Formula di Graßmann.

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

<u>N.B</u>: In generale se $\mathcal{B}_U$ base di U
$\mathcal{B}_W$ base di W $\Rightarrow$ $\mathcal{B}_U \cup \mathcal{B}_W$: generatori di $U+W$.

MA NON È VERO $\mathcal{B}_U \cap \mathcal{B}_W$ "base" di $U \cap W$.
Tutt'al più $\mathcal{B}_U \cap \mathcal{B}_W$ sequenza libera di ~~B~~ $U \cap W$.

Es. in $\mathbb{R}^3$ $\mathcal{B}_1 = ((1\,0\,0),(0\,1\,0),(0\,0\,1))$
$\mathcal{B}_2 = ((-1\,0\,0),(0\,-1\,0),(0\,0\,-1))$

conseguenze formula di Grassmann.

$$\max(\dim \mathcal{U}, \dim W) \leq \dim(\mathcal{U}+W) \leq \min(\dim \mathcal{U} + \dim W, \dim V_n).$$

$$\max(0, \dim \mathcal{U} + \dim W - \dim V_n) \leq \dim(\mathcal{U} \cap W) \leq \min(\dim \mathcal{U}, \dim W)$$

$\mathcal{U} \cap W \leq \mathcal{U}, W \leq \mathcal{U} + W$

Es. Siano $V, W \leq \mathbb{C}^{3,6}$: $\dim V = 8$, $\dim W = 12$
calcolare possibili valori $\dim V \cap W$

$V \cap W \leq V, W$ in particolare $\dim V \cap W \leq 8$

$\dim V \cap W = 8 \iff V \subseteq W.$ $\dim \mathbb{K}^{m,n} = m \cdot n$

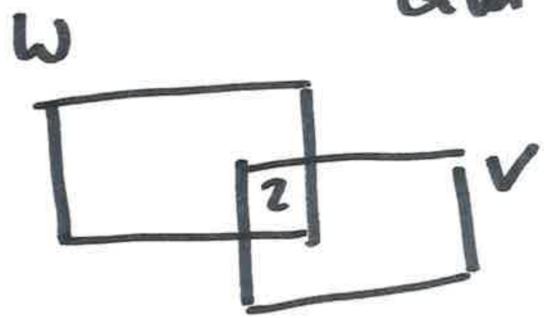
Altrimenti: $\boxed{\dim \mathbb{C}^{3,6} = 18}$

$$\dim \mathcal{U} + \dim W - \underbrace{\dim \mathcal{U} + W}_{\leq 18} = \dim V \cap W$$

$$8 \quad + \quad 12$$

$$8 + 12 - 18 \geq \dim V \cap W$$

$$\dim V \cap W \geq 2.$$



e in questo caso
$V + W = \mathbb{C}^{3,6}$

- Siano $\mathcal{U}, W \leq \mathbb{R}^{3,5}$ con $\dim \mathcal{U} = 6$, $\dim W = 4$.
  E' possibile $\mathcal{U} \oplus W$? Se sí trovare un esempio.

<u>OSS</u>: $U \oplus W \Leftrightarrow U \cap W = \{\underline{0}\} \Rightarrow \dim U + W = \dim U + \dim W.$

$\dim U + \dim W = 6 + 4 = 10 \leq 3 \cdot 5 = 15$

esistono sottospazi U, W di $\mathbb{R}^{3,5}$ con

le proprietà richieste.

$$\mathcal{B}_{\mathbb{R}^{3,5}} = \left(\underset{\bar{e}_{11}}{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \underset{\bar{e}_{12}}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}} \dots \underset{\bar{e}_{35}}{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}} \right)$$

$$U \oplus W = \mathcal{L}(\underbrace{\bar{e}_{11} \dots \bar{e}_{15}, \bar{e}_{21}}_{6} \overbrace{\dots \bar{e}_{25}})$$

$U = \mathcal{L}(\bar{e}_{11} \dots \bar{e}_{15}, \bar{e}_{21})$ $\qquad \dim U = 6$

$W = \mathcal{L}(\bar{e}_{22} \dots \bar{e}_{25})$ $\qquad \dim W = 4$

Si trovino $\mathcal{U}, W \leq \mathbb{R}^{3,3}$ con

$\dim \mathcal{U} = 2$, $\dim W = 4$, $\dim \mathcal{U} \cap W = 1$

$$\mathcal{B}_{\mathbb{R}^{33}} = \left(\underset{\bar{e}_{11}}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \underset{\bar{e}_{12}}{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \dots \underset{\bar{e}_{33}}{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \right).$$

$$\dim \mathcal{U} + W \leq 9$$

$$\| $$

$$\dim \mathcal{U} + \dim W - \dim \mathcal{U} \cap W =$$

$$= 2 + 4 - 1 = 5 < 9 \quad \underline{\text{ok}}$$

$$(\underbrace{\bar{e}_{11} \overbrace{\bar{e}_{12}}_{}}_{\mathcal{U}}\!\!\overbrace{\;\bar{e}_{13}\; \bar{e}_{21}\; \bar{e}_{22}}^{W}\; \bar{e}_{23}\; \bar{e}_{31}\; \bar{e}_{32}\; \bar{e}_{33})$$

dim $(\mathcal{U} \cap W) = 1$ perché

$\dim \mathcal{U} = 2 \quad \dim W = 4$ e $\dim(\mathcal{U}+W) = 5$

<u>N.B</u>

$$\mathcal{B}_U = (\bar{e}_{11}, \bar{e}_{12}) \qquad \mathcal{B}_W = (\bar{e}_{12}, \bar{e}_{13}\ \bar{e}_{21}\ \bar{e}_{22}).$$

Trovare basi DISGIUNTE DI $\mathcal{U}$ e W.

$$\mathcal{B}'_W = (-\bar{e}_{12}, \bar{e}_{13}\ \bar{e}_{21}\ \bar{e}_{22}).$$

$$\mathcal{U} = \mathcal{L}((1\,2\,3), (0\,1\,1))$$

$$W = \mathcal{L}((1\,1\,0), (0\,0\,1))$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\dim(\mathcal{U} \cap W) = ?$

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 \neq 0$$

$\dim \mathcal{U}+W = 3 \quad \dim \mathcal{U} = 2 \quad \dim W = 2 \quad \dim \mathcal{U} \cap W = 1$

$U = \mathcal{L}((1\,0\,1\,1\,0), (0\,1\,0\,0\,1))$

$W = \mathcal{L}((1\,1\,1\,1\,1), (1\,0\,0\,0\,0), (0\,0\,1\,0\,0))$

$\dim U = 2 \qquad \dim W = 3$

$$\rho\begin{bmatrix} \boxed{\begin{matrix}1&1&1\\1&0&0\\0&0&1\end{matrix}} & \begin{matrix}1&1\\0&0\\0&0\end{matrix} \end{bmatrix} = 3$$

$\dim U+W \geq 4$

$$A = \begin{matrix} 1&0&1&1&0 \\ 0&1&0&0&1 \\ 1&1&1&1&1 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \end{matrix}$$

$$\begin{vmatrix} 0&1&0&1 \\ 1&1&1&1 \\ 1&0&0&0 \\ 0&0&1&0 \end{vmatrix} = 0$$

$\det(A) = 0$

$$\begin{vmatrix} 0&1&0&0 \\ 1&1&1&1 \\ 1&0&0&0 \\ 0&0&1&0 \end{vmatrix} = \pm 1$$

$\dim U+W = 4$

$\dim U = 2, \ \dim W = 3$

$\dim U \cap W = 5 - 4 = 1$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \rho(A) \geq 4$$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = \pm \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \pm \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \pm 1 \neq 0$$

$\det(A)$ bo w A III wiersz = I wiersz + II wiersz

$\Rightarrow \det(A) = 0 \Rightarrow \rho(A) = 4$

$\dim(U + W) = 4$

DIM.

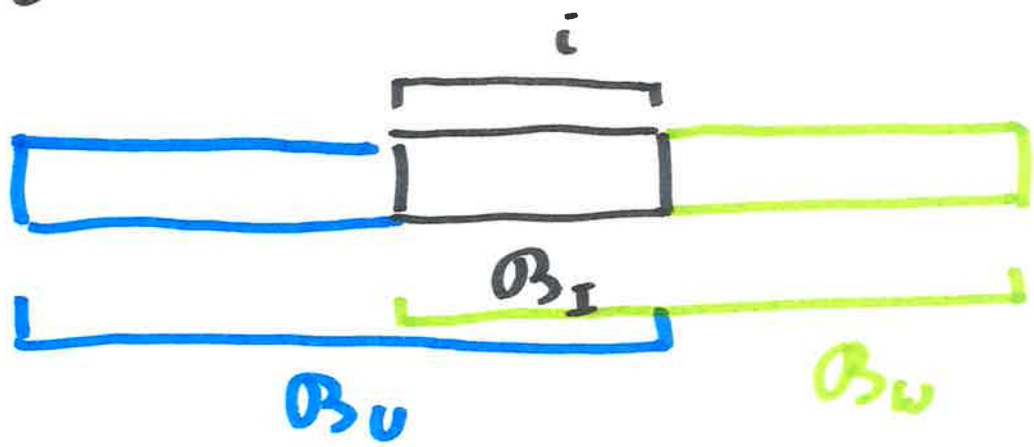
Idea: Se $\mathcal{U} \oplus W$ cioè $\dim(\mathcal{U} \cap W) = 0$

$\Rightarrow \dim(\mathcal{U} + W) = \dim(\mathcal{U}) + \dim(W) - 0 =$

$= \dim(\mathcal{U}) + \dim(W) - \underbrace{\dim(\mathcal{U} \cap W)}_{0}$

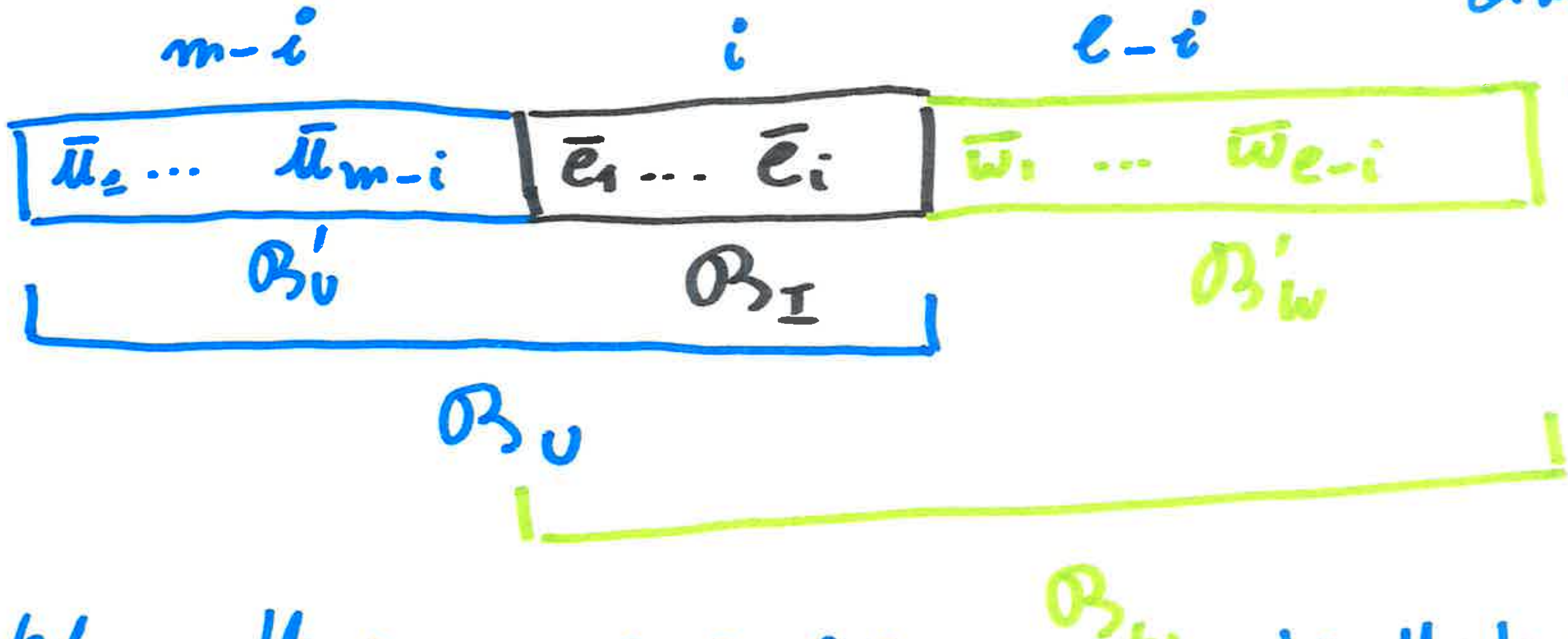
Supponiamo $\dim(\mathcal{U} \cap W) = i > 0$

$\Rightarrow$ Esiste una base $\mathcal{B}_I$ di $\mathcal{U} \cap W$ formata da vettori che stanno sia in $\mathcal{U}$ che in W e sono una seq. libera



Applicando completamento della base con vettori di U estendiamo $\mathcal{B}_I$ ad una base $\mathcal{B}_U$ di U (e poniamo $\mathcal{B}_U = \mathcal{B}_I \cup \mathcal{B}'_U$)
e similmente estendiamo $\mathcal{B}_I$ ad una base $\mathcal{B}_W$ di W (e poniamo $\mathcal{B}'_W = \mathcal{B}_W \setminus \mathcal{B}_I$).

$\dim U = m$
$\dim W = l$



totale vettori $= m-i+i+l-i = m+l-i = \dim U + \dim W - \dim U \cap W$

<u>OSS</u>: $\mathcal{B}_U \cup \mathcal{B}_W = \mathcal{B}_U' \cup \mathcal{B}_I \cup \mathcal{B}_W'$ è una seq. di generatori per $U + W$.

perché unione di una base di U e di una base di W.

Dobbiamo dimostrare che è una seq. libera!

Supponiamo $\exists (\alpha_1 \dots \alpha_{m-i}, \beta_1 \dots \beta_i, \gamma_1 \dots \gamma_{\ell-i}) \neq \underline{0}$ tali che

$$\alpha_1 \overline{u}_1 + \dots + \alpha_{m-i} \overline{u}_{m-i} + \beta_1 \overline{e}_1 + \dots + \beta_i \overline{e}_i + \gamma_1 \overline{w}_1 + \dots + \gamma_{\ell-i} \overline{w}_{\ell-i} = \underline{0}$$

$\Rightarrow$ è impossibile che sia $(\alpha_1 \dots \alpha_{m-i}) = (0\,0 \dots 0)$

perché altrimenti avremmo una c. lineare di vettori di $\mathcal{B}_W$ con coeff. non tutti $= 0$ che dà $\underline{0}$

$\Rightarrow$ possiamo scrivere

$$\underbrace{\beta_1 \bar{e}_1 + \dots + \beta_i \bar{e}_i + \gamma_1 \bar{w}_1 + \dots + \gamma_{\ell-i} \bar{w}_{\ell-i}}_{\in \mathcal{L}(\mathcal{B}_W) = W} = \underbrace{-\alpha_1 \bar{u}_1 \dots - \alpha_{m-i} \bar{u}_{m-i}}_{\in \mathcal{L}(\mathcal{B}'_U) \subseteq U} \neq \underline{0}$$

quindi il vettore $\bar{v} = -\alpha_1 \bar{u}_1 \dots - \alpha_{m-i} \bar{u}_{m-i} \in U \cap W$

$\Rightarrow \bar{v} = \delta_1 \bar{e}_1 + \dots + \delta_i \bar{e}_i$ con $(\delta_1 \dots \delta_i) \neq (0 0 \dots 0)$.

$$\beta_1\bar{e}_1 + \ldots + \beta_i\bar{e}_i + \gamma_1\bar{w}_1 + \ldots + \gamma_{\ell-i}\bar{w}_{\ell-i} =$$

$$= \delta_1\bar{e}_1 + \ldots + \delta_i\bar{e}_i$$

$$(\beta_1-\delta_1)\bar{e}_1 + \ldots + (\beta_i-\delta_i)\bar{e}_i + \gamma_1\bar{w}_1 + \ldots + \gamma_{\ell-i}\bar{w}_{\ell-i} = \underline{0}$$

c. lineare di vettori di $\mathcal{B}_w$ che è libera $\Rightarrow$

$$\beta_1-\delta_1 = 0, \quad \beta_2-\delta_2 = 0 \quad \ldots \quad \beta_i-\delta_i = 0$$

$$\gamma_1 = 0 \quad \gamma_2 = 0 \ldots \gamma_{\ell-i} = 0$$

$$\Rightarrow \underline{0} = \alpha_1\bar{u}_1 + \alpha_2\bar{u}_2 + \ldots + \alpha_{m-i}\bar{u}_{m-i} + \beta_1\bar{e}_1 + \ldots + \beta_i\bar{e}_i +$$

$$\cancel{\gamma_1\overline{ev}_1 + \ldots + \gamma_{\ell-i}\bar{w}_{\ell-i}} = \alpha_1\bar{u}_1 + \ldots + \alpha_{m-i}\bar{u}_{m-i} +$$

$$+ \beta_1\bar{e}_1 + \ldots + \beta_i\bar{e}_i$$

ma adesso abbiamo una c. lineare a coeff. non tutti nulli (perché almeno un $a_j \neq 0$) di vettori di $\mathcal{B}_U$ che dà $\underline{0}$ $\leadsto$ ASSURDO.

Ne segue che $\mathcal{B}'_U \cup \mathcal{B}_I \cup \mathcal{B}'_W$ è libera.

$\Rightarrow \dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$ $\square$

$\mathbb{R}^{2,3}$ $\qquad W = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \;\middle|\; \begin{array}{l} a_{11} + a_{13} = 0 \\ a_{22} + a_{12} = 0 \end{array} \right\}.$

U tale che $\dim(U + W) = 6$

$\dim(U \cap W) = 2$

$\dim \mathbb{R}^{2,3} = 6 \qquad \dim W = 4$

$$W = \left\{ \begin{pmatrix} a_{11} & a_{12} & -a_{11} \\ a_{21} & -a_{12} & a_{23} \end{pmatrix} \;\middle|\; a_{ij} \in \mathbb{R} \right\}.$$

$$\mathcal{B}_W = \left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

$$\mathcal{B}_{\mathbb{R}^{2,3}} = \Big(\underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{W}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big)$$

$\mathcal{U}$

$$\mathcal{U} = \mathcal{L}\left(\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$\mathcal{U} = \left\{ a + bx + cx^2 + dx^3 \;\middle|\; \begin{matrix} a + b + c = 0 \\ b - c - d = 1 \end{matrix} \right\}.$$

$$W_k = \langle k + (k+1)x^2 \rangle \qquad \dim\left(W_k + \langle \mathcal{U} \rangle\right) = 3$$

OSS: $\mathcal{U}$ non è s.vettoriale perché $0 \notin \mathcal{U}$

I vettori di U sono tutti del tipo

$$\left\{ \underbrace{(-b-c) + bx + cx^2 + (b-c)x^3}_{\substack{\text{risolve } a+b+c=0 \\ b-c-d=0}} + \underbrace{x}_{\substack{\text{risolve } a+b+c=0 \\ b-c-d=1 \\ \text{come sol. particolare}}} \;\middle|\; b, c \in \mathbb{R} \right\}.$$

$$\mathcal{L}(U) = \mathcal{L}\left(x,\ 1-x+x^3,\ 1-x^2+x^3 \right)$$

$$\dim(U) = 3$$

Fissate base $\mathbb{R}_{\leq 3}[x] = (1, x, x^2, x^3)$

osserviamo che in componenti i vettori

di U sono

$(-b-c \quad b+c \quad b-c)$

$\downarrow$

$\mathcal{L}(\mathcal{U})$ si ottiene in componenti dalle c. lineari di

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \qquad \rho = 3$$

$W_k = \mathcal{L}(k + (k+1)x^2) \rightarrow$ rapp. in componenti

$(k \quad 0 \quad 0 \quad k+1)$

$\mathcal{L}(\mathcal{U}) + W_k$ è in componenti generata da $\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ k & 0 & 0 & k+1 \end{bmatrix} = B$

$\dim \mathcal{L}(\mathcal{U}) + W_k = 3$ cioè $\rho(B) = 3$

$\det(B) = 0$

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ k & 0 & 0 & k+1 \end{vmatrix} = - \begin{vmatrix} -1 & 0 & 1 \\ -1 & 1 & -1 \\ k & 0 & k+1 \end{vmatrix} =$$

$$= - \begin{vmatrix} -1 & 1 \\ k & k+1 \end{vmatrix} =$$

$$= -k - 1 - k = -2k - 1$$

$k = -\frac{1}{2} \rightarrow \dim \mathcal{L}(U) + W_k = 3$

$k \neq -\frac{1}{2} \rightarrow \dim \mathcal{L}(U) + W_k = 4$

$$\begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \\ (a,b,c) \longrightarrow (a,b) \end{cases}$$

NON È ISOMORFISMO

$$\mathbb{R}_{\leq 3}[x] \longrightarrow \mathbb{R}^4$$

$$\{f(x) \in \mathbb{R}[x] : \deg f \leq 3\}$$

$$a + bx + cx^2 + dx^3 \longrightarrow (a \ b \ c \ d)$$

BASE DEL SOTT. GENERATO DA $\{x, x^2, x^2+3x, x^3-x^2+x, x^3\}$.

$$\begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 3 & 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0+1 & -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\mathcal{B} = (x, x^2, x^3)$$

$$\mathbb{K}^{2,3} \longrightarrow \mathbb{K}^6$$

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \longrightarrow (a\ b\ c\ d\ e\ f)$$

$$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$