then play proceeds as normal. Otherwise, they "take" that move. In which case, Player One then plays as if she were the second player. We could also consider a version which implements the pie rule that would allow the players to also switch roles (i.e., the original Jumper becomes the Merger and vice versa).

In several of Slater's papers on competitive graph parameters, he allows players to "pass" their turn. Under what conditions is this advantageous?

What other graphs are second player optimal? In particular, is there a family of graphs such that the difference $M^-(G) - J^+(G)$ can be made arbitrarily large?

Characterize those graphs for which $J^+(G) = M^-(G)$, $J^-(G) = M^+(G)$, $J^+(G) = J^-(G)$, and $M^+(G) = M^-(G)$.

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