Geometries arising from trilinear forms on low-dimensional vector spaces

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Abstract

Let $\mathcal{G}_k(V)$ be the k-Grassmannian of a vector space V with $\dim V = n$. Given a hyperplane H of $\mathcal{G}_k(V)$, we define in [3] a point-line subgeometry of $\mathrm{PG}(V)$ called the *geometry of poles of* H. In the present paper, exploiting the classification of alternating trilinear forms in low dimension, we characterize the possible geometries of poles arising for $k = 3$ and $n \leq 7$ and propose some new constructions. We also extend a result of [6] regarding the existence of line spreads of $\mathrm{PG}(5, \mathbb{K})$ arising from hyperplanes of $\mathcal{G}_3(V)$.

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1 Introduction

Denote by V a n-dimensional vector space over a field $\mathbb{K}$. For any fixed $1 \leq k < n$ the k-Grassmannian $\mathcal{G}_k(V)$ of V is the point-line geometry whose points are the k–dimensional vector subspaces of V and whose lines are the sets $\ell_{Y,Z} := \{X : Y \subset X \subset Z, \ \dim X = k\}$ where Y and Z are subspaces of V with $Y \subset Z$, $\dim Y = k-1$ and $\dim Z = k+1$. Incidence is containment.

It is well known that the geometry $\mathcal{G}_k(V)$ affords a full projective embedding, called Grassmann (or Plücker) embedding and denoted by ε_k, sending every k-subspace $\langle v_1, \dots, v_k\rangle$ of V to the point $[v_1 \wedge \cdots \wedge v_k]$ of $\mathrm{PG}(\bigwedge^k V)$, where we adopt the notation $[u]$ to refer to the projective point represented by the vector u. Also, for $X \subseteq V$ we put $[X] := \{[x] : x \in \langle X\rangle\}$ for the points of the projective space induced by $\langle X\rangle$.

A hyperplane H of $\mathcal{G}_k(V)$ is a proper subspace of $\mathcal{G}_k(V)$ such that any line of $\mathcal{G}_k(V)$ is either contained in H or it intersects H in just one point.

It is well known, see Shult [13] and also Havlicek [10], Havlicek and Zanella [11] and De Bruyn [5], that the hyperplanes of $\mathcal{G}_k(V)$ all arise from the Plücker embedding of $\mathcal{G}_k(V)$ in $\mathrm{PG}(\bigwedge^k V)$, i.e. they bijectively correspond to proportionality classes of non-zero linear functionals on $\bigwedge^k V$. More in detail, for any hyperplane H of $\mathcal{G}_k(V)$ there is a non-null linear functional h on $\bigwedge^k V$ such that $H = \varepsilon_k^{-1}([\ker(h)] \cap \varepsilon_k(\mathcal{G}_k(V)))$. Equivalently, if $\chi_h : V \times \ldots \times V \to \mathbb{K}$ is the alternating k-linear form on V associated to the linear functional h, defined by the clause $\chi_h(v_1, \ldots, v_k) := h(v_1 \wedge \cdots \wedge v_k)$, then the hyperplane H is the set of the k-subspaces of V where χ_h identically vanishes. So, the hyperplanes of $\mathcal{G}_k(V)$ bijectively correspond also to proportionality classes of non-trivial alternating k-linear forms of V.

In a recent paper [3] we introduced the notion of *i-radical* of a hyperplane H of $\mathcal{G}_k(V)$. In the present work we shall just consider the case of the *lower radical* $R_\downarrow(H)$, for $i = 1$, and that of the *upper radical* $R^\uparrow(H)$, for $i = k-1$. The lower radical $R_\downarrow(H)$ of H is the set of points $[p] \in \mathrm{PG}(V)$ such that all k-spaces X with $p \in X$ belong to H; the upper radical $R^\uparrow(H)$ of H is the set of $(k-1)$-subspaces Y of V such that all k-spaces through Y belong to H.

In the same paper [3] we investigated the problem of determining under which conditions the upper radical of a given hyperplane might be empty. Working in the case $k = 3$, we also defined a point-line subgeometry $\mathcal{P}(H) = (P(H), R^\uparrow(H))$ of $\mathrm{PG}(V)$ called *the geometry of poles* of H, whose points are called H-poles, a point $[p] \in P(H)$ and a line $\ell \in R^\uparrow(H)$ being incident when $p \in \ell$ (see Section 1.1). Some aspects of this geometry had already been studied, under slightly different settings, in [6, 7]. It has been shown in [6] that the set $P(H)$ is actually either $\mathrm{PG}(V)$ or an algebraic hypersurface in $\mathrm{PG}(V)$; for more details see Section 1.1.

In this paper we shall focus on the geometry $\mathcal{P}(H)$, providing explicit equations for its points and lines and also a geometric description in the cases where a complete classification of trilinear forms on a vector space V is available, namely $\dim(V) \leq 6$ with $\mathbb{K}$ arbitrary (see [12]) and $\dim(V) = 7$ with $\mathbb{K}$ a perfect field with cohomological dimension at most 1 (see [4]).

We shall briefly recall the definition of geometry of poles in the next section and we will state our main results in Section 1.2. The organization of the paper is outlined in Section 1.3.

1.1 The geometry of poles

Assume $k = 3$ and let H be a given hyperplane of $\mathcal{G}_3(V)$. For any (possibly empty) projective space $[X]$ we shall denote by $\dim(X)$ the vector dimension of X.

Let $[p]$ be a point of $\mathrm{PG}(V)$ and consider the point-line geometry $\mathcal{S}_p(H)$ having as points, the set of lines of $\mathrm{PG}(V)$ through $[p]$ and as lines, the set of planes $[\pi]$ of $\mathrm{PG}(V)$ through $[p]$ with $\pi \in H$. It is easy to see (see [3]) that $\mathcal{S}_p(H)$ is a polar space of symplectic type (possibly a trivial one). Let $R_p(H) := \mathrm{Rad}\,(\mathcal{S}_p(H))$ be the radical of $\mathcal{S}_p(H)$ and put $\delta(p) := \dim(\mathrm{Rad}\,(\mathcal{S}_p(H)))$.

We call $\delta(p)$ the *degree* of $[p]$ (relative to H). If $\delta(p) = 0$ then we say that $[p]$ is *smooth*, otherwise we call $[p]$ a *pole* of H or, also, a H-*pole* for short. Clearly, a point is a pole if and only if it belongs to a line of the upper radical $R^{\uparrow}(H)$ of H. So, $R^{\uparrow}(H) = \emptyset$ if and only if all points are smooth.

As the vector space underlying the symplectic polar space $\mathcal{S}_p(H)$ has dimension $n-1$, $\delta(p)$ is even when n is odd and it is odd if n is even. In particular, when n is even all the points are poles of degree at least 1. If they all have degree 1, then we say that H is *spread-like*.

We shall provide in Theorem 2 a direct geometric proof of the result of [3, Theorem 4], stating that for $n = 6$ there are spread-like hyperplanes if and only if the field $\mathbb{K}$ is not quadratically closed, extending a result of [7]; this is also implicit in the classification of [12]; observe that for $n = 8$ we prove in [3] that for quasi-algebraically closed fields there are no spread-like hyperplanes.

More in general, when all points of $\mathrm{PG}(V)$ are poles of the same degree δ and $(\delta+1)|n$, the set $\{\pi_p : [p] \in \mathrm{PG}(V)\}$ of all subspaces $\pi_p := \{[u] \in [\ell] : p \in \ell \text{ and } \ell \in R^{\uparrow}(H)\}$ as $[p]$ varies in $\mathrm{PG}(V)$ might in some case possibly be a spread of $\mathrm{PG}(V)$ in spaces of projective dimension δ. We are not currently aware of any case where this happens for $\delta > 1$ and we propose this as a problem which might be worthy of further investigation.

1.2 Main results

Before stating our main results we need to fix a terminology for linear functionals on $\bigwedge^3 V$ and to recall what is currently known from the literature regarding their classification. As already pointed out, a classification of alternating trilinear forms of V would determine a classification of the geometries of poles defined by hyperplanes of $\mathcal{G}_3(V)$.

We will first introduce the notion of isomorphism for hyperplanes and the notions of equivalence, near equivalence and geometrical equivalence for k-linear forms in general.

We say that two hyperplanes H and H' of $\mathcal{G}_k(V)$ are *isomorphic*, and we write $H \cong H'$, when $H' = g(H) := \{g(X)\}_{X\in H}$ for some $g \in \mathrm{GL}(V)$, where $g(X)$ is the natural action of g on the subspace X, i.e. $g(X) = \langle g(v_1), \ldots, g(v_k)\rangle$ for $X = \langle v_1, \ldots, v_k\rangle$.

Recall that two alternating k-linear forms χ and χ' on V are said to be *(linearly) equivalent* when

$$\chi'(x_1, \ldots, x_k) \;=\; \chi(g(x_1), \ldots, g(x_k)), \quad \forall x_1, \ldots, x_k \in V$$

for some $g \in \mathrm{GL}(V)$. Accordingly, if H and H' are the hyperplanes associated to χ and χ', we have $H \cong H'$ if and only if χ' is proportional to a form equivalent to χ. Note that if $\chi' = \lambda \cdot \chi$ for a scalar $\lambda \neq 0$ then χ and χ' are equivalent if and only if λ is a k-th power in $\mathbb{K}$.

We say that two forms χ and χ' are *nearly equivalent*, and we write $\chi \sim \chi'$, when each of them is equivalent to a non-zero scalar multiple of the other. Hence $H \cong H'$ if and only if $\chi \sim \chi'$.

We extend the above terminology to linear functionals of $\bigwedge^k V$ in a natural way, saying that two linear functionals $h, h' \in (\bigwedge^k V)^*$ are *nearly equivalent* and writing $h \sim h'$ when their corresponding k-alternating forms are nearly equivalent.

We say that two hyperplanes H and H' are *geometrically equivalent* if the incidence graphs of their geometries of poles are isomorphic; the forms defining geometrically equivalent hyperplanes are called *geometrically equivalent* as well.

Note that nearly equivalent forms are always geometrically equivalent but the converse does not hold in general. For example, let V be a vector space over a field $\mathbb{K}$ which is not quadratically closed and suppose $\dim(V) = 6$. To any quadratic extension of $\mathbb{K}$ there correspond a Desarguesian line-spread $\mathcal{S}$ of $\mathrm{PG}(V)$, and the geometry $(\mathrm{PG}(V), \mathcal{S})$ is a geometry of poles associated to a trilinear form. All hyperplanes inducing line-spreads are geometrically equivalent. If $\mathbb{K}$ is a finite field or $\mathbb{K} = \mathbb{R}$, it is easy to see that the hyperplanes inducing $\mathcal{S}$ must also be isomorphic. However, this is not the case when $\mathbb{K} = \mathbb{Q}$ or when $\mathbb{K}$ is a field of characteristic 2 which is not perfect. In particular, in the latter case hyperplanes arising from forms of type $T^{(1)}_{10,\lambda}$ and $T^{(2)}_{10,\lambda}$, see Table 1, are geometrically equivalent but not isomorphic.

Clearly, nearly equivalent or isomorphic hyperplanes are always geometrically equivalent.

1.2.1 Types for linear functionals of $\bigwedge^3 V$

Given a non-trivial linear functional $h \in (\bigwedge^3 V)^*$, let χ_h and H_h be respectively the alternating trilinear form and the hyperplane of $\mathcal{G}_3(V)$ associated to it. When no ambiguity might arise, we shall feel free to drop the subscript h in our notation.

By definition, $R_{\downarrow}(H) = [\mathrm{Rad}\,(\chi)]$, where $\mathrm{Rad}\,(\chi) = \{v \in V \colon \chi(x, y, v) = 0, \forall\, x, y \in V\}$. Define the *rank* of h as $\mathrm{rank}\,(h) := \mathrm{cod}_V(\mathrm{Rad}\,(\chi)) = \dim(V/\mathrm{Rad}\,(\chi))$. Obviously, functionals of different rank can never be nearly equivalent.

It is known that if h is a non-trivial trilinear form, then $\mathrm{rank}\,(h) \geq 3$ and $\mathrm{rank}\,(h) \neq 4$ (see [3, Proposition 19] for the latter result).

Fix now a basis $E := (e_i)_{i=1}^n$ of V. The dual basis of E in V^* is $E^* := (e^i)_{i=1}^n$, where $e^i \in V^*$ is the linear functional such that $e^i(e_j) = \delta_{i,j}$ (Kronecker symbol). The set $(e^i \wedge e^j \wedge e^k)_{1 \leq i < j < k \leq n}$ is the basis of $(\bigwedge^3 V)^*$ dual of the basis $(e_i \wedge e_j \wedge e_k)_{1 \leq i < j < k \leq n}$ of $\bigwedge^3 V$ canonically associated to E. We shall adopt the convention of writing $\underline{ijk}$ for $e^i \wedge e^j \wedge e^k$, thus representing linear functionals of $\bigwedge^3 V$ as linear combinations of symbols like $\underline{ijk}$.

In Table 1, see Appendix A, we list a number of possible types of linear functionals of $\bigwedge^3 V$ of rank at most 7, denoted by the symbols $T_1, \ldots, T_9$ and $T^{(1)}_{10,\lambda}$, $T^{(2)}_{10,\lambda}$, $T^{(1)}_{11,\lambda}$ and $T^{(2)}_{11,\lambda}$, $T_{12,\mu}$ where λ is a scalar subject to the conditions specified in the table. Whenever T is one of the types of Table 1, we say that $h \in (\bigwedge^3 V)^*$ is of *type* T if h is nearly equivalent to the linear functional described at row T of Table 1. The *type* of H_h or χ_h is the type of h. By definition, functionals of the same type are nearly equivalent. Note that the definitions of each of these types make sense for any n and for any field $\mathbb{K}$, provided that n is not smaller than the rank of (a linear functional admitting) that description and that $\mathbb{K}$ contains elements satisfying the special conditions there outlined.

In particular, Table 1 provides a complete classification (up to equivalence) in the case of perfect fields of cohomological dimension at most 1 and $\dim(V) \leq 7$, see [4, 12]. We recall that in [4] the authors also determine the full automorphism group associated to each form.

It is well known that a general classification of trilinear forms up to equivalence is hopeless; for instance for $\mathbb{K} = \mathbb{C}$ and $n = 9$ there are infinite families of linearly inequivalent trilinear forms; see e.g. [6].

When $n \leq 6$, Revoy [12] proves that all trilinear forms, up to equivalence, are of type T_i, $1 \leq i \leq 4$, $T^{(1)}_{10,\lambda}$, $T^{(2)}_{10,\lambda}$. In particular, if the field is quadratically closed, all forms of rank 6 are either of type T_3 or of type T_4. If the field is not quadratically closed, it is possible to distinguish two families of classes of forms linearly equivalent among themselves according as they are equivalent to T_3 or to T_4 over the quadratic closure $\mathbb{K}^\square$ of $\mathbb{K}$. More in detail, if $\mathrm{char}\,(\mathbb{K})$ is odd, $T^{(1)}_{10,\lambda} = T^{(2)}_{10,\lambda}$ and each form of a type in $T^{(1)}_{10,\lambda}$ is equivalent to T_3 over $\mathbb{K}^\square$. If $\mathrm{char}\,(\mathbb{K}) = 2$, then the classes $T^{(2)}_{10,\lambda}$ and $T^{(1)}_{10,\lambda}$ are in correspondence with respectively the separable and the inseparable quadratic extensions of the field $\mathbb{K}$; furthermore, any form of a type in $T^{(2)}_{10,\lambda}$ is equivalent to a form of type T_3 in $\mathbb{K}^\square$, while any form of type $T^{(1)}_{10,\lambda}$ is equivalent in $\mathbb{K}^\square$ to a form of type T_4.

If $n = 7$ and $\mathbb{K}$ is a perfect field of cohomological dimension at most 1, Cohen and Helmick [4] show that all trilinear forms, up to equivalence, are of a type described in Table 1.

Under the conditions of Table 1, we shall provide a classification for the geometries of poles.

We now present our main results; for the notions of extension, expansion and block decomposition as well as some of the notation, see Section 3. By the symbol $|x, y|_{ij}$ we mean the (i, j)-Plücker coordinate of the line $[x, y]$ spanned by the vectors $x = (x_i)^n_{i=1}$ and $y = (y_i)^n_{i=1}$ written in coordinates with respect to the basis E, i.e. $|x, y|_{ij} := x_i y_j - x_j y_i$ is the ij-coordinate of $e_i \wedge e_j$ with respect to the basis $(e_i \wedge e_j)_{1 \le i < j \le n}$.

Theorem 1. *Suppose* $\dim(V) \le 6$ *and let* h *be a non-trivial linear functional on* $\bigwedge^3 V$ *having type as described in Table 1, with associated alternating trilinear form* χ. *Denote by* H *the hyperplane of* $\mathcal{G}_3(V)$ *defined by* h. *Then one of the following occurs:*

1. *h has type T_1 (rank 3). In this case H is the trivial hyperplane centered at $\mathrm{Rad}(\chi)$ and $R^{\uparrow}(H)$ is the set of the lines of $\mathrm{PG}(V)$ that meet $[\mathrm{Rad}(\chi)]$ non-trivially.*

2. *h has type T_2 (rank 5), namely $\mathrm{Rad}(\chi)$ is 1-dimensional. In this case H is a trivial extension $\mathrm{Ext}_{\mathrm{Rad}(\chi)}(\mathrm{Exp}(H_0))$ of a symplectic hyperplane $\mathrm{Exp}(H_0)$, constructed in a complement V_0 of $\mathrm{Rad}(\chi)$ in V starting from the line-set H_0 of a symplectic generalized quadrangle. The elements of $R^{\uparrow}(H)$ are the lines of $\mathrm{PG}(V)$ that either belong to H_0 or pass through the point $[\mathrm{Rad}(\chi)] = R_{\downarrow}(H)$ or such that their projection onto V_0 is in H_0.*

3. *h has type T_3 (rank 6). Then H is a decomposable hyperplane $\mathrm{Dec}(H_0, H_1)$ arising form the hyperplanes H_0 and H_1 of $\mathcal{G}_3(V_0)$ and $\mathcal{G}_3(V_1)$ for a suitable decomposition $V = V_0 \oplus V_1$ with $\dim(V_0) = \dim(V_1) = 3$. Then $R^{\uparrow}(H) = \{[x, y] \colon x \in V_0 \setminus \{0\},\ y \in V_1 \setminus \{0\}\}$.*

4. *h has type T_4 (rank 6). Then $R^{\uparrow}(H) = \{[x, y] = [a+b, \omega(a)] \colon a \in V_0 \setminus \{0\}, b \in V_1\} \cup \{[x, y] \subseteq V_1\}$ for a decomposition $V = V_0 \oplus V_1$ with $\dim(V_0) = \dim(V_1) = 3$ and ω an isomorphism of V interchanging V_0 and V_1.*

5. *h has type $T^{(1)}_{10,\lambda}$ or $T^{(2)}_{10,\lambda}$ (rank 6). Then $R^{\uparrow}(H)$ is a Desarguesian line spread of $\mathrm{PG}(V)$ corresponding to the field extension $\mathbb{K}[\mu]$ with μ a root of $p_\lambda(t)$.*

Theorem 2. *Let* $V := V(6, \mathbb{K})$. *Line-spreads of* $\mathrm{PG}(V)$ *induced by hyperplanes of* $\mathcal{G}_3(V)$ *exist if and only if* $\mathbb{K}$ *is a non-quadratically closed field.*

Draisma and Shaw prove that when $\mathbb{K}$ is a finite field there always exist hyperplanes of $\mathcal{G}_3(V)$ with $\dim V = 6$ having a Desarguesian spread as upper radical [7, §3.1 and §3.2], while these hyperplanes do not exist when $\mathbb{K}$ is algebraically closed with characteristic 0, [7, Remark 9].

Our Theorem 2 generalizes their results to arbitrary fields $\mathbb{K}$ and provides necessary and sufficient conditions for the existence of spread-like hyperplanes for $n = 6$; its statement correspond to Theorem 20, point 5 in [3] (and also to Theorem 4 in [3]).

It also further clarifies the result of [12] linking the forms $T^{(i)}_{10,\lambda}$ with quadratic extensions of the field $\mathbb{K}$.

Theorem 3. *Suppose* $\dim(V) = 7$ *and let* h *be a non-trivial linear functional on* $\bigwedge^3 V$ *having type as described in Table 1. Denote by* χ *the associated alternating trilinear form and by* H *the hyperplane of* $\mathcal{G}_3(V)$ *defined by* h.

If $\mathrm{rank}\,(h) \leq 6$ *then* H *is a trivial extension* $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(H')$ *where* H' *is a hyperplane of* $\mathcal{G}_3(V')$ *with* $V = \mathrm{Rad}\,(h) \oplus V'$, $\dim(V') \leq 6$, *and* H' *is defined by a trilinear form* h' *of type as in Theorem 1.*

If $\mathrm{rank}\,(h) = 7$ *then one of the following occurs:*

1) h *has type* T_5. *Then, there are two non-degenerate symplectic polar spaces* $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *embedded as distinct hyperplanes in* $\mathrm{PG}(V)$ *such that both determine the same polar space* $\mathcal{S}_0$ *on their intersection. The radical of* $\mathcal{S}_0$ *is a point, say* p_0. *There are also two totally isotropic planes* A_1 *and* A_2 *of* $\mathcal{S}_0$ *such that* $A_1 \cap A_2 = \{p_0\}$. *The poles of* H *are the points of* $\mathcal{S}_1 \cup \mathcal{S}_2$, *the poles of degree* 4 *being the points of* $A_1 \cup A_2$. *The lines of* $\mathcal{P}(H)$ *are the totally isotropic lines of* $\mathcal{S}_i$ *that meet* A_i *non-trivially, for* $i = 1, 2$.

2) h *has type* T_6. *The poles of* H *lie in a hyperplane* $\mathcal{S}$ *of* $\mathrm{PG}(V)$. *A non-degenerate polar space of symplectic type is defined in* $\mathcal{S}$ *and a totally isotropic plane* A *of* $\mathcal{S}$ *is given. The lines of* $\mathcal{P}(H)$ *are the totally isotropic lines of* $\mathcal{S}$ *that meet* A *non-trivially. The points of* A *are the poles of* H *of degree* 4.

3) h *has type* T_7. *The poles of* H *are the points of a cone of* $\mathrm{PG}(V)$ *having as vertex a plane* A *and as basis a hyperbolic quadric* $\mathcal{Q}$. *A conic* $\mathcal{C}$ *is given in* A *such that the elements of* $\mathcal{C}$ *are the poles of degree* 4. *There is a correspondence mapping each point* $[x] \in \mathcal{C}$ *to a line* ℓ_x *contained in a regulus of* $\mathcal{Q}$. *For each* $[x] \in \mathcal{C}$ *it is possible to define a line spread* $\mathcal{S}_x$ *of* $\langle A, \ell_x \rangle / \langle x \rangle \cong \mathrm{PG}(3, \mathbb{K})$ *such that* $R^{\uparrow}(H) = \{\ell \subseteq \langle x, s \rangle : [x] \in \mathcal{C}, s \in \mathcal{S}_x\}$.

4) h *has type* T_8. *In this case* $H = \mathrm{Exp}(H_0)$ *is a symplectic hyperplane. In particular, the geometry* $\mathcal{P}(H)$ *is a non-degenerate polar space of symplectic type and rank* 3, *naturally embedded in a hyperplane* $[V_0]$ *of* $\mathrm{PG}(V)$. *All poles of* H *have degree* 4.

5) h *has type* T_9. *Then* $\mathcal{P}(H)$ *is a split-Cayley hexagon naturally embedded in a non-singular quadric of* $\mathrm{PG}(V)$. *All poles of* H *have degree* 2.

6) h *has type* $T^{(i)}_{11,\lambda}$, $i = 1, 2$. *The poles of* H *are the points of a subspace* $\mathcal{S}$ *of codimension* 2 *in* V. *There is only one point* $[p] \in \mathcal{S}$ *which is a pole of degree* 4. *Furthermore, there is a line-spread* $\mathcal{F}$ *of* $\mathcal{S}/\langle p \rangle \cong \mathrm{PG}(3, \mathbb{K})$ *such that* $R^{\uparrow}(H) := \{\ell \subseteq [\pi, p] : \pi \in \mathcal{F}\}$.

Theorems 1 and 3 correspond to Theorems 20 and 21 of [3], where they were presented without a detailed proof. In the present paper we have chosen to refine the results announced [3], by providing a fully geometric description of the geometries of poles arising for $n \leq 7$, without having to recourse to coordinates. In any case, the original statements for cases 3,5 and 7 of [3, Theorem 21] can be immediately deduced from Theorem 3 in light of the equations of Table 5.

1.3 Organization of the paper

In Section 2 we shall explain how to algebraically describe points and lines of a geometry of poles. Draisma and Shaw [6] have shown that either the set of H-poles is all of $\mathrm{PG}(V)$ or it determines an algebraic hypersurface in $\mathrm{PG}(V)$ described by an equation of degree $(n-3)/2$. In Section 2 we shall study such varieties. In particular, in Section 2.2 we shall explicitly determine their equations as determinantal varieties and in Section 2.3 describe some hyperplanes whose variety of poles is reducible in the product of distinct linear factors. In Section 3 we will present three families of hyperplanes of $\mathcal{G}_k(V)$ obtained by *extension*, *expansion* and *block decomposable construction*. Our main theorems will be proved in Section 4. For the ease of the reader, all the tables are collected in Appendix A.

2 Geometry of poles

Throughout this section we take $E = (e_i)_{1 \leq i \leq n}$ as a given basis of V and the coordinates of vectors of V will be given with respect to E.

2.1 Determination of points and lines

Let $h\colon \bigwedge^3 V \to \mathbb{K}$ be a linear functional associated to a given hyperplane H of $\mathcal{G}_3(V)$, where V is a n-dimensional vector space over a field $\mathbb{K}$.

For any $u \in V$ consider the bilinear alternating form

$$h_u\colon V/\langle u\rangle \times V/\langle u\rangle \to \mathbb{K},\ \ h_u(x+\langle u\rangle, y+\langle u\rangle) := h(u \wedge x \wedge y).$$

By definition of H-pole, a point $[u] \in \mathrm{PG}(V)$ is a H-pole if and only if the radical of h_u is not trivial. Consider also the bilinear alternating form on V

$$\chi_u\colon V \times V \to \mathbb{K},\ \ \chi_u(x,y) = h(u \wedge x \wedge y) = x^T M_u y \tag{1}$$

where M_u is the matrix associated to χ_u with respect to the basis E of V. Clearly, $\mathrm{Rad}\,(h_u) = (\mathrm{Rad}\,(\chi_u))/\langle u\rangle$; thus the rank of the matrix of h_u with respect to any basis of $V/\langle u\rangle$ and the rank of the matrix M_u are exactly the same.

Proposition 2.1. *Let $[u]$ be a point of $\mathrm{PG}(V)$ with $u = (u_i)_{i=1}^n$ and let M_u be the $n \times n$-matrix associated to the alternating bilinear form χ_u. If $u_i \neq 0$ then the i-th column (row) of M_u is a linear combination of the other columns (rows) of M_u.*

Proof. Denote by $C_1, \dots C_n$ the columns of the matrix M_u and let $x = (x_i)_{i=1}^n$ and $u = (u_i)_{i=1}^n$. Let $M_u^{(i)}$ be the $(n-1) \times (n-1)$-submatrix of M_u obtained by deleting its i-th column and i-th row. For any $x \in V$, the condition $x^T M_u u = 0$ is equivalent to

$$(x^T \cdot C_1)u_1 + (x^T \cdot C_2)u_2 + \cdots + (x^T \cdot C_i)u_i + \cdots + (x^T \cdot C_n)u_n = 0,$$

where $x^T \cdot C_i := \sum_{j=1}^n x_j c_{ji}$ with $C_i = (c_{ji})_{j=1}^n$. Since $u_i \neq 0$, we have

$$(x^T \cdot C_1)\frac{u_1}{u_i} + (x^T \cdot C_2)\frac{u_2}{u_i} + \cdots + x^T \cdot C_i + \cdots + (x^T \cdot C_n)\frac{u_n}{u_i} = 0,$$

i.e.

$$x^T \cdot (\frac{u_1}{u_i}C_1 + \cdots + C_i + \cdots + \frac{u_n}{u_i}C_n) = 0.$$

The previous condition holds for any $x \in V$, hence

$$C_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{u_j}{u_i} C_j.$$

As M_u is antisymmetric, the same argument can be applied also to the i-th row of M_u. □

The following corollaries are straightforward.

Corollary 2.2. $\text{rank}\,(M_u) \leq n-1$.

Corollary 2.3. *The point* $[u]$ *is a* H*-pole if and only if* $\text{rank}\,(M_u) \leq n-2$.

Note that the matrix M_u is antisymmetric; hence its rank must be an even number. By Corollary 2.3 it is clear that if n is even then every point of $\text{PG}(V)$ is a H-pole and this holds for any hyperplane H of $\mathcal{G}_3(V)$. More precisely, the degree of the point $[u]$ is $\delta(u) = (n-1) - \text{rank}\,(M_u)$. So, it is straightforward to see that the set of all the H-poles of degree at least t is either the whole of $\text{PG}(V)$ or the determinantal variety of $\text{PG}(V)$ described by the condition $\text{rank}\,(M_u) \leq (n-1) - t \leq n-2$; see [9, Lecture 9] for some properties of these varieties. Furthermore all entries of M_u are linear homogeneous polynomials in the coordinates of u; so the condition $\text{rank}\,(M_u) \leq n-2$ provides algebraic conditions on the coordinates of u for $[u]$ to be a H-pole.

In Table 2 and Table 3 of Appendix A we have explicitly written the matrices M_u associated to the trilinear forms h of type T_i of Table 1 where $u = (u_i)_{i=1}^n$. To simplify the notation, when $\text{rank}\,(h) < \dim(V)$, we have just written the $\text{rank}\,(h) \times \text{rank}\,(h)$-matrix associated to $h_u|_{V/\text{Rad}\,(h)}$.

To determine the elements of the upper radical of H, namely the lines $\ell = [x, y]$ of $\mathrm{PG}(V)$ with the property that any plane through them is in H, we need to determine conditions on x and y such that the linear functional

$$\tilde{h}_{xy} \colon V \to \mathbb{K}, \ \tilde{h}_{xy}(u) = h(u \wedge x \wedge y) \tag{2}$$

is null. To do this, it is sufficient to require that $\tilde{h}_{xy}$ annihilates on the basis vectors of V, i.e. $\tilde{h}_{xy}(e_i) = 0$, for every $i = 1, \ldots, n$.

2.2 A determinantal variety

Let h be a trilinear form associated to the hyperplane H and for any $u \in V$, let χ_u be the alternating bilinear form as in Equation (1) whose representative matrix is M_u. With $1 \le i \le n$, denote by $M_u^{(i)}$ the principal submatrix of M_u obtained by deleting its i-th row and its i-th column. The matrix $M_u^{(i)}$ is a $(n-1) \times (n-1)$-antisymmetric matrix whose entries are linear functionals defined over $\mathbb{K}$; so, its determinant is a polynomial of degree $n-1$ in the unknowns $u_1, \ldots, u_n$ which is a square in the ring $\mathbb{K}[u_1, \ldots, u_n]$, that is there exists a polynomial $d_i(u_1, \ldots, u_n)$ with $\deg d_i(u_1, \ldots, u_n) = (n-1)/2$ such that

$$\det M_u^{(i)} = (d_i(u_1, \ldots, u_n))^2. \tag{3}$$

Define $g_i(u_1, \ldots, u_n)$ to be the polynomial in $\mathbb{K}[u_1, \ldots, u_n]$ such that

$$d_i(u_1, \ldots, u_n) = u_i^{\alpha_i} g_i(u_1, \ldots, u_n)$$

where $\alpha \in \mathbb{N}$ and $u_i^{\alpha_i+1}$ does not divide $d_i(u_1, \ldots, u_n)$.

Theorem 2.4. *The set $P(H)$ of H-poles is either the whole pointset of $\mathrm{PG}(V)$ or there exists an index i, $1 \le i \le n$, such that $P(H)$ is an algebraic hypersurface of $\mathrm{PG}(V)$ with equation $g_i(u_1, \ldots, u_n) = 0$.*

Proof. Suppose $P(H) \neq \mathrm{PG}(V)$. By [6], $P(H)$ is an algebraic hypersurface admitting an equation of degree $(n-3)/2$; so $P(H)$ contains at most $(n-3)/2$ coordinate hyperplanes of the form $\Pi_j : u_j = 0$. Then, there exists i with $1 \le i \le n$ such that $\Pi_i \not\subseteq P(H)$. By Corollary 2.3, $[u] \in \mathrm{PG}(V)$ is a H-pole if and only if $\mathrm{rank}\,(M_u) \le n-2$. If n is even, this always happens. So, assume n odd. We shall work over the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$. Take $u = (u_1, \ldots, u_n)$ with $u_j \in \overline{\mathbb{K}}$. We can regard u as a vector in coordinates with respect to the basis induced by E on $\overline{V} := V \otimes \overline{\mathbb{K}}$. In any case, the matrix M_u is antisymmetric and its entries are homogeneous linear functionals in $u_1, \ldots, u_n$ defined over the field $\mathbb{K}$.

Consider the points $[u]$ with $u_i \neq 0$. By Proposition 2.1, the i-th row/column of M_u is a linear combination of the remaining $n-1$ rows/columns. So, $\mathrm{rank}\, M_u = \mathrm{rank}\, M_u^{(i)}$.

Let $d_i(u_1,\ldots,u_n)$ be as in Equation (3). We now show that

$$d_i(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_n)=0$$

for all $u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n\in\overline{\mathbb{K}}$. Indeed, when $u_i=0$, by Proposition 2.1, there exists a column C_j of M_u, $u_j\neq 0$, such that

$$C_j=-\sum_{\substack{k=1\\k\neq j}}^{n}\frac{u_k}{u_j}C_k.$$

So, the j-th column of $M_u^{(i)}$ is also a linear combination of the other columns of $M_u^{(i)}$. Hence $\det M_u^{(i)}=0=(d_i(u_1,\ldots,0,\ldots,u_n))^2$.

Since $\overline{\mathbb{K}}$ is algebraically closed, we have

$$d_i(u_1,\ldots,u_n)=u_i d_i'(u_1,\ldots,u_n),$$

with $d_i'(u_1,\ldots,u_n)$ a polynomial in $\mathbb{K}[u_1,\ldots,u_n]$ with $\deg d_i'=(n-3)/2$. We remark that the unknowns of the polynomials may assume their values in $\overline{\mathbb{K}}$ but the coefficients are all in $\mathbb{K}$. By Corollary 2.3, a point $[u]\in \mathrm{PG}(\overline{V})\setminus\Pi_i$ (i.e. $u_i\neq 0$) is a H-pole if and only if $\det M_u^{(i)}=0$, i.e. $d_i'(u_1,\ldots,u_n)=0$.

Denote by $\overline{\Gamma}_i$ the algebraic variety (over $\overline{\mathbb{K}}$) of equation $d_i'(u_1,\ldots,u_n)=0$. Since we are assuming $\Pi_i\not\subseteq P(H)$ and $P(H)\setminus\Pi_i=\overline{\Gamma}_i\setminus\Pi_i$, we have

$$P(H)=C(P(H)\setminus\Pi_i)=C(\overline{\Gamma}_i\setminus\Pi_i),$$

where $C(X)$ denotes the projective closure of X in $\mathrm{PG}(\overline{V})$ with Π_i regarded as the hyperplane at infinity. Note that since $P(H)$ is an algebraic variety of $\mathrm{PG}(\overline{V})$ which does not contain Π_i, the points of $P(H)\cap\Pi_i$ are exactly those of the projective closure $C(P(H)\setminus\Pi_i)\cap\Pi_i$. The same applies to the points at infinity of the affine variety $\overline{\Gamma}_i\setminus\Pi_i$.

Suppose $u_i^{\beta}|d_i'(u_1,\ldots,u_n)$ and $u_i^{\beta+1}\nmid d_i'(u_1,\ldots,u_n)$ for $\beta\in\mathbb{N}$.

Then, $d_i'(u_1,\ldots,u_n)=u_i^{\beta}g_i(u_1,\ldots,u_n)$ and $g_i(u_1,\ldots,u_n)=0$ is an equation for $C(\overline{\Gamma}_i\setminus\Pi_i)=P(H)$. This completes the proof. □

Observe that if $\alpha_i=1$, then $g_i(u_1,\ldots,u_n)=0$ is exactly an equation of degree $(n-3)/2$ for $P(H)$.

If n is even then each point of $\mathrm{PG}(V)$ is a H-pole. Likewise, when n is odd and H is defined by a trilinear form χ of rank less than n, then also each point of $\mathrm{PG}(V)$ is a H-pole. Indeed, whenever $R_{\downarrow}(H)=\mathrm{Rad}\,(\chi)$ is non-trivial, for any $[u]\in\mathrm{PG}(V)$ there exists a line ℓ through $[u]$ in $R^{\uparrow}(H)$ which meets $\mathrm{Rad}\,(\chi)$ and so $[u]$ is a H-pole. In any case the above conditions, while sufficient, are not in general necessary; in fact, in Section 3.3 we shall provide a construction which might lead to alternating forms of rank n with n odd and $R_{\downarrow}(H) = \emptyset$ such that all points of $\mathrm{PG}(V)$ are H-poles.

Note that Theorem 2.4 shows that the variety of the H-poles in $\mathrm{PG}(V)$ admits at least one equation over $\mathbb{K}$ of degree at most $(n-3)/2$; this does not mean that $(n-3)/2$ is the minimum degree for such an equation or that the polynomial $g_i(u_1, \dots, u_n)$ generates the radical ideal of such variety (even over $\mathbb{K}$). For instance, in the case of symplectic hyperplanes, see Section 3.2.1, the variety of poles is always a hyperplane of $\mathrm{PG}(V)$ and so it admits an equation of degree 1 for any odd n.

2.3 A family of reducible hyperplanes

It is an interesting problem to investigate which algebraic varieties might arise as set of H-poles; as noted before such varieties will always be skew-symmetric determinantal varieties [8]. We leave the development of this study to a further paper. However, with this aim in mind, we give here a construction for hyperplanes whose variety of poles are reducible.

Theorem 2.5. *Suppose $V = V_1 + V_2$ with $(e_1, \dots, e_{n_1})$ and $(e_{n_1}, \dots, e_n)$ bases of V_1 and V_2 respectively. For $i = 1, 2$ let H_i be a hyperplane of $\mathcal{G}_3(V_i)$ whose H_i-poles in $\mathrm{PG}(V_i)$ satisfy respectively the equations $f_1(u_1, \dots, u_{n_1}) = 0$ and $f_2(u_{n_1}, \dots, u_n) = 0$. Then there exists a hyperplane H of $\mathcal{G}_3(V)$ whose set of H-poles defines a variety of $\mathrm{PG}(V)$ with equation $f(u_1, \dots, u_n) = 0$ where*

$$f(u_1, \dots, u_n) := u_{n_1} \cdot f_1(u_1, \dots, u_{n_1}) f_2(u_{n_1}, \dots, u_n).$$

Proof. For $i = 1, 2$ denote by $\overline{h}_i$ the trilinear form on V_i defining H_i and consider the extension $h_i \colon V \times V \times V \to \mathbb{K}$ given by

$$h_i(x, y, z) = \overline{h}_i(\pi_i(x), \pi_i(y), \pi_i(z))$$

where $\pi_1 \colon V \to V_1$ is the projection on V_1 along $\langle e_{n_1+1}, \dots e_n \rangle$ and $\pi_2 \colon V \to V_2$ is the projection on V_2 along $\langle e_1, \dots e_{n_1-1} \rangle$.

Put $h := h_1 + h_2$ and let H be the hyperplane of $\mathcal{G}_3(V)$ defined by h. For any $[u] \in \mathrm{PG}(V)$, let χ_u as in Equation (1) be the bilinear alternating form induced by h and represented by the matrix M_u with respect to the basis $(e_1, \dots, e_{n_1}, \dots, e_n)$ of V. By construction, the matrix M_u has the following structure:

$$M_u = \begin{pmatrix} M_1 & (v^1_{n_1}) & 0 \\ -(v^1_{n_1})^T & 0 & -(v^2_{n_1})^T \\ 0 & (v^2_{n_1}) & M_2 \end{pmatrix}$$

where $\bar{M}_1 := \begin{pmatrix} M_1 & (v^1_{n_1}) \\ -(v^1_{n_1})^T & 0 \end{pmatrix}$ is the $n_1 \times n_1$-matrix representing $\overline{h}_1$ and $\bar{M}_2 = \begin{pmatrix} 0 & -(v^2_{n_1})^T \\ (v^2_{n_1}) & M_2 \end{pmatrix}$ is the $(n - n_1 + 1) \times (n - n_1 + 1)$-matrix representing $\overline{h}_2$. Note that $(v^1_{n_1})$ and $(v^2_{n_1})$ are suitable columns with entries respectively in the rings $\mathbb{K}[u_1, \dots, u_{n_1}]$ and $\mathbb{K}[u_{n_1}, \dots, u_n]$.

Suppose $u_{n_1} = 0$. Since the entries of $\bar{M}_1$ and $\bar{M}_2$ are respectively linear functionals in $u_1, \dots, u_{n_1}$ and $u_{n_1}, \dots, u_n$ if all entries u_i of u with $i \geq n_1$ are null, then $\bar{M}_2$ is a zero matrix and $\operatorname{rank} M_u \leq n - 2$. Likewise if all entries u_i of u with $i \leq n_1$ are zero, then $\bar{M}_1$ is the zero matrix and $\operatorname{rank} M_u \leq n - 2$. Assume now that $u_{n_1} = 0$ and that there exist i, j with $i < n_1$ and $j > n_1$ such that $u_i \neq 0 \neq u_j$. Then, by Proposition 2.1, the first $(n_1 - 1)$ columns of $\bar{M}_1$ and the last $n - n_1$ columns of $\bar{M}_2$ are a linearly dependent set; in particular, $\operatorname{rank} M_u \leq n - 2$. So we have proved that the hyperplane $\Pi_{n_1} : u_{n_1} = 0$ is contained in the variety of poles $P(H)$.

By construction, $\operatorname{rank} \bar{M}_1 \leq n_1 - 2$ if and only if $f_1(u_1, \dots, u_{n_1}) = 0$ and $\operatorname{rank} \bar{M}_2 \leq n - n_1 - 1$ if and only if $f_2(u_{n_1}, \dots, u_n) = 0$. Let Δ be the variety of equation $f_1(u_1, \dots, u_{n_1}) f_2(u_{n_1}, \dots, u_n) = 0$. Observe that $\Delta \setminus \Pi_{n_1} \subseteq P(H) \setminus \Pi_{n_1}$, as $[u] \in \Delta \setminus \Pi_{n_1}$ implies that either $\operatorname{rank} \bar{M}_1^{(n_1)} \leq \operatorname{rank} \bar{M}_1 \leq n_1 - 2$ or $\operatorname{rank} \bar{M}_2^{(n_1)} \leq \operatorname{rank} \bar{M}_2 \leq n - n_1 - 1$ and, by Proposition 2.1 ($u_{n_1} \neq 0$), the n_1-th column of M is a linear combinations of the columns of $\bar{M}_1^{(n_1)}$ as well as a linear combination of the columns of $\bar{M}_2^{(n_1)}$ (so it does not contribute to the rank); so $\operatorname{rank} M_u \leq n - 2$ and $[u] \in P(H) \setminus \Pi_{n_1}$. Conversely, suppose $[u] \in P(H)$ with $u_{n_1} \neq 0$. Then, by Theorem 2.4,

$$0 = \det M_u^{(n_1)} = \det M_1 \cdot \det M_2 = \det \bar{M}_1^{(n_1)} \cdot \det \bar{M}_2^{(n_1)}.$$

If $[u] \in \mathrm{PG}(V) \setminus \Pi_{n_1}$, again by Theorem 2.4 applied to V_1 and V_2 we have $\det \bar{M}_1^{(n_1)} = 0$ if and only if $f_1(u_1, \dots, u_{n_1}) = 0$ and $\det \bar{M}_2^{(n_1)} = 0$ if and only if $f_2(u_{n_1}, \dots, u_n) = 0$. So $P(H) \setminus \Pi_{n_1} \subseteq \Delta \setminus \Pi_{n_1}$, whence

$$P(H) \setminus \Pi_{n_1} = \Delta \setminus \Pi_{n_1}$$

Since $\Pi_{n_1} \subseteq P(H)$, we have

$$P(H) = (P(H) \setminus \Pi_{n_1}) \cup \Pi_{n_1} = (\Delta \setminus \Pi_{n_1}) \cup \Pi_{n_1}$$

and, consequently $f(u_1, \dots, u_n) = u_{n_1} \cdot f_1(u_1, \dots, u_{n_1}) \cdot f_2(u_{n_1}, \dots, u_n)$ is an equation for $P(H)$. $\square$

Corollary 2.6. *Let V be a vector space of odd dimension $n \geq 5$ over a field $\mathbb{K}$. Then there exists a hyperplane H of $\mathcal{G}_3(V)$ whose set of H-poles is the union of $(n-3)/2$ distinct hyperplanes of* $\mathrm{PG}(V)$.

Proof. We proceed by induction on n odd. If $n = 5$, consider the hyperplane of $\mathcal{G}_3(V)$ defined by the trilinear form $h := \underline{123} + \underline{345}$. It is easy to verify that its poles are all the points of the hyperplane of equation $u_3 = 0$.

By induction hypothesis, suppose that the thesis holds for vector spaces of odd dimension n. We shall prove it also holds for vector spaces of (odd) dimension $n+2$. Let V with $\dim(V) = n+2$ and let $(e_i)_{i=1,\dots,n+2}$ be a given basis of V. Put $V_1 := \langle e_i \rangle_{1 \le i \le n}$ and $V_2 := \langle e_n, e_{n+1}, e_{n+2} \rangle$. Clearly $V = V_1 + V_2$. As $\dim(V_1) = n$ we can apply the induction hypothesis. So, there exists a trilinear form $\overline{h}_1$ on V_1 (defining a hyperplane H_1 of $\mathcal{G}_3(V_1)$) such that its set of poles is the union of $(n-3)/2$ distinct hyperplanes of $\mathrm{PG}(V)$. Equivalently, we can assume without loss of generality that any H_1-pole satisfies the equation $g_1(u_1, \dots, u_n) = 0$ where $g_1(u_1, \dots, u_n) := \prod_{i=1}^{(n-3)/2} u_{2i+1}$.

Let $\overline{h}_2$ be the trilinear form on V_2 defined by $\overline{h}_2 = \underline{(n)(n+1)(n+2)}$. Clearly $\overline{h}_2$ has no pole in $\mathrm{PG}(V_2)$.

By (the proof of) Theorem 2.5, we can consider the hyperplane H of $\mathcal{G}_3(V)$ defined by the sum of the extensions h_1 and h_2 to V of $\overline{h}_1$ and $\overline{h}_2$ (see the beginning of the proof of Theorem 2.5 for the definition of $\overline{h}_1$ and $\overline{h}_2$). Then, the set of H-poles is a variety of $\mathrm{PG}(V)$ with equation $g(u_1, \dots, u_{n+2}) = 0$ where

$$g(u_1, \dots, u_{n+2}) := g_1(u_1, \dots, u_n) \cdot u_n = \left(\prod_{i=1}^{(n-3)/2} u_{2i+1} \right) \cdot u_n = \prod_{i=1}^{(n-1)/2} u_{2i+1}.$$

□

3 Constructions of families of hyperplanes

In this section we will explain some general constructions yielding large families of hyperplanes of k-Grassmannians. More precisely, in Sections 3.1 and 3.2 we shall briefly recall (without proofs) two constructions already introduced in [3] while in Section 3.3 we will present a new one.

We first need to give the following definition which extends the definition of $\mathcal{S}_p(H)$ given in Section 1.1. For a $(k-2)$-subspace X of V, let $(X)G_k$ be the set of k-subspaces of V containing X. This is a subspace of $\mathcal{G}_k(V)$. Let $(X)\mathcal{G}_k$ be the geometry induced by $\mathcal{G}_k(V)$ on $(X)G_k$ and put $(X)H := (X)G_k \cap H$. Then $(X)\mathcal{G}_k \cong \mathcal{G}_2(V/X)$ and either $(X)H = (X)G_k$ or $(X)H$ is a hyperplane of $(X)\mathcal{G}_k$. In either case, the point-line geometry $\mathcal{S}_X(H) = ((X)G_{k-1}, (X)H)$ is a polar space of symplectic type (possibly a trivial one, when $(X)H = (X)G_k$)). Let $R_X(H) := \mathrm{Rad}\,(\mathcal{S}_X(H))$ be the radical of $\mathcal{S}_X(H)$.

3.1 Extensions and trivial extensions

Let $V = V_0 \oplus V_1$ be a decomposition of V as the direct sum of two non-trivial subspaces V_0 and V_1. Put $n_0 := \dim(V_0)$ and assume that $n_0 \geq k$ (≥ 3). Let $\chi_0 : V_0 \times \cdots \times V_0 \to \mathbb{K}$ be a non-trivial k-linear alternating form on V_0. The form χ_0 can naturally be extended to a k-linear alternating form χ of V by setting

$$\left.\begin{array}{lcll} \chi(x_1,\ldots,x_k) & = & 0 & \text{if } x_i \in V_1 \text{ for some } 1 \leq i \leq k, \\ \chi(x_1,\ldots,x_k) & = & \chi_0(x_1,\ldots,x_k) & \text{if } x_i \in V_0 \text{ for all } 1 \leq i \leq k, \end{array}\right\} \tag{4}$$

and then extending by (multi)linearity. Let H_χ be the hyperplane of $\mathcal{G}_k(V)$ defined by χ. Then, the following properties hold:

Theorem 3.1 ([3]). *Let χ_0 be a k-alternating linear form on V_0 and $n_0 = \dim V_0$; define χ as in (4). For $n_0 = k$, put $H_0 = \emptyset$; otherwise, let H_0 be the hyperplane of $\mathcal{G}_k(V_0)$ defined by χ_0. Let also $\pi : V \to V_0$ be the projection of V onto V_0 along V_1. Then,*

(1) $H_\chi \; = \; \{X \in \mathcal{G}_k(V) \; : \;$ *either* $X \cap V_1 \neq 0$ *or* $\pi(X) \in H_0\}$.

(2) $R_\downarrow(H_\chi) \; = \; \langle R_\downarrow(H_0) \cup [V_1]\rangle$ *where the span in taken in* $\mathrm{PG}(V)$.

(3) $R^\uparrow(H_\chi) \; = \; \{X \in G_{k-1}(V) \; : \;$ *either* $X \cap V_1 \neq 0$ *or* $\pi(X) \in R^\uparrow(H_0)\}$.

(4) When $k = 3$, the points $[p] \notin [V_1]$ have degree $\delta(p) = \delta_0(\pi(p)) + n - n_0$, where $\delta_0(\pi(p))$ is the degree of $[\pi(p)]$ with respect to H_0. The points $p \in [V_1]$ have degree $n - 1$.

We call H_χ the *trivial extension of H_0 centered at V_1* (also *extension of H_0 by V_1*, for short) and we denote it by the symbol $\mathrm{Ext}_{V_1}(H_0)$. When $k = n_0$ we have $H_0 = \emptyset$; we shall call $\mathrm{Ext}_{V_1}(\emptyset)$ the *trivial hyperplane centered at V_1*. In this case Theorem 3.1 can also be rephrased as follows, with no direct mention of H_0.

Proposition 3.2 ([3]). *Let $H = \mathrm{Ext}_{V_1}(\emptyset)$ be a trivial hyperplane of $\mathcal{G}_k(V)$. Then*

$$H \; = \; \{X \in G_k(V) \; : \; X \cap V_1 \neq 0\}.$$

Moreover, $R_\downarrow(H) \; = \; [V_1]$, $R^\uparrow(H) \; = \; \{X \in \mathcal{G}_{k-1}(V) \; : \; X \cap V_1 \neq 0\}$ and, for $X \in \mathcal{G}_{k-2}(V)$, if $X \cap V_1 \neq 0$ then $R_X(H) = [V/X]$, otherwise $R_X(H) = [(V_1 + X)/X]$.

By construction, the lower radical of a trivial extension is never empty. The following theorem shows that the converse is also true, namely if $R_\downarrow(H) \neq \emptyset$ then H is a trivial extension, possibly a trivial hyperplane.

If S is a subspace of V with $\dim(S) > k$, denote by $H(S) := \mathcal{G}_k(S) \cap H$ the hyperplane of $\mathcal{G}_k(S)$ induced by H.

Theorem 3.3 ([3])**.** *Suppose* $R_{\downarrow}(H) \neq \emptyset$ *and let* S, S' *be complements in* V *of the subspace* $R < V$ *such that* $[R] = R_{\downarrow}(H)$*. Then*

(1) $H \ = \ \mathrm{Ext}_R(H(S))$*;*

(2) $H(S) \cong H(S')$*;*

(3) $R_{\downarrow}(H(S)) = \emptyset$*.*

Each hyperplane H of $\mathcal{G}_k(V)$ defined by a k-linear alternating form h with $\mathrm{rank}\,(h) < \dim V$ is clearly a trivial extension of a hyperplane H' of $\mathcal{G}_k(V')$ with $\dim V' = \mathrm{rank}\,(h)$, since $V = \mathrm{Rad}\,(h) \oplus V'$.

3.2 Expansions

Let V_0 be a hyperplane of V and H_0 a given hyperplane of $\mathcal{G}_{k-1}(V_0)$. Assume $k \geq 3$; hence V has dimension $n \geq 4$. Put

$$\mathrm{Exp}(H_0) \ := \ \{X \in \mathcal{G}_k(V) \ : \ \text{either } X \subset V_0 \text{ or } X \cap V_0 \in H_0\}.$$

Theorem 3.4 ([3])**.** *The set* $\mathrm{Exp}(H_0)$ *is a hyperplane of* $\mathcal{G}_k(V)$*. Moreover,*

(1) $R_{\downarrow}(\mathrm{Exp}(H_0)) \ = \ R_{\downarrow}(H_0)$*.*

(2) $R^{\uparrow}(\mathrm{Exp}(H_0)) \ = \ H_0 \cup \{X \in \mathcal{G}_{k-1}(V) \setminus \mathcal{G}_{k-1}(V_0) \ : \ X \cap V_0 \in R^{\uparrow}(H_0)\}$*.*

(3) For $X \in \mathcal{G}_{k-2}(V)$*, if* $X \subseteq V_0$ *with* $X \in R^{\uparrow}(H_0)$ *then* $R_X(\mathrm{Exp}(H_0)) = \mathcal{S}_X(\mathrm{Exp}(H_0)) = (X)G_{k-1}$ *(the latter being computed in* V*). If* $X \subseteq V_0$ *but* $X \notin R^{\uparrow}(H_0)$ *then* $R_X(\mathrm{Exp}(H_0)) = (X)H_0$ *(a subspace of* $\mathrm{PG}(V_0/X)$*). Finally, if* $X \not\subseteq V_0$*, then* $R_X(\mathrm{Exp}(H_0)) \ = \ \{\langle x, Y\rangle \ : \ Y \in R_{X\cap V_0}(H_0)\}$ *for a given* $x \in X \setminus V_0$*, no matter which.*

We call $\mathrm{Exp}(H_0)$ the *expansion* of H_0. A form $h : \bigwedge^k V \to \mathbb{K}$ associated to $\mathrm{Exp}(H_0)$ can be constructed as follows. Suppose $h_0 : \bigwedge^{k-1} V_0 \to \mathbb{K}$ is the $(k-1)$-alternating linear form defining H_0. Suppose $V_0 = \langle e_1, \ldots, e_{n-1}\rangle$ where $E = (e_i)_{i=1}^n$ is the given basis of V. Recall that $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \ : \ 1 \leq i_1 < \ldots < i_k \leq n\}$ is a basis of $\bigwedge^k V$. Put

$$h(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \begin{cases} 0 \text{ if } i_k < n, \\ h_0(e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}) \text{ if } i_k = n. \end{cases} \tag{5}$$

and extend it by linearity. It is easy to check that the form h defines $\mathrm{Exp}(H_0)$.

We now recall some properties linking expansions and trivial extensions which might be of use in investigating the geometries involved.

Theorem 3.5 ([3])**.** *Let H_0 be a hyperplane of $\mathcal{G}_k(V_0)$; then*

1. $R_\downarrow(\mathrm{Exp}(H_0)) = \emptyset$ *if and only if* $R_\downarrow(H_0) = \emptyset$*;*

2. *denote by S_0 a complement of $R_0 \leq V$ such that $[R_0] = R_\downarrow(H_0)$; then,* $\mathrm{Exp}(H_0) = \mathrm{Ext}_{R_0}(\mathrm{Exp}(H_0(S_0)))$ *where $H_0(S_0)$ is the hyperplane induced on S_0 by H_0;*

3. *if H_0 is trivial, then* $\mathrm{Exp}(H_0)$ *is also trivial with center $R_\downarrow(H_0)$.*

3.2.1 Symplectic hyperplanes

Assume now $k = 3$ and take H_0 to be a hyperplane of $\mathcal{G}_2(V_0)$ (hence defined by a bilinear alternating form of V_0). The point-line geometry $\mathcal{S}(H_0) = (G_1(V_0), H_0)$ having as points all points of $[V_0]$ and as lines all elements in H_0, is a polar space of symplectic type. The upper and lower radical of H_0 are mutually equal and coincide with the radical $R(H_0)$ of $\mathcal{S}(H_0)$.

First suppose that $\mathcal{S}(H_0)$ is non-degenerate. Then $n - 1$ is even, whence $n \geq 5$. Claims (1) and (2) of Theorem 3.4 imply that $R_\downarrow(\mathrm{Exp}(H_0)) = \emptyset$ and $R^\uparrow(\mathrm{Exp}(H_0)) = H_0$; thus the geometry of poles $\mathcal{P}(\mathrm{Exp}(H_0))$ of $\mathrm{Exp}(H_0)$ coincides precisely with the symplectic polar space $\mathcal{S}(H_0)$. In particular, the points of $[V]\backslash[V_0]$ are smooth while those of $[V_0]$ are poles of degree 1. Motivated by the above remark we call $\mathrm{Exp}(H_0)$ a *symplectic hyperplane* whenever $R(H_0) = \emptyset$.

Assume now that $\mathcal{S}(H_0)$ is degenerate, i.e. $\mathrm{Rad}\,(\mathcal{S}(H_0)) \neq 0$. In this case, $R_\downarrow(\mathrm{Exp}(H_0)) \neq 0$ since $R_\downarrow(\mathrm{Exp}(H_0)) = \mathrm{Rad}\,(\mathcal{S}(H_0))$; so $\mathrm{Exp}(H_0)$ is either a trivial extension of a symplectic hyperplane by $\mathrm{Rad}\,(\mathcal{S}(H_0))$ (this happens when $\dim(\mathrm{Rad}\,(\mathcal{S}(H_0)) < n - 3)$ or a trivial hyperplane centered at $\mathrm{Rad}\,(\mathcal{S}(H_0))$ (this happens when $\dim(\mathrm{Rad}\,(\mathcal{S}(H_0)) = n - 3)$.

3.3 Block decomposable hyperplanes

The construction of block decomposable hyperplanes can be done for general $k \geq 3$ but we will give the details for the case $k = 3$.

Suppose $V = V_0 \oplus V_1$. Any vector $x \in V$ can then be uniquely written as $x = x_0 + x_1$ with $x_0 \in V_0$ and $x_1 \in V_1$. For $i = 0, 1$ let $\bar{h}_i : \bigwedge^3 V_i \to \mathbb{K}$ be a linear functional defining the hyperplane H_i of $\mathcal{G}_3(V_i)$. Consider the extension $h_i : \bigwedge^3 V \to \mathbb{K}$ of $\bar{h}_i$ to V given by

$$h_i(x \wedge y \wedge z) = \bar{h}_i(x_i \wedge y_i \wedge z_i)$$

where $x = x_0 + x_1, y = y_0 + y_1, z = z_0 + z_1 \in V$ and $x_i, y_i, z_i \in V_i$.

Let $h := h_0 + h_1$ be the trilinear form of V defined by the sum of h_0 and h_1. So,

$$h((x_0 + x_1) \wedge (y_0 + y_1) \wedge (z_0 + z_1)) = \bar{h}_0(x_0 \wedge y_0 \wedge z_0) + \bar{h}_1(x_1 \wedge y_1 \wedge z_1).$$

Then the hyperplane of $\mathcal{G}_3(V)$ defined by h is called a *block decomposable hyperplane* arising from H_0 and H_1 and it will be denoted by $\mathrm{Dec}(H_0, H_1)$.

Theorem 3.6. *Let $H := \mathrm{Dec}(H_0, H_1)$ be a block decomposable hyperplane of $\mathcal{G}_3(V)$. Then the following hold:*

1. *The poles of H are all the points of $\mathrm{PG}(V_0 \oplus V_1)$;*
2. *$R^{\uparrow}(H) = \{\ell \in \mathcal{G}_2(V) : (\pi_0(\ell) \in R^{\uparrow}(H_0)$ or $\dim(\pi_0(\ell)) < 2)$ and $(\pi_1(\ell) \in R^{\uparrow}(H_1)$ or $\dim(\pi_1(\ell)) < 2)\}$ where $\pi_i \colon V \to V_i$ is the projection of V onto V_i along V_j ($j \neq i$, $i = 0, 1$). Denote by $\varepsilon_2 \colon \mathcal{G}_2(V) \to \mathrm{PG}(\bigwedge^2 V)$ the Plücker embedding of the 2-Grassmannian $\mathcal{G}_2$. We have*

   $$\varepsilon_2(R^{\uparrow}(H)) = [\varepsilon_2(R^{\uparrow}(H_0)) + \varepsilon_2(R^{\uparrow}(H_1)) + V_0 \wedge V_1] \cap \varepsilon_2(\mathcal{G}_2)$$

   where $V_0 \wedge V_1 := \langle v_0 \wedge v_1 : v_0 \in V_0, v_1 \in V_1 \rangle$.

Proof. Put $n_0 = \dim V_0$ and $n_1 = \dim V_1$. Let $u \in V$ where $u = u_0 + u_1 \in V$, $u_0 \in V_0$ and $u_1 \in V_1$. Denote by M_u the matrix of the bilinear form $\chi_u(x, y) := h(u \wedge x \wedge y)$. Then, M_u is a block matrix of the form

$$M_u = \begin{pmatrix} M_{u_0} & 0 \\ 0 & M_{u_1} \end{pmatrix}$$

where M_{u_i} is the matrix representing the form $\chi_u^i(x, y) := \bar{h}_i(u \wedge x \wedge y)$ associated to the hyperplane H_i of $\mathcal{G}_3(V_i)$.

For any $x, y, u \in V$ with $x = x_0 + x_1$, $y = y_0 + y_1$, $u = u_0 + u_1$ and $x_i, y_i, u_i \in V_i$, we have by definition of decomposable hyperplane,

$$\chi_u(x, y) = x^T M_u y = x_0^T M_{u_0} y_0 + x_1^T M_{u_1} y_1.$$

By Corollary 2.2, $\mathrm{rank}\,(M_{u_i}) \leq n_i - 1$. So, $\mathrm{rank}\, M_u \leq (n_0 - 1) + (n_1 - 1) = n - 2$. By Corollary 2.3, $[u] = [u_0 + u_1]$ is a pole.

A line $\ell = \langle x, y \rangle$ is in the upper radical $R^{\uparrow}(H)$ if, and only if, for any choice of $u \in V$ we have $\chi_u(x, y) = 0$. Since

$$\chi_u(x, y) = \chi_{u_0}(x_0, y_0) + \chi_{u_1}(x_1, y_1),$$

where $x_i, y_i \in V_i$ and $x = x_0 + x_1$, $y = y_0 + y_1$, we have $\ell \in R^{\uparrow}(H)$ if and only if for all $u_i \in V_i$ and $i = 0, 1$, $\chi_{u_i}(x_i, y_i) = 0$. This holds if and only if $(\pi_0(\ell) \in R^{\uparrow}(H_0)$ or $\dim(\pi_0(\ell)) < 2)$ and $(\pi_1(\ell) \in R^{\uparrow}(H_1)$ or $\dim(\pi_1(\ell)) < 2)$. The first part of claim 2 is proved.

Consider the following subspace of $\mathrm{PG}(\bigwedge^3 V)$

$$\mathcal{L} := [\varepsilon_2(R^{\uparrow}(H_0)) + \varepsilon_2(R^{\uparrow}(H_1)) + V_0 \wedge V_1]$$

where $V_0 \wedge V_1 := \langle v_0 \wedge v_1 : v_0 \in V_0, v_1 \in V_1 \rangle$.

We claim that any point in $\varepsilon_2(R^{\uparrow}(H_0)) + \varepsilon_2(R^{\uparrow}(H_1)) + V_0 \wedge V_1$ is in $\varepsilon_2(R^{\uparrow}(H))$. Since $R^{\uparrow}(H_i) \subseteq R^{\uparrow}(H)$ (for $i = 0, 1$) and $\varepsilon_2^{-1}((V_0 \wedge V_1) \cap \varepsilon_2(\mathcal{G}_2(V))) \subseteq R^{\uparrow}(H)$, by the first part of claim 2, the inclusion $\mathcal{L} \subseteq \varepsilon_2(R^{\uparrow}(H))$ is immediate.

Suppose now $\langle (x_0 + x_1) \wedge (y_0 + y_1) \rangle \in \varepsilon_2(R^{\uparrow}(H))$. We can write (by the first part of claim 2)

$$(x_0 + x_1) \wedge (y_0 + y_1) = \underbrace{(x_0 \wedge y_0)}_{\in \varepsilon_2(R^{\uparrow}(H_0))} + \underbrace{(x_0 \wedge y_1) + (x_1 \wedge y_0)}_{\in V_0 \wedge V_1} + \underbrace{(x_1 \wedge y_1)}_{\in \varepsilon_2(R^{\uparrow}(H_1))} \in \mathcal{L}.$$

The thesis follows. □

We remark that, with some slight abuse of notation, the extension $\mathrm{Ext}_{V_1}(H_0)$ of an hyperplane H_0 can always be regarded as a special case of a block decomposable hyperplane, where the form $\bar{h}_1$ defined over V_1 is identically null.

The definition given for block decomposable hyperplane arising from two hyperplanes H_0 and H_1 can be extended by induction to the definition of block decomposable hyperplane $\mathrm{Dec}(H_0, \cdots, H_{n-1})$ arising from n hyperplanes H_i $(0 \leq i \leq n-1)$, where H_i is a hyperplane of $\mathcal{G}_3(V_i)$ and $V = \oplus_i V_i$.

In general, given two linear subspaces V_0, V_1 of V such that $V = V_0 \oplus V_1$, and given two hyperplanes H_0 and H_1 of $\mathcal{G}_3(V_0)$ and $\mathcal{G}_3(V_1)$, there exist several possible hyperplanes H of $\mathcal{G}_3(V_0 \oplus V_1)$ which are block decomposable and arise from H_0 and H_1, namely all of those induced by the forms $h_{\alpha,\beta} := \alpha h_0 + \beta h_1$ with $\alpha, \beta \in \mathbb{K} \setminus \{0\}$. Even if all these hyperplanes are in general neither equivalent nor nearly equivalent, they turn out to be always geometrically equivalent and their geometry of poles depends only on the geometries of H_0 and H_1.

4 Characterization of the geometry of poles

In this section we will prove our Theorems 1, 2 and 3. As in the previous sections, let $E := (e_i)_{i=1}^n$ be a given basis of V. Let H be a given hyperplane of $\mathcal{G}_3(V)$ and $\mathcal{P}(H) = (P(H), R^{\uparrow}(H))$ be the geometry of poles of H. In Section 2.1 we have explained how to algebraically describe the pointset $P(H)$ and the lineset $R^{\uparrow}(H)$ of the geometry of poles associated to H. The main steps to describe $\mathcal{P}(H)$ are the following:

- Consider the bilinear form χ_u associated to the trilinear form defining H and write the matrix M_u representing χ_u. This is done in Table 2 for forms of rank up to 6 and in Table 3 for forms of rank 7. Recall that for $\dim(V) \leq 7$ and $\mathbb{K}$ perfect with cohomological dimension at most 1, all trilinear forms are classified: they are listed in Table 1.

- By Theorem 2.4, we know that the set of poles is either the pointset of $\mathrm{PG}(V)$ or an algebraic variety. If $\dim(V) = 6$ all points of $\mathrm{PG}(V)$ are poles, for any hyperplane H of $\mathcal{G}_3(V)$. If $\dim(V) = 7$, to get the equations describing the variety $P(H)$ we rely on Corollary 2.3, which gives algebraic conditions on $(u_i)_{i=1}^n$ for $[(u_i)_{i=1}^n]$ to be a H-pole. In the second column of Table 5 we have written down those equations, according to the type of H.

- To describe the lines of $\mathcal{P}(H)$ we rely on the last part of Section 2.1. In particular, $\ell := [x, y] \in R^{\uparrow}(H)$ if and only if the functional $\tilde{h}_{xy}$ described in Equation (2) is the null functional. This immediately reads as some linear equations in the Plücker coordinates $|x, y|_{ij}$ of the line ℓ. The results of these (straightforward) computations are reported in the third column of Tables 4 for forms of rank at most 6 and Table 5 for forms of rank 7.

So, Tables 1, 2, 3, 4, 5 provide an algebraic description of points and lines of $\mathcal{P}(H)$, for any hyperplane H of $\mathcal{G}_3(V)$. In the remainder of this section we shall also provide a geometrical description of $\mathcal{P}(H)$.

4.1 Hyperplanes arising from forms of rank at most 6

4.1.1 Hyperplanes of type T_1

A hyperplane H of $\mathcal{G}_3(V)$ of type T_1 is defined by a trilinear form of rank 3 equivalent to $h = \underline{123}$, see the first row of Table 1.

Suppose $\dim(V) \geq 4$ and let $(e_i)_{i=1}^n$ be a given basis of V. Then $\mathrm{Rad}\,(h) = \langle e_i \rangle_{i \geq 4}$. According to Section 3.1, H is a trivial hyperplane $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(\emptyset)$ centered at $\mathrm{Rad}\,(h)$. By Proposition 3.2, the set of $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(\emptyset)$-poles is the whole pointset of $\mathrm{PG}(V)$ and the lines of the geometry of poles, i.e. the elements in the upper radical $R^{\uparrow}(\mathrm{Ext}_{\mathrm{Rad}\,(h)}(\emptyset))$, are those lines of $\mathrm{PG}(V)$ meeting $\mathrm{Rad}\,(h)$ non-trivially. When $n = \dim(V) \leq 6$, this proves part 1 of Theorem 1.

4.1.2 Hyperplanes of type T_2

A hyperplane H of $\mathcal{G}_3(V)$ of type T_2 is defined by a trilinear form of rank 5 equivalent to $h = \underline{123} + \underline{145}$, see the second row of Table 1.

Suppose $\dim(V) > 5$. Then $\dim(\mathrm{Rad}\,(h)) \geq 1$. Let $V = \mathrm{Rad}\,(h) \oplus V'$. By Section 3.1, H is a trivial extension $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(H')$ of a hyperplane H' of $\mathcal{G}_3(V')$. By Theorem 3.3, we can assume without loss of generality $V' = \langle e_i \rangle_{i=1}^5$. Put $V_0 := \langle e_i \rangle_{i=2}^5$ and consider the hyperplane H_0 of $\mathcal{G}_2(V_0)$ defined by the functional $\underline{23} + \underline{45}$. By Section 3.2, H' is the expansion $\mathrm{Exp}(H_0)$ of H_0. By Subsection 3.2.1, since the geometry $\mathcal{S}(H_0) = (G_1(V_0), H_0)$ is a non-degenerate symplectic polar space, the geometry of poles of $\mathrm{Exp}(H_0)$ coincides precisely with $\mathcal{S}(H_0)$. Hence, if

$\dim(V) > 5$, H is a trivial extension $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(\mathrm{Exp}(H_0))$ of a symplectic hyperplane $\mathrm{Exp}(H_0)$. The H-poles are all the points of $\mathrm{PG}(V)$ and the lines of the geometry of poles are all the lines ℓ of $\mathrm{PG}(V)$ meeting $\mathrm{Rad}\,(h)$ non-trivially or such that $\pi(\ell) \in H_0$ where π is the projection onto V_0 along $\mathrm{Rad}\,(h)$.

If $\dim(V) = 5$ then H is the expansion $\mathrm{Exp}(H_0)$ of a non-degenerate symplectic polar space $\mathcal{S}(H_0)$ defined by the functional $\underline{23} + \underline{45}$ in $V_0 = \langle e_i \rangle_{i=2}^{5}$. By Subsection 3.2.1, the geometry of poles of H coincides with the symplectic polar space $\mathcal{S}(H_0)$.

Part 2 of Theorem 1 is proved.

4.1.3 Hyperplanes of type T_3

A hyperplane H of $\mathcal{G}_3(V)$ of type T_3 is defined by a trilinear form of rank 6 equivalent to $h = \underline{123} + \underline{456}$, see the third row of Table 1.

Suppose $\dim(V) = 6$. Let $(e_i)_{i=1}^{6}$ be a given basis of V. Put $V_0 = \langle e_1, e_2, e_3 \rangle$ and $V_1 = \langle e_4, e_5, e_6 \rangle$. Clearly, $V = V_0 \oplus V_1$. For $i = 0, 1$, denote by $\bar{h}_i := \underline{(3i+1)(3i+2)(3i+3)}$ the trilinear form induced by the restriction of h to $\bigwedge^3 V_i$. By Section 3.3, H is a decomposable hyperplane $\mathrm{Dec}(H_0, H_1)$ of $\mathcal{G}_3(V)$ arising from the hyperplanes H_i, $i = 0, 1$, of $\mathcal{G}_3(V_i)$ defined by the forms $\bar{h}_i$. Since $R^{\uparrow}(H_i) = \emptyset$, by Theorem 3.6, all points of $\mathrm{PG}(V)$ are elements of the geometry of poles $\mathcal{P}(H)$ and the lines of $\mathcal{P}(H)$ are exactly those lines of $\mathrm{PG}(V)$ intersecting both $\mathrm{PG}(V_0)$ and $\mathrm{PG}(V_1)$. This proves part 3 of Theorem 1.

Suppose $\dim(V) > 6$. Let $(e_i)_{i \geq 1}$ be a given basis of V. Then $\mathrm{Rad}\,(h) = \langle e_i \rangle_{i \geq 7}$. By last part of Section 3.1, H is a trivial extension $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(\mathrm{Dec}(H_0, H_1))$ of a decomposable hyperplane $\mathrm{Dec}(H_0, H_1)$ of $\mathcal{G}_3(V')$ where $V = \mathrm{Rad}\,(h) \oplus V'$ and H_i, $i = 0, 1$, are the hyperplanes of $\mathcal{G}_3(V_i)$, $V_i = \langle e_{3i+1}, e_{3i+2}, e_{3i+3} \rangle$, $V' = V_1 \oplus V_2$, defined by $\underline{(3i+1)(3i+2)(3i+3)}$.

4.1.4 Hyperplanes of type T_4

A hyperplane H of $\mathcal{G}_3(V)$ of type T_4 is defined by a trilinear form of rank 6 equivalent to $h = \underline{162} + \underline{243} + \underline{135}$, see the fourth row of Table 1.

Suppose $\dim(V) = 6$. For any $u \in V$, $\mathrm{rank}\,(M_u) \leq 4$ (see Table 2 for the description of the matrix M_u). If $(e_i)_{i=1}^{6}$ is a basis of V and $V = V_0 \oplus V_1$ with $V_0 = \langle e_1, e_2, e_3 \rangle$ and $V_1 = \langle e_4, e_5, e_6 \rangle$, by a direct computation we have that the elements of V_1 are poles of degree 3 while all remaining poles have degree 1.

Let $\ell = [u, v]$ be a line of $\mathrm{PG}(V)$. By the forth row of Table 4, $\ell \in R^{\uparrow}(H)$ if and only if its Plücker coordinates satisfy 6 linear equations. More explicitly, we have that $\ell = [u, v] \in R^{\uparrow}(H)$ if and only if $u = \bar{u}_0 + \bar{u}_1$ and $v = \omega(\bar{u}_0) \in V_1$ with $\bar{u}_0 \in V_0$, $\bar{u}_1 \in V_1$ and $\omega \colon V \to V$, $\omega(u_1, u_2, u_3, u_4, u_5, u_6) = (u_4, u_5, u_6, u_1, u_2, u_3)$. Note that ω interchanges V_0 and V_1. Indeed, if $u = \bar{u}_0 + \bar{u}_1$ and $v = \bar{v}_0 + \bar{v}_1$ with

$\bar{u}_0, \bar{v}_0 \in V_0$ and $\bar{u}_1, \bar{v}_1 \in V_1$, the Plücker coordinates of the line $\ell = [u, v]$, satisfy the equations $|u, v|_{12} = 0, |u, v|_{13} = 0, |u, v|_{23} = 0$ if and only if $\bar{v}_0 = \lambda \bar{u}_0$ with $\lambda \in \mathbb{K}$. Hence $\ell = [u, v] = [u, v - \lambda u] = [\bar{u}_0 + \bar{u}_1, \bar{v}'_1]$ with $\bar{v}'_1 = \bar{v}_1 - \lambda \bar{u}_1 \in V_1$.

The remaining three equations $|x, y|_{26} - |x, y|_{35} = 0, |x, y|_{16} - |x, y|_{34} = 0, |x, y|_{24} - |x, y|_{15} = 0$ are satisfied by the Plücker coordinates of $\ell = [u, v] = [\bar{u}_0 + \bar{u}_1, \bar{v}'_1]$ if and only if either $\bar{u}_0$ is the null vector and in this case $\ell = [\bar{u}_1, \bar{v}'_1] \subset V_1$ or $[\bar{v}'_1] = [(0, 0, 0, u_1, u_2, u_3)]$ where $[u] = [(u_1, u_2, u_3, u_4, u_5, u_6)]$.

This proves part 4 of Theorem 1.

Suppose $\dim(V) > 6$. Let $(e_i)_{i \geq 1}$ be a given basis of V. Then $\mathrm{Rad}\,(h) = \langle e_i \rangle_{i \geq 7}$. By last part of Section 3.1, H is a trivial extension $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(H')$ of a hyperplane H' of $\mathcal{G}_3(V')$ where $V = \mathrm{Rad}\,(h) \oplus V'$, $V' = \langle e_i \rangle_{i=1}^{6}$ and H' is defined by a trilinear form equivalent to $\underline{162} + \underline{243} + \underline{135}$. A description of the geometry of poles of H follows from Theorem 3.1 and the already done case of hyperplanes of type T_4 of $\mathcal{G}_3(V)$ for $\dim(V) = 6$.

4.1.5 Hyperplanes of type $T^{(i)}_{10,\lambda}$

A hyperplane H of $\mathcal{G}_3(V)$ of type $T^{(i)}_{10,\lambda}$ is defined by a trilinear form of rank 6 as written in row 10 or 11 of Table 1, according as $\mathrm{char}\,(\mathbb{K})$ is odd or even. We remark that forms of type $T^{(1)}_{10,\lambda}$ make sense also in even characteristic, provided that the field $\mathbb{K}$ is not perfect.

Let $\dim(V) = 6$. For any $u \in V$, the matrix M_u representing the bilinear alternating form χ_u associated to H is written in Table 2. We have that $\mathrm{rank}\,(M_u) \leq 4$ for every $u \in V$, i.e. every point of $\mathrm{PG}(V)$ is a pole. Actually, we will prove that $\mathrm{rank}\,(M_u) = 4$ for any $u \in V$, i.e. every point of $\mathrm{PG}(V)$ is a pole of degree 1, equivalently, $R^{\uparrow}(H)$ is a line spread of $\mathrm{PG}(V)$.

Consider first a hyperplane of type $T^{(1)}_{10,\lambda}$. Suppose by way of contradiction that $\mathrm{rank}\,(M_u) < 4$, i.e. all minors of order 4 vanish. With $u = (u_1, \ldots, u_6)$, take the three 4×4 principal minors of M_u given by

Case $i = 1$

Principal submatrix of M_u corresponding to rows/columns	Value of the minor
$1, 2, 4, 5$	$\lambda^2(\lambda u_6^2 - u_3^2)^2$
$1, 3, 4, 6$	$\lambda^2(\lambda u_5^2 - u_2^2)^2$
$2, 3, 5, 6$	$\lambda^2(\lambda u_4^2 - u_1^2)^2$.

By the third column of Table 1 corresponding to $T^{(1)}_{10,\lambda}$, we have that $p_\lambda(t) = t^2 - \lambda$ is an irreducible polynomial in $\mathbb{K}[t]$. Hence the minors mentioned are null if and only if $u_i = 0$ for all $i = 1, \ldots, 6$.

We argue in a similar way for case $T^{(2)}_{10,\lambda}$ with $\mathrm{char}\,(\mathbb{K}) = 2$, by choosing the minors of M_u given by

Case $i = 2$

Principal submatrix of M_u corresponding to rows/columns	Value of the minor
$1, 2, 4, 5$	$(u_6^2 + \lambda u_3 u_6 + u_3^2)^2$
$1, 3, 4, 6$	$(u_5^2 + \lambda u_2 u_5 + u_2^2)^2$
$2, 3, 5, 6$	$(u_4^2 + \lambda u_1 u_4 + u_1^2)^2$.

By the third column of Table 1 corresponding to $T^{(2)}_{10,\lambda}$, we have that $p_\lambda(t) := t^2 + \lambda t + 1$ is irreducible in $\mathbb{K}[t]$; hence $\mathrm{rank}\, M_u = 4$ unless $u = (0, \ldots, 0)$. So $R^{\uparrow}(H)$ is a line-spread of $\mathrm{PG}(V)$.

Lemma 4.1. *Let H be a hyperplane of $\mathcal{G}_3(V)$ with $\dim V = 6$ whose upper radical $R^{\uparrow}(H)$ is a line-spread of $\mathrm{PG}(V)$. Then $R^{\uparrow}(H)$ is a Desarguesian line-spread of $\mathrm{PG}(V)$.*

Proof. For simplicity of notation, denote by $\mathcal{S}$ the line-spread of $\mathrm{PG}(V)$ induced by H. Then, by duality, H induces also a line spread $\mathcal{S}^*$ in the dual space $\mathrm{PG}(V^*)$, where V^* is the dual of V. In particular, a 4-dimensional vector space Σ is in $\mathcal{S}^*$ if and only if all planes contained in Σ are elements of H. We will prove that $\mathcal{S}$ is a *normal spread*, i.e. given any two distinct elements $\ell_1, \ell_2 \in \mathcal{S}$ and $\Sigma = \ell_1 + \ell_2$, the set $\mathcal{S}_\Sigma := \{\ell \in \mathcal{S} : \ell \subseteq \Sigma\}$ is a line-spread of Σ.

Let π be a plane contained in Σ. Then, $\pi \cap \ell_1 \neq \emptyset \neq \pi \cap \ell_2$ and we can write $\pi = [p_1, p_2, q_1 + q_2]$ with $p_1, q_1 \in \ell_1$, $p_2, q_2 \in \ell_2$ suitably chosen. Denote by h a form defining the hyperplane H. Then $h(p_1 \wedge p_2 \wedge (q_1 + q_2)) = h(p_1 \wedge p_2 \wedge q_1) + h(p_1 \wedge p_2 \wedge q_2) = 0$ as h is identically zero on all planes through either ℓ_1 or ℓ_2. Hence $\pi \in H$ and, consequently, $\Sigma \in \mathcal{S}^*$.

So the 3-dimensional projective space spanned by any two elements of $\mathcal{S}$ is in $\mathcal{S}^*$; by duality, the intersection of any two elements of $\mathcal{S}^*$ is in $\mathcal{S}$. Take now $\Sigma \in \mathcal{S}^*$ and $p \in \Sigma$; denote by ℓ_p the unique line of $\mathcal{S}$ with $p \in \ell_p$. Let $\ell' \in \mathcal{S}$ such that ℓ' is not contained in Σ. Then, $\Sigma' = \ell_p + \ell' \in \mathcal{S}^*$ and, by the argument above, $\Sigma' \cap \Sigma \in \mathcal{S}$. Since $p \in \Sigma' \cap \Sigma$, it follows that $\Sigma' \cap \Sigma = \ell_p$. This for all points $p \in \Sigma$; so $\mathcal{S}_\Sigma$ is a spread of Σ. Hence $\mathcal{S}$ is a normal spread and by [1, Theorem 2] $\mathcal{S}$ is a Desarguesian line spread of $\mathrm{PG}(V)$. □

Part 5 of Theorem 1 is now proved.

Proof of Theorem 2. By Lemma 4.1, line-spreads of $\mathrm{PG}(6, V)$ induced by hyperplanes of $\mathcal{G}_3(V)$ are Desarguesian. As Desarguesian line-spreads are coordinatized over division rings, there must exist a division ring $\mathbb{D}$ having dimension

2 over $\mathbb{K}$ (see [2]) with either $\mathbb{D}$ commutative or $\mathbb{K}$ being the center of $\mathbb{D}$. We show that $\mathbb{D}$ is commutative. Take $a \in \mathbb{D} \setminus \mathbb{K}$. Then the algebraic extension $\mathbb{K}(a)$ is a proper field extension of $\mathbb{K}$ contained in $\mathbb{D}$; so $2 = [\mathbb{D} : \mathbb{K}] \geq [\mathbb{K}(a) : \mathbb{K}] \geq 2$. It follows that $\mathbb{D} = \mathbb{K}(a)$ and $\mathbb{D}$ is a field which is an algebraic extension of degree 2 of $\mathbb{K}$. So, for $\mathbb{D}$ to exist, $\mathbb{K}$ must not be quadratically closed. $\square$

Remark 1. The commutativity of $\mathbb{D}$ in the proof of Theorem 2 is also a consequence of the Artin-Wedderburn theorem. We have provided a short argument.

Suppose $\dim(V) > 6$. Let $(e_i)_{i\geq 1}$ be a given basis of V. Then $\mathrm{Rad}\,(h) = \langle e_i \rangle_{i\geq 7}$. By last part of Section 3.1, H is a trivial extension $\mathrm{Ext}_{\mathrm{Rad}\,(h)}(H')$ of a hyperplane H' of $\mathcal{G}_3(V')$ where $V = \mathrm{Rad}\,(h) \oplus V'$, $V' = \langle e_i \rangle_{i=1}^{6}$ and H' is defined by a trilinear form of type $T_{10,\lambda}^{(i)}$. A description of the geometry of poles of H follows from Theorem 3.1 and the already done case of hyperplanes of type $T_{10,\lambda}^{(i)}$ of $\mathcal{G}_3(V)$ with $\dim(V) = 6$.

4.2 Hyperplanes arising from forms of rank 7

Throughout this section let $[u] = [(u_i)_{i=1}^{7}]$.

4.2.1 Hyperplanes of type T_5

A hyperplane H of $\mathcal{G}_3(V)$ of type T_5 is defined by a trilinear form of rank 7 equivalent to $h = \underline{123} + \underline{456} + \underline{147}$, see the fifth row of Table 1.

Suppose $\dim(V) = 7$. Straightforward computations shows that $\mathrm{rank}\,(M_u) \leq 4$ (see Table 3 for the description of the matrix M_u) if and only if $u_1 = 0$ or $u_4 = 0$ and $\mathrm{rank}\,(M_u) = 2$ if and only if $u_1 = u_4 = u_5 = u_6 = 0$ or $u_1 = u_2 = u_3 = u_4 = 0$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ the hyperplanes of $\mathrm{PG}(V)$ with equations respectively $u_1 = 0$ and $u_4 = 0$. Denoted by $P(H)$ the set of poles of H, we then have $P(H) = \mathcal{S}_1 \cup \mathcal{S}_2$. A point $[u]$ has degree 4 if and only if $[u] \in A_1 \cup A_2$ where A_1 is the plane of $\mathrm{PG}(V)$ of equation $u_1 = u_4 = u_5 = u_6 = 0$ and A_2 is the plane of $\mathrm{PG}(V)$ of equation $u_1 = u_2 = u_3 = u_4 = 0$.

Take $[u] = [(0, u_2, \ldots, u_7)] \in \mathcal{S}_1$. In this case, by the equations of Table 5, a line $\ell := [u, v]$ of $\mathrm{PG}(V)$ through $[u]$ is in $R^{\uparrow}(H)$ if and only if $\ell \subseteq \mathcal{S}_1$, intersects A_1 non-trivially and it is totally isotropic for the non-degenerate bilinear alternating form of $\mathcal{S}_1$ defined by $\beta(u, v) = u_2v_3 - u_3v_2 + u_4v_7 - u_7v_4 + u_5v_6 - u_6v_5$. A similar argument shows that if $[u]$ is taken in $\mathcal{S}_2$, then any line of $R^{\uparrow}(H)$ through it meets A_2 and it is totally isotropic for the non-degenerate bilinear alternating form of $\mathcal{S}_2$ defined by $\beta(u, v) = u_1v_7 - u_7v_1 + u_2v_3 - u_3v_2 + u_5v_6 - u_6v_5$.

This proves part 1 of Theorem 3.

4.2.2 Hyperplanes of type T_6

A hyperplane H of $\mathcal{G}_3(V)$ of type T_6 is defined by a trilinear form of rank 7 equivalent to $h = \underline{152} + \underline{174} + \underline{163} + \underline{243}$, see the sixth row of Table 1. Suppose $\dim V = 7$; straightforward computations show that $\mathrm{rank}\,(M_u) \leq 4$ (see Table 3 for the description of the matrix M_u) if and only if $u_1 = 0$. Let $\mathcal{S}$ be the hyperplane of $\mathrm{PG}(V)$ of equation $u_1 = 0$. Then the set of the H-poles is precisely the point-set of $\mathcal{S}$. Also, a point $[u]$ has degree 4 if and only if $[u] \in A$ where A is the plane of equation $u_1 = u_2 = u_3 = u_4 = 0$. Take $u = [(0, u_2, \dots, u_7)] \in \mathcal{S}$. By the equations $|u, v|_{34} = |u, v|_{24} = |u, v|_{23}$, of Table 5, each line of the upper radical must intersect the plane A. By the equation $|u, v|_{25} + |u, v|_{36} + |u, v|_{47} = 0$, we have that a line $\ell = [u, v]$ is in $R^{\uparrow}(H)$ if and only if $\ell \subseteq \mathcal{S}$, $\ell \cap A \neq 0$ and ℓ is totally isotropic for the non-degenerate alternating form $\beta(u, v) := |u, v|_{25} + |u, v|_{36} + |u, v|_{47} = 0$. This proves part 2 of Theorem 3.

4.2.3 Hyperplanes of type T_7

A hyperplane H of $\mathcal{G}_3(V)$ of type T_7 is defined by a trilinear form of rank 7 equivalent to $h = \underline{146} + \underline{157} + \underline{245} + \underline{367}$, see the seventh row of Table 1. Suppose $\dim V = 7$; straightforward computations show that $\mathrm{rank}\,(M_u) \leq 4$ (see Table 3 for the description of the matrix M_u) if and only if $u_5u_7 + u_4u_6 = 0$. Let A be the plane of $\mathrm{PG}(V)$ of equations $u_4 = u_5 = u_6 = u_7 = 0$ and denote by $\mathcal{Q}$ the hyperbolic quadric of equation $u_5u_7 + u_4u_6 = 0$ embedded in the subspace W of equations $u_1 = u_2 = u_3 = 0$. The set of H-poles is the point-set of the quadratic cone of $\mathrm{PG}(V)$ with vertex A and basis $\mathcal{Q}$. Also, a point $[u]$ has degree 4 if and only if $[u] \in \mathcal{C}$ where $\mathcal{C}$ is the conic of A with equation $u_1^2 - u_2u_3 = 0$. Using the equations of Table 5, it is possible to associate to any $[x] = [(st, t^2, s^2, 0, 0, 0, 0)] \in \mathcal{C}$ a unique line $\ell_x = [(0, 0, 0, s, 0, 0, t), (0, 0, 0, 0, s, -t, 0)]$ of $\mathcal{Q}$.

With $[x] \in \mathcal{C}$, denote by $\mathrm{Res}_{\langle A, \ell_x \rangle}(x)$ the projective geometry whose points are all the 2-dimensional vector spaces through x in the 5-dimensional vector space $\langle A, \ell_x \rangle$ and whose lines are the 3-dimensional vector spaces of $\langle A, \ell_x \rangle$ through x. Note that $\mathrm{Res}_{\langle A, \ell_x \rangle}(x) \cong \mathrm{PG}(3, \mathbb{K})$.

Proposition 4.2. *For any $[x] \in \mathcal{C}$ there exists a line spread $\mathcal{S}_x$ of $\mathrm{Res}_{\langle A, \ell_x \rangle}(x)$ such that $\ell \in R^{\uparrow}(H)$ if and only if $\ell \subseteq \langle x, s \rangle$ for some $x \in \mathcal{C}$ and $s \in \mathcal{S}_x$.*

Proof. Suppose $x, x' \in \mathcal{C}$ with $x \neq x'$. Then, $\langle A, \ell_x \rangle \cap \langle A, \ell_{x'} \rangle = A$. We shall now define $\mathcal{S}_x$ as follows

$$\mathcal{S}_x := \{\pi_p : p \in \langle A, \ell_x \rangle \setminus A\} \cup \{A\}$$

where π_p is the unique plane of $\mathrm{PG}(V)$ spanned by all the lines through p in $R^{\uparrow}(H)$. Note that $\delta(p) = 2$ and $x \in \pi_p \subseteq \langle A, \ell_x \rangle$.

Let now $[q]$ be a point in π_p with $\pi_p \in \mathcal{S}_x$. If $[q] \notin [x,p]$, then $[x,q]$ and $[p,q]$ are both lines in the upper radical of H through $[q]$; since $\delta(q) = 2$ we have $\pi_q = \pi_p$. If $[q] \in [x,p]$ we consider a point $[r] \notin [x,p]$ and apply the same argument to show that $\pi_q = \pi_r = \pi_p$. Hence all lines of $\mathrm{PG}(V)$ in any plane $\pi_p \in \mathcal{S}_x$ are in the upper radical and for any pole $p \notin A$, the lines of the upper radical through p are contained in π_p.

Suppose π_p and π_q are two lines of $\mathcal{S}_x$ with non-trivial intersection, i.e. $\pi_p \cap \pi_q$ is a line through x not contained in A. Any point on this line would have degree at least 3 — a contradiction.

Finally all lines contained in A are elements of the upper radical and A reads as one line of $\mathcal{S}_x$. □

This proves part 3 of Theorem 3.

4.2.4 Hyperplanes of type T_8

A hyperplane H of $\mathcal{G}_3(V)$ of type T_8 is defined by a trilinear form of rank 7 equivalent to $h = \underline{123} + \underline{145} + \underline{167}$, see the eighth row of Table 1. Suppose $\dim V = 7$. Put $V_0 = \langle e_i \rangle_{i=2}^7$ and consider the hyperplane H_0 of $\mathcal{G}_2(V_0)$ define by the functional $\underline{23} + \underline{45} + \underline{67}$. By Subsection 3.2, H is the expansion $\mathrm{Exp}(H_0)$ of H_0 and since the geometry $S(H_0) := (G_1(V_0), H_0)$ is a non-degenerate symplectic polar space, the geometry of poles of $\mathrm{Exp}(H_0)$ coincides with $S(H_0)$.

This proves part 4 of Theorem 3.

4.2.5 Hyperplanes of type T_9 and $T_{12,\mu}$

Hyperplanes H of $\mathcal{G}_3(V)$ of types either T_9 or $T_{12,\mu}$ are defined by trilinear forms of rank 7 nearly equivalent to $h = \underline{123} + \underline{456} + \underline{147} + \underline{257} + \underline{367}$, see rows 9 and 14 of Table 1. Suppose $\dim V = 7$; straightforward computations show that $\mathrm{rank}\,(M_u) \leq 4$ (see Table 3 for the description of the matrix M_u) if and only if $[u]$ is a point of a non-degenerate parabolic quadric $\mathcal{Q}$ of $\mathrm{PG}(V)$. More precisely, each point of $\mathcal{Q}$ has degree 2.

The equations appearing in Table 5 for cases T_9 (and $T_{12,\mu}$) are the same as those in [15, §2.4.13] for the standard embedding of the Split-Cayley hexagon $H(\mathbb{K})$ in $\mathrm{PG}(6,\mathbb{K})$; see also [14]. Hence, the geometry of the poles of H is precisely a Split-Cayley hexagon. For these reasons, hyperplanes of this type are called *hexagonal.* This proves part 5 of Theorem 3.

4.2.6 Hyperplanes of type $T_{11,\lambda}^{(i)}$

A hyperplane H of $\mathcal{G}_3(V)$ of type $T_{11,\lambda}^{(i)}$ $(i = 1, 2)$ is defined by a trilinear form of rank 7 as written in row 12 or 13 of Table 1, according as $\mathrm{char}\,(\mathbb{K})$ is odd or even.

As in the case $T^{(1)}_{10,\lambda}$, we remark that forms of type $T^{(1)}_{11,\lambda}$ may also be considered in even characteristic, provided that the field is not perfect. The geometries of poles arising in both cases $i = 1, 2$ afford a similar description.

Suppose $\dim V = 7$. By the proof of Theorem 2.4, straightforward computations show that $\mathrm{rank}\,(M_u) \leq 4$ if and only if $\det(M_u^{(7)})/u_7^2 = 0$ where by $M_u^{(7)}$ is the submatrix of M_u (see Table 5 for the description of M_u) obtained by deleting its last row and column.

First consider the case $T^{(1)}_{11,\lambda}$. Then $[u]$ is a pole if and only if its coordinates satisfy the equation $\lambda u_4^2 - u_1^2 = 0$. Since the polynomial $p_\lambda(x) = x^2 - \lambda$ is irreducible in $\mathbb{K}$, the points satisfying the above equation have coordinates with $u_1 = u_4 = 0$. Hence, the set of poles is $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ where $\mathcal{S}_1$ is the hyperplane of $\mathrm{PG}(V)$ of equation $u_1 = 0$ and $\mathcal{S}_2$ is the hyperplane of $\mathrm{PG}(V)$ of equation $u_4 = 0$. Considering the 4×4 principal minors of M_u given by

Case $i = 1$

Principal submatrix of M_u corresponding to rows/columns	Value of the minor
$1, 2, 4, 5$	$\lambda^2(\lambda u_6^2 - u_3^2)^2$
$1, 3, 4, 6$	$\lambda^2(\lambda u_5^2 - u_2^2)^2$,

we have that the only point of degree 4 is $[e_7]$.

In the case $T^{(2)}_{11,\lambda}$, then $[u]$ is a pole if and only if its coordinates satisfy the equation $u_4^2 + \lambda u_1 u_4 + u_1^2 = 0$. Since the polynomial $p_\lambda(x) = x^2 + \lambda x + 1$ is irreducible in $\mathbb{K}$, the points satisfying the above equation have coordinates with $u_1 = u_4 = 0$. Hence, the set of poles is $\mathcal{S} := \mathcal{S}_1 \cap \mathcal{S}_2$ where $\mathcal{S}_1$ is the hyperplane of $\mathrm{PG}(V)$ of equation $u_1 = 0$ and $\mathcal{S}_2$ is the hyperplane of $\mathrm{PG}(V)$ of equation $u_4 = 0$. Considering the 4×4 principal minors of M_u given by

Case $i = 2$

Principal submatrix of M_u corresponding to rows/columns	Value of the minor
$1, 2, 4, 5$	$(u_6^2 + \lambda u_3 u_6 + u_3^2)^2$
$1, 3, 4, 6$	$(u_5^2 + \lambda u_2 u_5 + u_2^2)^2$

so we have that the only point of degree 4 is $[e_7]$ as above.

We now provide a geometric description of $R^\uparrow(H)$ holding for both $i = 1$ and $i = 2$. Denote by $\mathrm{Res}_{\mathcal{S}}(e_7)$ the projective geometry whose points are all the lines of $\mathcal{S}$ through $[e_7]$ and whose lines are the planes of $\mathcal{S}$ through $[e_7]$. It is well-known that $\mathrm{Res}_{\mathcal{S}}(e_7) \cong \mathrm{PG}(3, \mathbb{K})$. Consider the following set

$$\mathcal{F} := \{\pi_p : p \in \mathcal{S} \setminus [e_7]\}$$

where π_p is the plane of $\mathrm{PG}(V)$ spanned by the lines in $R^\uparrow(H)$.

Proposition 4.3. *The set $\mathcal{F}$ is a line-spread of* $\mathrm{Res}_{\mathcal{S}}(e_7)$ *and*

$$R^{\uparrow}(H) = \{\ell \subseteq \pi_p : \pi_p \in \mathcal{F}\}.$$

Proof. Take $[p] \in \mathcal{S} \setminus [e_7]$. Since $[p]$ a pole of degree 2, then there exists a plane π_p with $[p] \in \pi_p$ spanned by lines of $R^{\uparrow}(H)$. Note that $[p, e_7] \in R^{\uparrow}(H)$; so $[e_7] \in \pi_p$. Any line $\ell \subseteq \pi_p$ is in the upper radical of H. Indeed, let now $[q]$ be a point in π_p with $\pi_p \in \mathcal{F}$. If $[q] \notin [x, p]$, then $[x, q]$ and $[p, q]$ are both lines in the upper radical of H through $[q]$; since $\delta(q) = 2$ we have $\pi_q = \pi_p$. If $[q] \in [x, p]$ we consider a point $[r] \notin [x, p]$ and apply the same argument to show that $\pi_q = \pi_r = \pi_p$. Hence all lines of $\mathrm{PG}(V)$ in any plane $\pi_p \in \mathcal{F}$ are in the upper radical of H.

Suppose now $\pi_p, \pi_q \in \mathcal{F}$ and $\pi_p \cap \pi_q = r$ where r is a line through $[e_7]$. Then any point on r has degree at least 3, a contradiction. We have proved that the set $\mathcal{F}$ is a line-spread of $\mathrm{Res}_{\mathcal{S}}(e_7)$. The characterization of the upper radical is now straightforward. □

This proves part 6 of Theorem 3.

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Appendix A Tables

Table 1: Types for linear functionals on $\bigwedge^3 V$.

Type	Description	Rank	Special conditions, if any
T_1	$\underline{123}$	3	
T_2	$\underline{123} + \underline{145}$	5	
T_3	$\underline{123} + \underline{456}$	6	
T_4	$\underline{162} + \underline{243} + \underline{135}$	6	
T_5	$\underline{123} + \underline{456} + \underline{147}$	7	
T_6	$\underline{152} + \underline{174} + \underline{163} + \underline{243}$	7	
T_7	$\underline{146} + \underline{157} + \underline{245} + \underline{367}$	7	
T_8	$\underline{123} + \underline{145} + \underline{167}$	7	
T_9	$\underline{123} + \underline{456} + \underline{147} + \underline{257} + \underline{367}$	7	
$T^{(1)}_{10,\lambda}$	$\underline{123} + \lambda(\underline{156} + \underline{345} + \underline{426})$	6	$p_\lambda(t) := t^2 - \lambda$ irreducible in $\mathbb{K}[t]$.
$T^{(2)}_{10,\lambda}$	$\underline{126} + \underline{153} + \underline{234} +$ $(\lambda^2 + 1)\underline{456} + \lambda(\underline{156} + \underline{345} + \underline{426})$	6	char $(\mathbb{K}) = 2$ and $p_\lambda(t) := t^2 + \lambda t + 1$ irreducible in $\mathbb{K}[t]$.
$T^{(1)}_{11,\lambda}$	[the same as at $T^{(1)}_{10,\lambda}$] + $\underline{147}$	7	same conditions as for $T^{(1)}_{10,\lambda}$
$T^{(2)}_{11,\lambda}$	[the same as at $T^{(2)}_{10,\lambda}$] + $\underline{147}$	7	same conditions as for $T^{(2)}_{10,\lambda}$
$T_{12,\mu}$	μ[the same as at T_9]	7	$p_\mu(t) = t^3 - \mu$ irreducible in $\mathbb{K}[t]$

According to the clauses assumed on λ, types $T^{(r)}_{s,\lambda}$ ($r \in \{1, 2\}$, $s \in \{10, 11\}$) can be considered only when $\mathbb{K}$ is not quadratically closed. Moreover, when $\lambda \neq \lambda'$ the types $T^{(r)}_{s,\lambda}$ and $T^{(r)}_{s,\lambda'}$ are different up to linear and near equivalence, even if they might describe geometrically equivalent forms.

It follows from Revoy [12] and Cohen and Helminck [4] that two functionals of types T_i and T_j with $1 \leq i < j \leq 9$ are never nearly equivalent; a functional of type T_i with $i \leq 9$ is never nearly equivalent to a functional of type $T^{(r)}_{s,\lambda}$; two functionals of type $T^{(r)}_{s,\lambda}$ and $T^{(r')}_{s',\lambda'}$ with $(r, s) \neq (r', s')$ are never nearly equivalent while two functionals of type $T^{(r)}_{s,\lambda}$ and $T^{(r)}_{s,\lambda'}$ are nearly equivalent if and only if, denoted by μ and μ' respectively a root of $p_\lambda(t)$ and a root of $p_{\lambda'}(t)$ in the algebraic closure of $\mathbb{K}$, we have $\mathbb{K}(\mu) = \mathbb{K}(\mu')$ (see the fourth column of Table 1 for the definition of $p_\lambda(t)$). The forms T_9 and $T_{12,\mu}$ are not linearly equivalent; however they are, by construction, nearly equivalent.

Also, the forms $T^{(i)}_{10,\lambda}$ and $T^{(i)}_{10,\lambda'}$ as well as $T^{(i)}_{11,\lambda}$ and $T^{(i)}_{11,\lambda'}$ are not in general nearly-equivalent however they are geometrically equivalent. In particular both $T^{(1)}_{10,\lambda}$ and $T^{(2)}_{10,\lambda'}$ induce a Desarguesian spread on $\mathrm{PG}(V)$. Note that the forms $T^{(1)}_{10,\lambda}$ exist only if $\mathrm{char}\,(\mathbb{K}) \neq 2$ or if $\mathrm{char}\,(\mathbb{K}) = 2$ and $\mathbb{K}$ is not perfect. The forms of type $T^{(2)}_{10,\lambda}$ are equivalent to form of type $T^{(1)}_{10,\lambda}$ if $\mathrm{char}\,(\mathbb{K}) \neq 2$; however, they are not equivalent if $\mathrm{char}\,(\mathbb{K}) = 2$.

Table 2: Matrices associated to forms of rank up to 6.

Type	Rk	Matrix
T_1	3	$\begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}$
T_2	5	$\begin{pmatrix} 0 & u_3 & -u_2 & u_5 & -u_4 \\ -u_3 & 0 & u_1 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & 0 \\ -u_5 & 0 & 0 & 0 & u_1 \\ u_4 & 0 & 0 & -u_1 & 0 \end{pmatrix}$
T_3	6	$\begin{pmatrix} 0 & u_3 & -u_2 & 0 & 0 & 0 \\ -u_3 & 0 & u_1 & 0 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_6 & -u_5 \\ 0 & 0 & 0 & -u_6 & 0 & u_4 \\ 0 & 0 & 0 & u_5 & -u_4 & 0 \end{pmatrix}$
T_4	6	$\begin{pmatrix} 0 & -u_6 & u_5 & 0 & -u_3 & u_2 \\ u_6 & 0 & -u_4 & u_3 & 0 & -u_1 \\ -u_5 & u_4 & 0 & -u_2 & u_1 & 0 \\ 0 & -u_3 & u_2 & 0 & 0 & 0 \\ u_3 & 0 & -u_1 & 0 & 0 & 0 \\ -u_2 & u_1 & 0 & 0 & 0 & 0 \end{pmatrix}$
$T^{(1)}_{10,\lambda}$	6	$\begin{pmatrix} 0 & u_3 & -u_2 & 0 & \lambda u_6 & -\lambda u_5 \\ -u_3 & 0 & u_1 & -\lambda u_6 & 0 & \lambda u_4 \\ u_2 & -u_1 & 0 & \lambda u_5 & -\lambda u_4 & 0 \\ 0 & \lambda u_6 & -\lambda u_5 & 0 & \lambda u_3 & -\lambda u_2 \\ -\lambda u_6 & 0 & \lambda u_4 & -\lambda u_3 & 0 & \lambda u_1 \\ \lambda u_5 & -\lambda u_4 & 0 & \lambda u_2 & -\lambda u_1 & 0 \end{pmatrix}$
$T^{(2)}_{10,\lambda}$	6	$\begin{pmatrix} 0 & u_6 & -u_5 & 0 & \lambda u_6 + u_3 & -\lambda u_5 - u_2 \\ -u_6 & 0 & u_4 & -\lambda u_6 - u_3 & 0 & \lambda u_4 + u_1 \\ u_5 & -u_4 & 0 & \lambda u_5 + u_2 & -\lambda u_4 - u_1 & 0 \\ 0 & \lambda u_6 + u_3 & -\lambda u_5 - u_2 & 0 & (\lambda^2+1)u_6 + \lambda u_3 & (-\lambda^2-1)u_5 - \lambda u_2 \\ -\lambda u_6 - u_3 & 0 & \lambda u_4 + u_1 & -(\lambda^2+1)u_6 - \lambda u_3 & 0 & (\lambda^2+1)u_4 + \lambda u_1 \\ \lambda u_5 + u_2 & -\lambda u_4 - u_1 & 0 & (\lambda^2+1)u_5 + \lambda u_2 & -(\lambda^2+1)u_4 - \lambda u_1 & 0 \end{pmatrix}$

Table 3: Matrices associated to forms of rank 7.

Type	Matrix
T_5	$\begin{pmatrix} 0 & u_3 & -u_2 & u_7 & 0 & 0 & -u_4 \\ -u_3 & 0 & u_1 & 0 & 0 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & 0 & 0 & 0 \\ -u_7 & 0 & 0 & 0 & u_6 & -u_5 & u_1 \\ 0 & 0 & 0 & -u_6 & 0 & u_4 & 0 \\ 0 & 0 & 0 & u_5 & -u_4 & 0 & 0 \\ u_4 & 0 & 0 & -u_1 & 0 & 0 & 0 \end{pmatrix}$
T_6	$\begin{pmatrix} 0 & -u_5 & -u_6 & -u_7 & u_2 & u_3 & u_4 \\ u_5 & 0 & -u_4 & u_3 & -u_1 & 0 & 0 \\ u_6 & u_4 & 0 & -u_2 & 0 & -u_1 & 0 \\ u_7 & -u_3 & u_2 & 0 & 0 & 0 & -u_1 \\ -u_2 & u_1 & 0 & 0 & 0 & 0 & 0 \\ -u_3 & 0 & u_1 & 0 & 0 & 0 & 0 \\ -u_4 & 0 & 0 & u_1 & 0 & 0 & 0 \end{pmatrix}$
T_7	$\begin{pmatrix} 0 & 0 & 0 & u_6 & u_7 & -u_4 & -u_5 \\ 0 & 0 & 0 & u_5 & -u_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_7 & -u_6 \\ -u_6 & -u_5 & 0 & 0 & u_2 & u_1 & 0 \\ -u_7 & u_4 & 0 & -u_2 & 0 & 0 & u_1 \\ u_4 & 0 & -u_7 & -u_1 & 0 & 0 & u_3 \\ u_5 & 0 & u_6 & 0 & -u_1 & -u_3 & 0 \end{pmatrix}$

Type	Matrix
T_8	$\begin{pmatrix} 0 & u_3 & -u_2 & u_5 & -u_4 & u_7 & -u_6 \\ -u_3 & 0 & u_1 & 0 & 0 & 0 & 0 \\ u_2 & -u_1 & 0 & 0 & 0 & 0 & 0 \\ -u_5 & 0 & 0 & 0 & u_1 & 0 & 0 \\ u_4 & 0 & 0 & -u_1 & 0 & 0 & 0 \\ -u_7 & 0 & 0 & 0 & 0 & 0 & u_1 \\ u_6 & 0 & 0 & 0 & 0 & -u_1 & 0 \end{pmatrix}$
T_9	$\begin{pmatrix} 0 & u_3 & -u_2 & u_7 & 0 & 0 & -u_4 \\ -u_3 & 0 & u_1 & 0 & u_7 & 0 & -u_5 \\ u_2 & -u_1 & 0 & 0 & 0 & u_7 & -u_6 \\ -u_7 & 0 & 0 & 0 & u_6 & -u_5 & u_1 \\ 0 & -u_7 & 0 & -u_6 & 0 & u_4 & u_2 \\ 0 & 0 & -u_7 & u_5 & -u_4 & 0 & u_3 \\ u_4 & u_5 & u_6 & -u_1 & -u_2 & -u_3 & 0 \end{pmatrix}$
$T^{(1)}_{11,\lambda}$	$\begin{pmatrix} 0 & u_3 & -u_2 & u_7 & \lambda u_6 & -\lambda u_5 & -u_4 \\ -u_3 & 0 & u_1 & -\lambda u_6 & 0 & \lambda u_4 & 0 \\ u_2 & -u_1 & 0 & \lambda u_5 & -\lambda u_4 & 0 & 0 \\ -u_7 & \lambda u_6 & -\lambda u_5 & 0 & \lambda u_3 & -\lambda u_2 & u_1 \\ -\lambda u_6 & 0 & \lambda u_4 & -\lambda u_3 & 0 & \lambda u_1 & 0 \\ \lambda u_5 & -\lambda u_4 & 0 & \lambda u_2 & -\lambda u_1 & 0 & 0 \\ u_4 & 0 & 0 & -u_1 & 0 & 0 & 0 \end{pmatrix}$

Type	Matrix
$T^{(2)}_{11,\lambda}$	$\begin{pmatrix} 0 & u_6 & -u_5 & u_7 & \lambda u_6 + u_3 & -\lambda u_5 - u_2 & -u_4 \\ -u_6 & 0 & u_4 & -\lambda u_6 - u_3 & 0 & \lambda u_4 + u_1 & 0 \\ u_5 & -u_4 & 0 & \lambda u_5 + u_2 & -\lambda u_4 - u_1 & 0 & 0 \\ -u_7 & \lambda u_6 + u_3 & -\lambda u_5 - u_2 & 0 & (\lambda^2+1)u_6 + \lambda u_3 & (-\lambda^2-1)u_5 - \lambda u_2 & u_1 \\ -\lambda u_6 - u_3 & 0 & \lambda u_4 + u_1 & -(\lambda^2+1)u_6 - \lambda u_3 & 0 & (\lambda^2+1)u_4 + \lambda u_1 & 0 \\ \lambda u_5 + u_2 & -\lambda u_4 - u_1 & 0 & (\lambda^2+1)u_5 + \lambda u_2 & -(\lambda^2+1)u_4 - \lambda u_1 & 0 & 0 \\ u_4 & 0 & 0 & -u_1 & 0 & 0 & 0 \end{pmatrix}$

Table 4: Description of the geometries of poles associated to forms of rank up to 6.

Type	Poles	Upper radical
T_1	PG(V)	$\lvert x,y\rvert_{12}=0, \lvert x,y\rvert_{13}=0, \lvert x,y\rvert_{23}=0.$
T_2	PG(V)	$\lvert x,y\rvert_{12}=0, \lvert x,y\rvert_{13}=0, \lvert x,y\rvert_{15}=0,$ $\lvert x,y\rvert_{14}=0, \lvert x,y\rvert_{23}+\lvert x,y\rvert_{45}=0.$
T_3	PG(V)	$\lvert x,y\rvert_{12}=0, \lvert x,y\rvert_{13}=0, \lvert x,y\rvert_{23}=0,$ $\lvert x,y\rvert_{45}=0, \lvert x,y\rvert_{46}=0, \lvert x,y\rvert_{56}=0.$
T_4	PG(V)	$\lvert x,y\rvert_{26}-\lvert x,y\rvert_{35}=0, \lvert x,y\rvert_{16}-\lvert x,y\rvert_{34}=0$ $\lvert x,y\rvert_{24}-\lvert x,y\rvert_{15}=0, \lvert x,y\rvert_{23}=0,$ $\lvert x,y\rvert_{13}=0, \lvert x,y\rvert_{12}=0.$
$T^{(1)}_{10,\lambda}$	PG(V)	$\lvert x,y\rvert_{23}+\lambda\lvert x,y\rvert_{56}=0, \lvert x,y\rvert_{26}-\lvert x,y\rvert_{35}=0$ $\lvert x,y\rvert_{13}+\lambda\lvert x,y\rvert_{46}=0, \lvert x,y\rvert_{16}-\lvert x,y\rvert_{34}=0$ $\lvert x,y\rvert_{12}+\lambda\lvert x,y\rvert_{45}=0, \lvert x,y\rvert_{15}-\lvert x,y\rvert_{24}=0$
$T^{(2)}_{10,\lambda}$	PG(V)	$\lvert x,y\rvert_{26}-\lvert x,y\rvert_{35}+\lambda\lvert x,y\rvert_{56}=0,$ $-\lvert x,y\rvert_{16}+\lvert x,y\rvert_{34}-\lambda\lvert x,y\rvert_{46}=0,$ $\lvert x,y\rvert_{15}-\lvert x,y\rvert_{24}+\lambda\lvert x,y\rvert_{45}=0,$ $\lvert x,y\rvert_{23}+\lambda\lvert x,y\rvert_{26}-\lambda\lvert x,y\rvert_{35}+(\lambda^2+1)\lvert x,y\rvert_{56}=0,$ $-\lvert x,y\rvert_{13}-\lambda\lvert x,y\rvert_{16}+\lambda\lvert x,y\rvert_{34}-(1+\lambda^2)\lvert x,y\rvert_{46}=0,$ $\lvert x,y\rvert_{12}+\lambda\lvert x,y\rvert_{15}-\lambda\lvert x,y\rvert_{24}+(1+\lambda^2)\lvert x,y\rvert_{45}=0.$

Table 5: Description of the geometry of poles associated to forms of rank 7.

Type	Poles	Upper radical
T_5	$x_1x_4 = 0$	$\|x,y\|_{23} + \|x,y\|_{47} = 0, \|x,y\|_{13} = 0, \|x,y\|_{12} = 0,$ $\|x,y\|_{56} - \|x,y\|_{17} = 0, \|x,y\|_{14} = 0, \|x,y\|_{45} = 0,$ $\|x,y\|_{46} = 0$
T_6	$x_1^2 = 0$	$\|x,y\|_{25} + \|x,y\|_{36} + \|x,y\|_{47} = 0, \|x,y\|_{14} = 0,$ $\|x,y\|_{15} - \|x,y\|_{34} = 0, \|x,y\|_{16} + \|x,y\|_{24} = 0,$ $\|x,y\|_{17} - \|x,y\|_{23} = 0, \|x,y\|_{12} = 0, \|x,y\|_{13} = 0.$
T_7	$x_5x_7 + x_4x_6 = 0$	$\|x,y\|_{46} + \|x,y\|_{57} = 0, \|x,y\|_{45} = 0, \|x,y\|_{67} = 0,$ $\|x,y\|_{16} + \|x,y\|_{25} = 0, \|x,y\|_{24} - \|x,y\|_{17} = 0,$ $\|x,y\|_{14} - \|x,y\|_{37} = 0, \|x,y\|_{15} + \|x,y\|_{36} = 0.$
T_8	$x_1^2 = 0$	$\|x,y\|_{23} + \|x,y\|_{45} + \|x,y\|_{67} = 0, \|x,y\|_{13} = 0,$ $\|x,y\|_{12} = 0, \|x,y\|_{14} = 0, \|x,y\|_{15} = 0,$ $\|x,y\|_{16} = 0, \|x,y\|_{17} = 0.$
T_9	$x_7^2 - x_3x_6 - x_2x_5 - x_1x_4 = 0$	$\|x,y\|_{23} + \|x,y\|_{47} = 0, \|x,y\|_{57} - \|x,y\|_{13} = 0$ $\|x,y\|_{12} + \|x,y\|_{67} = 0, \|x,y\|_{56} - \|x,y\|_{17} = 0$ $\|x,y\|_{27} + \|x,y\|_{46} = 0, \|x,y\|_{45} - \|x,y\|_{37} = 0$ $\|x,y\|_{14} + \|x,y\|_{25} + \|x,y\|_{36} = 0.$
$T_{11,\lambda}^{(1)}$	$\lambda x_4^2 - x_1^2 = 0$	$\|x,y\|_{13} + \lambda\|x,y\|_{46} = 0, \|x,y\|_{12} + \lambda\|x,y\|_{45} = 0$ $\|x,y\|_{23} + \|x,y\|_{47} + \lambda\|x,y\|_{56} = 0, \|x,y\|_{14} = 0,$ $-\|x,y\|_{17} + \lambda(\|x,y\|_{26} - \|x,y\|_{35}) = 0,$ $\|x,y\|_{15} - \|x,y\|_{24} = 0, \|x,y\|_{16} - \|x,y\|_{34} = 0.$
$T_{11,\lambda}^{(2)}$	$x_4^2 + \lambda x_1x_4 + x_1^2 = 0$	$\|x,y\|_{26} - \|x,y\|_{35} + \|x,y\|_{47} + \lambda\|x,y\|_{56} = 0,$ $\|x,y\|_{16} - \|x,y\|_{34} + \lambda\|x,y\|_{46} = 0, \|x,y\|_{14} = 0,$ $\|x,y\|_{15} - \|x,y\|_{24} + \lambda\|x,y\|_{45} = 0,$ $\|x,y\|_{17} - \|x,y\|_{23} - \lambda(\|x,y\|_{26} - \|x,y\|_{35}) -$ $-(\lambda^2 + 1)\|x,y\|_{56} = 0,$ $\|x,y\|_{13} + \lambda(\|x,y\|_{16} - \|x,y\|_{34}) +$ $+(\lambda^2 + 1)\|x,y\|_{46} = 0,$ $\|x,y\|_{12} + \lambda\|x,y\|_{15} - \lambda\|x,y\|_{24} +$ $+(\lambda^2 + 1)\|x,y\|_{45} = 0.$