Down-linking (K_v, Γ)-designs to P_3-designs

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Abstract

Let Γ' be a subgraph of a graph Γ. We define a *down-link* from a (K_v, Γ)-design $\mathcal{B}$ to a (K_n, Γ')-design $\mathcal{B}'$ as an injective map $f : \mathcal{B} \to \mathcal{B}'$ mapping any block of $\mathcal{B}$ into one of its subgraphs. This is a new concept, closely related with both the classical notion of *embedding* and the more recent one of *sampling*. In the present paper we study down-links in general and prove that any (K_v, Γ)-design might be down-linked to a (K_n, Γ')-design, provided that n is suitably chosen. We also investigate in detail down-links to P_3-designs, P_3 being the path with 3 vertices, proving that it is always possible to take $n \leq v + 3$. Furthermore, for several classes of graphs Γ, we provide explicit constructions of down-links from balanced Γ-designs to P_3-designs, improving on the aforementioned bound.

Keywords: down-link; embedding; (K_v, Γ)-design.
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1 Introduction

Let K_v be the complete undirected graph on v vertices, and assume $\Gamma \leq K_v$ to be a subgraph of K_v. For any graph Γ, write $V(\Gamma)$ for the set of vertices of Γ and $E(\Gamma)$ for the set of its edges. A (K_v, Γ)-*design*, also called a Γ-*design of order* v, is a set $\mathcal{B}$ of graphs, called *blocks*, isomorphic to Γ and whose edges partition $E(K_v)$. A (K_v, Γ)-design is *balanced* if each vertex of K_v occurs in the same number of blocks. The problem of determining the existence of (K_v, Γ)-designs for a given graph Γ has been extensively studied; for surveys on this topic see, for instance, [3, 4, 20].

We propose the following new definition.

Definition 1.1. *Given a (K_v, Γ)-design $\mathcal{B}$ and a (K_n, Γ')-design $\mathcal{B}'$ with $\Gamma' \leq \Gamma$, a* down-link *from $\mathcal{B}$ to $\mathcal{B}'$ is an injective function $f : \mathcal{B} \to \mathcal{B}'$ such that $f(B) \leq B$, for any $B \in \mathcal{B}$.*

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If such a function f exists, that is if each block of $\mathcal{B}$ contains at least an element of $\mathcal{B}'$ as a subgraph, we will say that it is possible to *down-link* $\mathcal{B}$ to $\mathcal{B}'$.

In Section 2 we investigate the relationship between the new concept of down-link, the classical notion of embedding, see [21], and the recently defined concept of sampling, see [13]. In Section 3 we will introduce, in close analogy to embeddings, the main problems on the spectra of down-links and determine bounds on their minima. In Section 4 down-links from any (K_v, Γ)-design to P_3-designs of order at most $v + 3$ are constructed; this improves, for the case under consideration, on the values found in Section 3. In further sections 5, 6, 7, 8 we investigate the existence and construct down-links to P_3-designs from, respectively, balanced star-designs, kite-designs, cycle systems and path-designs; the problem of determining the spectrum of down-links to P_3-designs is fully solved, by providing suitable functions, in the case of stars and kites, as well as, for 4-cycle systems and P_4-designs.

2 Down-linking, embedding and sampling

The notions of down-link, embedding and sampling are closely related.

Recall that an *embedding* of a design $\mathcal{B}'$ into a design $\mathcal{B}$ is a function $\psi : \mathcal{B}' \to \mathcal{B}$ such that $\Gamma \leq \psi(\Gamma)$, for any $\Gamma \in \mathcal{B}'$; see [21]. An injective embedding is called *strict*. Existence of embeddings of designs has been widely investigated. In particular, a great deal of interesting results are known on strict embeddings of path-designs; see, for instance, [8, 9, 10, 12, 16, 18, 19, 22, 23]. If $\psi : \mathcal{B}' \to \mathcal{B}$ is a *bijective* embedding, clearly ψ^{-1} is a down-link from $\mathcal{B}$ to $\mathcal{B}'$. The straightforward necessary condition for the existence of a bijective embedding of $\mathcal{B}'$ into $\mathcal{B}$, namely that $\mathcal{B}$ and $\mathcal{B}'$ have the same number of blocks, is quite restrictive. However, as the following example shows, this does not necessarily lead to trivial embeddings.

Example 2.1. *Consider the* (K_4, P_3)*-design*

$$\mathcal{B}' = \{\Gamma_1' = [1, 2, 3], \Gamma_2' = [1, 3, 0], \Gamma_3' = [2, 0, 1]\}$$

and the balanced (K_6, P_6)*-design*

$$\mathcal{B} = \{\Gamma_1 = [4, 0, 5, 1, 2, 3], \Gamma_2 = [2, 5, 4, 1, 3, 0], \Gamma_3 = [5, 3, 4, 2, 0, 1]\}.$$

Define $\psi : \mathcal{B}' \to \mathcal{B}$ *by* $\psi(\Gamma_i') = \Gamma_i$ *for* $i = 1, 2, 3$. *Then,* ψ *is a bijective embedding; consequently,* ψ^{-1} *is a down-link from* $\mathcal{B}$ *to* $\mathcal{B}'$.

Obviously, a necessary condition for the existence of an embedding of a (K_n, Γ')-design into a (K_v, Γ)-design is $v \geq n$. On the contrary, the existence of a down-link from a (K_v, Γ)-design to a (K_n, Γ')-design implies neither $v \geq n$ nor $n \geq v$, as we can deduce out of examples 2.1 and 2.2.

Example 2.2. *Consider the balanced* (K_7, P_4)*-design*

$$\mathcal{B} = \{[0,3,1,2],[1,0,2,3],[2,6,1,4],[3,6,4,5],[4,3,5,6],[5,2,4,0],[6,0,5,1]\},$$

and take the (K_8, P_3)*-design*

$$\mathcal{B}' = \{[3,1,2],[0,2,3],[6,1,4],[6,4,5],[3,5,6],[2,4,0],[0,5,1],\\ [7,0,3],[7,1,0],[7,2,6],[7,3,6],[7,4,3],[7,5,2],[7,6,0]\}.$$

It is easy to see that $\mathcal{B}$ *can be down-linked to* $\mathcal{B}'$*. Clearly, as* $|\mathcal{B}'| > |\mathcal{B}|$*, an embedding of* $\mathcal{B}'$ *into* $\mathcal{B}$ *cannot exist.*

Down-links are also related to *samplings.* Samplings have been introduced in [13] for complete designs and studied extensively therein. In the case of graph decompositions, the analogous definition is the following.

Definition 2.3. *Given a* (K_v, Γ)*-design* $\mathcal{B}$ *and a* (K_n, Γ')*-design* $\mathcal{B}'$*, with* $\Gamma' \leq \Gamma$*, a* sampling ξ *is any surjective function* $\xi : \mathcal{B} \to \mathcal{B}'$ *with* $\xi(B) \leq B$ *for all* $B \in \mathcal{B}$*.*

Indeed, both down-links and samplings act on elements of a design $\mathcal{B}$ in a similar way — namely, they map a graph Γ into one of its subgraphs, suitably chosen in $\mathcal{B}'$. However, as the following Proposition 2.4 shows, any sampling, in the case of graph decompositions, is a bijective down-link and, as such, also the inverse of an embedding.

Proposition 2.4. *Any sampling* ξ *of a* (K_v, Γ)*-design* $\mathcal{B}$ *into a* (K_n, Γ')*-design* $\mathcal{B}'$ *is bijective.*

Proof. By definition of sampling, ξ is surjective. For any $e \in E(K_n)$, there is exactly one graph $\Gamma' \in \mathcal{B}'$ with $e \in E(\Gamma')$. On the other hand, there is also exactly one graph $\Gamma \in \mathcal{B}$ with $e \in E(\Gamma)$. It follows that the set of the preimages of Γ' is $\{\Gamma\}$; hence, ξ is injective. □

3 Spectrum problems

Let $\Gamma' \leq \Gamma$. The following spectrum problems about existence of embeddings have been considered.

(A) For each admissible n, determine the set $\mathcal{S}_1(n)$ of all integers v such that there exists *some* Γ'-design of order n embedded into a Γ-design of order v.

(B) For each admissible n, determine the set $\mathcal{S}_2(n)$ of all integers v such that *every* Γ'-design of order n can be embedded into a Γ-design of order v.

For some classes of designs, problems (A) and (B) have been fully solved, respectively in [9, 12, 18, 22, 23] and [11, 12].

In the present paper we pose the following analogous questions about down-links:

(I) For each admissible v, determine the set $\mathcal{L}_1(v)$ of all integers n such that there exists *some* Γ-design of order v down-linked to a Γ'-design of order n.

(II) For each admissible v, determine the set $\mathcal{L}_2(v)$ of all integers n such that *every* Γ-design of order v can be down-linked to a Γ'-design of order n.

In general, write $\eta_i(v; \Gamma, \Gamma') = \inf \mathcal{L}_i(v)$. When the graphs Γ and Γ' are easily understood from the context, we shall simply use $\eta_i(v)$ instead of $\eta_i(v; \Gamma, \Gamma')$.

The problem of the actual existence of down-links, for given $\Gamma' \leq \Gamma$, is addressed in Proposition 3.2. We recall the following lemma on the existence of finite embeddings for partial decompositions, a straightforward consequence of an asymptotic result by R.M. Wilson [28, Lemma 6.1]; see also [6].

Lemma 3.1. *Any partial* (K_v, Γ)*-design can be embedded into a* (K_n, Γ)*-design with* $n = O((v^2/2)^{v^2})$.

Proposition 3.2. *For any* v *such that there exists a* (K_v, Γ)*-design and any* $\Gamma' \leq \Gamma$*, the sets* $\mathcal{L}_1(v)$ *and* $\mathcal{L}_2(v)$ *are non-empty.*

Proof. Fix first a (K_v, Γ)-design $\mathcal{B}$. Denote by $K_v(\Gamma')$ the so called *complete* (K_v, Γ')*-design*, that is the set of all subgraphs of K_v isomorphic to Γ', and let $\zeta : \mathcal{B} \to K_v(\Gamma')$ be any function such that $\zeta(\Gamma) \leq \Gamma$ for all $\Gamma \in \mathcal{B}$. Clearly, the image of ζ is a partial (K_v, Γ')-design $\mathcal{P}$; see [7]. By Lemma 3.1, there is an integer n such that $\mathcal{P}$ embeds into a (K_n, Γ')-design $\mathcal{B}'$. Let $\psi : \mathcal{P} \to \mathcal{B}'$ be such an embedding; then, $\xi = \psi\zeta$ is, clearly, a down-link from $\mathcal{B}$ to a Γ'-design $\mathcal{B}'$ of order n. Thus, we have shown that for any (K_v, Γ)-design and for any $\Gamma' \leq \Gamma$ the set $\mathcal{L}_1(v)$ is non-empty.

To show that $\mathcal{L}_2(v)$ is also non-empty, proceed as follows. Let ω be the number of distinct (K_v, Γ)-designs $\mathcal{B}_i$. For any $i = 0, \ldots, \omega - 1$, write $V(\mathcal{B}_i) = \{0, \ldots, v-1\} + i \cdot v$. Consider now $\Omega = \bigcup_{i=0}^{\omega-1} \mathcal{B}_i$. Clearly, Ω is a partial Γ-design of order $v\omega$. As above, take $K_{v\omega}(\Gamma')$ and construct a function $\zeta : \Omega \to K_{v\omega}(\Gamma')$ associating to each $\Gamma \in \mathcal{B}_i$ a $\zeta(\Gamma) \leq \Gamma$. The image $\bigcup_i \zeta(\mathcal{B}_i)$ is a partial Γ'-design Ω'. Using Lemma 3.1 once more, we determine an integer n and an embedding ψ of Ω' into a (K_n, Γ')-design $\mathcal{B}'$. For any i, let ζ_i be the restriction of ζ to $\mathcal{B}_i$. It is straightforward to see that $\psi\zeta_i : \mathcal{B}_i \to \mathcal{B}'$ is a down-link from $\mathcal{B}_i$ to a (K_n, Γ')-design. It follows that $n \in \mathcal{L}_2(v)$. □

Notice that the order of magnitude of n is v^{2v^2}; yet, it will be shown that, in several cases, it is possible to construct down-links from (K_v, Γ)-designs to (K_n, Γ')-designs with $n \approx v$.

Furthermore, it is easy to prove the following lower bound on $\eta_1(v; \Gamma, \Gamma')$:

$$(v-1)\sqrt{\frac{|E(\Gamma')|}{|E(\Gamma)|}} \leq \eta_1(v; \Gamma, \Gamma').$$

4 Down-linking (K_v, Γ)-designs to P_3-designs

In this section we shall focus our attention on the existence of down-links from (K_v, Γ)-designs to (K_n, P_3)-designs. Observe first that it has been shown in [26] that a (K_n, P_3)-design exists if, and only if, $n \equiv 0, 1 (\mathrm{mod}\, 4)$.

In order to prove our results we need some technical preliminaries. The *star* on k vertices S_k is the complete bipartite graph with one part having a single vertex, say c, called the *center* of the star, and the other part having $k-1$ vertices, say x_i for $i = 0, \ldots, k-2$, called *external vertices*. In general, we shall write $S_k = [c; x_0, x_1, \ldots, x_{k-2}]$. Consider the set

$$\mathcal{P}_3(S_k) = \left\{ [x_{2\ell}, c, x_{2\ell+1}] \mid \ell = 0, \ldots, \left\lfloor \frac{k-3}{2} \right\rfloor \right\}. \tag{1}$$

Note that if k is even, $E(S_k) = E(\mathcal{P}_3(S_k)) \cup \{[c, x_{k-2}]\}$, while if k is odd, $E(S_k) = E(\mathcal{P}_3(S_k))$.

Let now $\mathcal{B}$ be a (K_v, Γ)-design with $P_3 \leq \Gamma$. For any $B \in \mathcal{B}$, denote by $\mathcal{P}_3(B)$ a maximal partial decomposition of B in P_3's. Define also

$$\mathcal{R}(B) = E(B) \setminus \bigcup_{P \in \mathcal{P}_3(B)} E(P).$$

By the maximality of $\mathcal{P}_3(B)$, distinct edges in $\mathcal{R}(B)$ have no vertex in common.

Write

$$\mathcal{P}_3(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{P}_3(B); \qquad \mathcal{R}(\mathcal{B}) = \bigcup_{B \in \mathcal{B}} \mathcal{R}(B).$$

Clearly, $\mathcal{P}_3(\mathcal{B})$ is a partial, yet not necessarily maximal, (K_v, P_3)-design. If not all edges in $\mathcal{R}(\mathcal{B})$ are disjoint, extract as many P_3's from $\mathcal{R}(\mathcal{B})$ as possible, thus determining a new set $\mathcal{P}_3(\mathcal{R}(\mathcal{B}))$. Let

$$\mathcal{R}^\flat(\mathcal{B}) = \{[x_{2i}, x_{2i+1}] \mid i = 0, \ldots, \mu - 1\}$$

be the remaining disjoint edges and write $\mathcal{V} = V(K_v) \setminus V(\mathcal{R}^\flat(\mathcal{B}))$.

We distinguish several cases:

a) $v \equiv 0 (\mathrm{mod}\ 4)$. In this case μ is even. Consider $V(K_{v+1}) = V(K_v) \cup \{\alpha\}$ and let S^α be the star with center α and external vertices the elements of $\mathcal{V}$. Define $\mathcal{P}^\alpha$ as

$$\Big\{[x_{4i}, x_{4i+1}, \alpha], [x_{4i}, \alpha, x_{4i+3}], [\alpha, x_{4i+2}, x_{4i+3}] \mid i = 0, \dots, \left\lfloor \frac{\mu}{2} \right\rfloor - 1\Big\} \\ \cup\ \mathcal{P}_3(S^\alpha). \qquad (2)$$

It is easy to see that

$$\mathcal{B}' = \mathcal{P}_3(\mathcal{B}) \ \cup\ \mathcal{P}_3(\mathcal{R}(\mathcal{B})) \ \cup\ \mathcal{P}^\alpha$$

is a (K_{v+1}, P_3)-design.

b) $v \equiv 1 (\mathrm{mod}\ 4)$. Also under this assumption μ is even. Write $V(K_{v+3}) = V(K_v) \cup \{\alpha, \beta, \gamma\}$. Take $\mathcal{P}^\alpha$ as in (2). Note that

$$\mathcal{P}_3(\mathcal{B}) \ \cup\ \mathcal{P}_3(\mathcal{R}(\mathcal{B})) \ \cup\ \mathcal{P}^\alpha$$

is a (K_{v+1}, P_3)-design minus an edge, say $e = [\alpha, x]$. Let S^β and S^γ respectively be the stars having center β and γ and external vertices $V(K_v) \setminus \{x\}$. Put

$$\mathcal{P}^{\beta,\gamma} = \{[\beta, x, \gamma], [x, \alpha, \beta], [\alpha, \gamma, \beta]\} \ \cup\ \mathcal{P}_3(S^\beta) \ \cup \mathcal{P}_3(S^\gamma).$$

It might be easily verified that

$$\mathcal{B}' = \mathcal{P}_3(\mathcal{B}) \ \cup\ \mathcal{P}_3(\mathcal{R}(\mathcal{B})) \ \cup\ \mathcal{P}^\alpha \ \cup\ \mathcal{P}^{\beta,\gamma}$$

is a (K_{v+3}, P_3)-design.

c) $v \equiv 2 (\mathrm{mod}\ 4)$, which implies μ odd. Write $V(K_{v+2}) = V(K_v) \cup \{\alpha, \beta\}$ and take $\mathcal{P}^\alpha$ as in (2). Let S^β be the star with center β and external vertices the elements of $V(K_v)$. Define

$$\mathcal{P}^\beta = \{[x_{2\mu-2}, x_{2\mu-1}, \alpha], [x_{2\mu-2}, \alpha, \beta]\} \ \cup\ \mathcal{P}_3(S^\beta).$$

It is possible to directly check that

$$\mathcal{B}' = \mathcal{P}_3(\mathcal{B}) \ \cup\ \mathcal{P}_3(\mathcal{R}(\mathcal{B})) \ \cup\ \mathcal{P}^\alpha \ \cup\ \mathcal{P}^\beta$$

is a (K_{v+2}, P_3)-design.

d) $v \equiv 3 (\mathrm{mod}\ 4)$, so μ is odd. Write $V(K_{v+1}) = V(K_v) \cup \{\alpha\}$. Take S^α and $\mathcal{P}^\alpha$ as in (2). Note that since S^α has an odd number of external vertices, we have $E(S^\alpha) \setminus E(\mathcal{P}_3(S^\alpha)) = [\alpha, y]$. Let

$$\mathcal{P} = \{[x_{2\mu-2}, x_{2\mu-1}, \alpha], [x_{2\mu-2}, \alpha, y]\}.$$

It is not hard to verify that

$$\mathcal{B}' = \mathcal{P}_3(\mathcal{B}) \ \cup\ \mathcal{P}_3(\mathcal{R}(\mathcal{B})) \ \cup\ \mathcal{P}^\alpha \ \cup\ \mathcal{P}$$

is a (K_{v+1}, P_3)-design.

Any $f : \mathcal{B} \to \mathcal{B}'$ mapping $B \in \mathcal{B}$ to a $B' \in \mathcal{P}_3(B)$ is a down-link.

We may now state the main theorem of this section.

Theorem 4.1. *For any (K_v, Γ)-design with $P_3 \leq \Gamma$,*

$$\eta_1(v) \leq \eta_2(v) \leq v + 3.$$

The down-links constructed above are not, in general, to designs whose order is as small as possible; thus, Theorem 4.1 does not provide, unless further assumptions are made, $\eta_1(v)$ exactly.

Remark 4.2. *In general, a (K_n, P_3)-design can be trivially embedded into P_3-designs of any admissible order $m \geq n$. Thus, for down-links to P_3-designs, given any $n \in \mathcal{L}_i(v)$,*

$$\{m \geq n \mid m \equiv 0, 1 (\bmod 4)\} \subseteq \mathcal{L}_i(v) \subseteq \{m \geq \eta_i(v) \mid m \equiv 0, 1 (\bmod 4)\}.$$

Hence, solving problems (I) *and* (II) *for (K_v, Γ)-designs is actually equivalent to determining exactly the values $\eta_1(v; \Gamma, P_3)$ and $\eta_2(v; \Gamma, P_3)$.*

5 Balanced star-designs

In this section the existence of down-links from balanced star-designs to P_3-designs is investigated; in particular, we shall use the notations introduced at the beginning of Section 4.

In [27], Tarsi proved that a (K_v, S_k)-design exists if, and only if, $v \geq 2k-2$ and $v(v-1) \equiv 0 (\bmod 2k-2)$. It is not hard to verify that a *balanced* (K_v, S_k)-design exists if, and only if, $v > 1$ and $v \equiv 1 (\bmod 2k-2)$.
Suppose then $v \equiv 1 (\bmod 2k-2)$, $v > 1$. Denote by $\mathcal{LS}_1(v)$ the set of all the integers $n \equiv 0, 1 (\bmod 4)$ such that there exists *some* balanced S_k-design of order v down-linked to a P_3-design of order n. Write $\mathcal{LS}_2(v)$ for the set of all the integers $n \equiv 0, 1 (\bmod 4)$ such that *every* balanced S_k-design of order v can be down-linked to a P_3-design of order n. We shall fully determine the sets $\mathcal{LS}_1(v)$ and $\mathcal{LS}_2(v)$, solving problems (I) and (II).

Theorem 5.1. *Assume $k \geq 3$. For every $v \equiv 1 (\bmod 2k-2)$, $v > 1$,*

$$\mathcal{LS}_2(v) \subseteq \mathcal{LS}_1(v) \subseteq \{n \mid n \geq v,\ n \equiv 0, 1 (\bmod 4)\}.$$

Proof. Let $\mathcal{B}$ be a balanced (K_v, S_k)-design and let $\mathcal{B}'$ be a (K_n, P_3)-design. A direct computation proves that any vertex of K_v is contained in exactly $\frac{(v-1)k}{2(k-1)}$ blocks of $\mathcal{B}$. Let $x \in V(K_v)$. Write m for the number of blocks of $\mathcal{B}$ whose center is x and p for the number of blocks of $\mathcal{B}$ where x is external, so that $m + p = \frac{(v-1)k}{2(k-1)}$. Since in S_k the center has degree $k-1$, while an external vertex has degree 1, we have $(k-1)m + p = v - 1$. Thus, $m = \frac{v-1}{2(k-1)}$.

Hence, any vertex of K_v is the center of $\frac{v-1}{2(k-1)} > 0$ stars of $\mathcal{B}$. Let now $f : \mathcal{B} \to \mathcal{B}'$ be a down-link. For every $S \in \mathcal{B}$, $f(S)$ contains the center of S. Hence, the image of $\mathcal{B}$ by f contains every vertex of K_v. Thus, $n \geq v$. □

Lemma 5.2. *Assume $v \equiv 1 (\mathrm{mod}\, 4)$ admissible. Then, every balanced (K_v, S_k)-design can be down-linked to a (K_v, P_3)-design.*

Proof. Let $\mathcal{B}$ be a balanced (K_v, S_k)-design. Note that k is odd, since $v \equiv 1 (\mathrm{mod}\, 4)$ and $v \equiv 1 (\mathrm{mod}\, 2k-2)$. It is easy to see that

$$\mathcal{B}' = \bigcup_{S_k \in \mathcal{B}} \mathcal{P}_3(S_k),$$

where $\mathcal{P}_3(S_k)$ is as in (1), is a P_3-design of order v. A function $f : \mathcal{B} \to \mathcal{B}'$, sending any S_k to a $P_3 \in \mathcal{P}_3(S_k)$, is a down-link. □

Lemma 5.3. *Assume $v \equiv 3 (\mathrm{mod}\, 4)$ admissible. Then, every balanced (K_v, S_k)-design can be down-linked to a (K_{v+1}, P_3)-design.*

Proof. Let $\mathcal{B}$ be a balanced (K_v, S_k)-design. Note that, since $v \equiv 3 (\mathrm{mod}\ 4)$, k is even and every $c \in V(K_v)$ is the center of an odd number of stars, say m. Let $V(K_{v+1}) = V(K_v) \cup \{\alpha\}$. We can write $\mathcal{B} = \bigcup_{c \in V} \mathcal{B}_c$, where $\mathcal{B}_c$ is the set of the stars of $\mathcal{B}$ with center c. For any $c \in V(K_v)$ consider

$$\mathcal{P}_3(\mathcal{B}_c) = \{\mathcal{P}_3(S_k) \mid S_k \in \mathcal{B}_c\}.$$

Observe that the edges of $\mathcal{B}_c$ not contained in $\mathcal{P}_3(\mathcal{B}_c)$ give a star $S_c^* = [c; y_0, \ldots, y_{m-1}]$ of center c and with m edges. Define

$$\mathcal{B}_c^* = \{[\alpha, c, y_{m-1}]\}.$$

It is easy to see that

$$\mathcal{B}' = \bigcup_{c \in V} \mathcal{P}_3(\mathcal{B}_c) \cup \mathcal{P}_3(S_c^*) \cup \mathcal{B}_c^*$$

is a (K_{v+1}, P_3)-design. Note that if $m = 1$, then $\mathcal{P}_3(S_c^*)$ is empty for all $c \in V$. Any function $f : \mathcal{B} \to \mathcal{B}'$ mapping S_k to a $P_3 \in \mathcal{P}_3(S_k)$ is a down-link. □

In general, provided the conditions hold, any given (K_v, S_k)-design $\mathcal{B}$ can be down-linked to several non-equivalent (K_n, P_3)-designs $\mathcal{B}'$. Even if $\mathcal{B}'$ is fixed, the construction of a down-link $f : \mathcal{B} \to \mathcal{B}'$ is not unique. Actually, it is easy to see that lemmas 5.2 and 5.3 do provide with $\lfloor \frac{k-1}{2} \rfloor^{\frac{v(v-1)}{2(k-1)}}$ distinct down-links, since each star contains exactly $\lfloor \frac{k-1}{2} \rfloor$ paths $P_3 \in \mathcal{B}'$.

Example 5.4. *We apply the previous construction with $v = 11$ and $k = 6$. Let $V(K_{11}) = \mathbb{Z}_{11}$. It can be directly verified that $\mathcal{B} = \{S_t = [t; 1+t, 2+t, 3+t, 4+t, 5+t] \mid t \in \mathbb{Z}_{11}\}$ is a balanced (K_{11}, S_6)-design. Any vertex of K_{11} is the center of exactly one star of $\mathcal{B}$; actually, in this case $m = 1$. Now we construct the following sets of P_3's:*

$$\begin{aligned}
\bigcup_{c\in\mathbb{Z}_{11}} \mathcal{P}_3(\mathcal{B}_c) = \{ \; & [1,0,2],[3,0,4],[2,1,3],[4,1,5],[3,2,4],[5,2,6],[4,3,5],\\
& [6,3,7],[5,4,6],[7,4,8],[6,5,7],[8,5,9],[7,6,8],[9,6,10],\\
& [8,7,9],[10,7,0],[9,8,10],[0,8,1],[10,9,0],[1,9,2],\\
& [0,10,1],[2,10,3] \;\};\\
\bigcup_{c\in\mathbb{Z}_{11}} \mathcal{P}_3(S_c^*) = \; & \emptyset;\\
\bigcup_{c\in\mathbb{Z}_{11}} \mathcal{B}_c^* = \{ \; & [\alpha,0,5],[\alpha,1,6],[\alpha,2,7],[\alpha,3,8],[\alpha,4,9],[\alpha,5,10],\\
& [\alpha,6,0],[\alpha,7,1],[\alpha,8,2],[\alpha,9,3],[\alpha,10,4] \;\}.
\end{aligned}$$

One can check that $\mathcal{B}' = \bigcup_{c\in\mathbb{Z}_{11}} \mathcal{P}_3(\mathcal{B}_c) \cup \mathcal{B}_c^*$ is a (K_{12}, P_3)-design on the vertex-set $\mathbb{Z}_{11} \cup \{\alpha\}$. A down-link f acts, for instance, by mapping any $S_t \in \mathcal{B}$ to $f(S_t) = [1+t, t, 2+t] \in \mathcal{B}'$.

Theorem 5.5. *Let $k \geq 3$. For every admissible v,*

$$\mathcal{LS}_1(v) = \mathcal{LS}_2(v) = \{n \mid n \geq v,\; n \equiv 0, 1 (\mathrm{mod}\, 4)\}.$$

Proof. The assertion follows directly from Theorem 5.1, Lemma 5.2, Lemma 5.3 and Remark 4.2. □

6 Balanced kite-designs

Denote by $D = [a, b, c \bowtie d]$ the *kite*, a triangle with an attached edge, having vertices $\{a, b, c, d\}$ and edges $[c, a], [c, b], [c, d], [a, b]$.

In [2], Bermond and Schönheim proved that a kite-design of order v exists if, and only if, $v \equiv 0, 1(\mathrm{mod}\, 8)$. In particular, a *balanced* kite-design of order v exists if, and only if, $v \equiv 1(\mathrm{mod}\ 8)$; see [14].
Let $v \equiv 1(\mathrm{mod}\ 8)$, $v > 1$. Denote by $\mathcal{LD}_1(v)$ the set of all the integers $n \equiv 0, 1(\mathrm{mod}\ 4)$ such that there exists *some* balanced D-design of order v down-linked to a P_3-design of order n. Denote by $\mathcal{LD}_2(v)$ the set of all the integers $n \equiv 0, 1(\mathrm{mod}\ 4)$ such that *every* balanced D-design of order v can be down-linked to a P_3-design of order n. In this section we determine the sets $\mathcal{LD}_1(v)$ and $\mathcal{LD}_2(v)$.

Theorem 6.1. *For every $v \equiv 1(\mathrm{mod}\ 8)$, $v > 1$,*

$$\mathcal{LD}_2(v) \subseteq \mathcal{LD}_1(v) \subseteq \{n \mid n \geq v - 1,\; n \equiv 0, 1(\mathrm{mod}\ 4)\}.$$

Proof. Let $\mathcal{B}$ be a balanced (K_v, D)-design and $\mathcal{B}'$ a (K_n, P_3)-design. Note that, since $\mathcal{B}$ is balanced, the number b of blocks in which a vertex $x \in V(K_v)$ has degree 3 is the same as the number of blocks in which x has degree 1. By hypothesis, suppose there exists a down-link $f : \mathcal{B} \to \mathcal{B}'$. Observe that for every $D = [a, b, c \bowtie d] \in \mathcal{B}$, $f(D)$ must contain c. Hence, $V(K_n)$ contains each vertex of $V(K_v)$ having degree different from 2 in at least one block of $\mathcal{B}$. Let now $x, y \in V(K_v)$ be two vertices having degree 2 in all the blocks of $\mathcal{B}$ in which they appear. Since $\mathcal{B}$ is a (K_v, D)-design, there must be a block $D = [x, y, c \bowtie d] \in \mathcal{B}$. Hence, $f(D)$ contains either x or y. Thus, we can state that $V(K_n)$ must contain all $V(K_v)$, apart from, at most, one vertex, having degree 2 in all the blocks in which it appears. It follows $n \geq v - 1$. □

Lemma 6.2. *Let $v = 1 (\text{mod } 8)$, $v > 1$. Every balanced (K_v, D)-design can be down-linked to a (K_v, P_3)-design.*

Proof. Let $\mathcal{B} = \{D_j = [a_j, b_j, c_j \bowtie d_j] \mid j = 1, \dots, \frac{v(v-1)}{8}\}$ be a balanced (K_v, D)-design. It is immediate to see that

$$\mathcal{B}' = \left\{ P_{0j} = [a_j, b_j, c_j],\ P_{1j} = [a_j, c_j, d_j] \mid j = 1, \dots, \frac{v(v-1)}{8} \right\}$$

is a (K_v, P_3)-design and that $f : \mathcal{B} \to \mathcal{B}'$, mapping D_j into P_{0j}, is a down-link. □

Lemma 6.3. *Let $v = 1 (\text{mod } 8)$, $v > 1$. Suppose there exists a balanced (K_v, D)-design $\mathcal{B}$ having a vertex $x \in V(K_v)$ with degree 2 in all the blocks in which it appears. Then, $\mathcal{B}$ can be down-linked to a (K_{v-1}, P_3)-design.*

Proof. Let $\mathcal{B}$ be a balanced (K_v, D)-design satisfying the hypothesis in the statement. By the choice of x, we can write

$$\begin{aligned} \mathcal{B} = &\left\{ D_i^x = [x, y_i, z_i \bowtie t_i] \mid i = 1, \dots, \frac{v-1}{2} \right\} \\ &\cup \left\{ D_j = [a_j, b_j, c_j \bowtie d_j] \mid j = 1, \dots, \frac{(v-4)(v-1)}{8} \right\}. \end{aligned}$$

Now consider

$$\begin{aligned} \mathcal{B}' = &\left\{ P_i = [y_i, z_i, t_i] \mid i = 1, \dots, \frac{v-1}{2} \right\} \\ &\cup \left\{ P_{0j} = [a_j, b_j, c_j],\ P_{1j} = [a_j, c_j, d_j] \mid j = 1, \dots, \frac{(v-4)(v-1)}{8} \right\}. \end{aligned}$$

It is easy to see that $\mathcal{B}'$ is a P_3-design of order $v - 1$ whose vertex-set is $V(K_v) \setminus \{x\}$. Let now $f : \mathcal{B} \to \mathcal{B}'$, mapping D_j into P_{0j} for any $j = 1, \dots, \frac{(v-4)(v-1)}{8}$, and mapping D_i^x into P_i for any $i = 1, \dots, \frac{v-1}{2}$. The assertion follows. □

Theorem 6.4. *For every $v \equiv 1 (\bmod 8)$, $v > 1$,*

$$\mathcal{LD}_2(v) = \{n \mid n \geq v,\ n \equiv 0, 1 (\text{mod } 4)\}.$$

If there exists a (K_v, D)-design satisfying the hypothesis of Lemma 6.3 then

$$\mathcal{LD}_1(v) = \{n \mid n \geq v - 1,\ n \equiv 0, 1 (\text{mod } 4)\};$$

otherwise,

$$\mathcal{LD}_1(v) = \{n \mid n \geq v,\ n \equiv 0, 1 (\text{mod } 4)\}.$$

Proof. The theorem follows from Theorem 6.1, Lemma 6.2, Lemma 6.3 and Remark 4.2. □

Example 6.5. *We provide an application of Lemma 6.3 with $v = 9$. One can check that*

$$\mathcal{B} = \{[2, 3, 1 \bowtie 4], [6, 9, 1 \bowtie 5], [2, 9, 7 \bowtie 1], [3, 7, 8 \bowtie 1], [8, 9, 4 \bowtie 2], [6, 8, 2 \bowtie 5], [3, 6, 4 \bowtie 7], [3, 9, 5 \bowtie 4], [6, 7, 5 \bowtie 8]\}$$

is a balanced (K_9, D)-design with three vertices, $3, 6, 9$, having degree 2 in all the blocks in which they appear. So we are able to down-link $\mathcal{B}$ to a P_3-design of order 8. Choose, for instance, $V(K_8) = V(K_9) \setminus \{9\}$. We have

$$\mathcal{B}' = \{[2, 3, 1], [2, 1, 4], [6, 1, 5], [2, 7, 1], [3, 7, 8], [3, 8, 1], [8, 4, 2], [6, 8, 2], [6, 2, 5], [3, 6, 4], [3, 4, 7], [3, 5, 4], [6, 7, 5], [6, 5, 8]\}.$$

It is easy to see that $\mathcal{B}$ can be down-linked to $\mathcal{B}'$.

Example 6.6. *In this example we consider a balanced (K_9, D)-design $\mathcal{B}$ not satisfying the hypothesis of Lemma 6.3; thus, it is not possible to down-link $\mathcal{B}$ to a P_3-design whose vertex-set is smaller than $V(K_9)$. We construct a down-link from $\mathcal{B}$ to a P_3-design of order 9. Let*

$$\mathcal{B} = \{[0, 1, 4 \bowtie 6], [1, 2, 5 \bowtie 7], [2, 3, 6 \bowtie 8], [3, 4, 7 \bowtie 0], [4, 5, 8 \bowtie 1], [5, 6, 0 \bowtie 2], [6, 7, 1 \bowtie 3], [7, 8, 2 \bowtie 4], [8, 0, 3 \bowtie 5]\}.$$

Following the construction of Lemma 6.2,

$$\mathcal{B}' = \{[0, 1, 4], [0, 4, 6], [1, 2, 5], [1, 5, 7], [2, 3, 6], [2, 6, 8], [3, 4, 7], [3, 7, 0], [4, 5, 8], [4, 8, 1], [5, 6, 0], [5, 0, 2], [6, 7, 1], [6, 1, 3], [7, 8, 2], [7, 2, 4], [8, 0, 3], [8, 3, 5]\}.$$

The existence of a down-link $f : \mathcal{B} \to \mathcal{B}'$ is immediate.

7 Cycle systems

Let C_k be the cycle on k vertices. It is well known that there exists a k-cycle system of order v, namely a (K_v, C_k)-design, if, and only if, $k \leqslant v$, v is odd and $v(v-1) \equiv 0 \pmod{2k}$. The *if part* of this theorem was solved by Alspach and Gavlas [1] for k odd (see [5] for a simpler proof) and by Šajna [24] and [25] for k even.
Let now $v \geq k$ be odd and $v(v-1) \equiv 0 \pmod{2k}$. Denote by $\mathcal{LC}_1^k(v)$ the set of all the integers $n \equiv 0, 1 (\mathrm{mod}\, 4)$ such that there exists *some* C_k-design of order v down-linked to a P_3-design of order n. Write $\mathcal{LC}_2^k(v)$ for the set of all the integers $n \equiv 0, 1 (\mathrm{mod}\, 4)$ such that *every* C_k-design of order v can be down-linked to a P_3-design of order n.
For any cycle $C_k = (c_0, c_1, \ldots, c_{k-1}) \in \mathcal{B}$, define

$$\mathcal{P}_3(C_k) = \left\{ [c_{2\ell}, c_{2\ell+1}, c_{2\ell+2}] \mid \ell = 0, \ldots, \left\lfloor \frac{k-2}{2} \right\rfloor \right\}, \tag{3}$$

where the indexes are taken $\mathrm{mod}\, k$. Note that if k is odd, $E(C_k) = E(\mathcal{P}_3(C_k)) \cup [c_{k-1}, c_0]$; if k is even, $E(C_k) = E(\mathcal{P}_3(C_k))$, instead. Observe also that, for k even, the graph C_k^x obtained from C_k by removing a vertex $x \in C_k$ is a path on $k-1$ vertices and, consequently, has an even number of edges. Given a path $P_{2h+1} = [a_0, a_1, \ldots, a_{2h}]$ with an even number of edges, let

$$\mathcal{P}_3(P_{2h+1}) = \{[a_{2\ell}, a_{2\ell+1}, a_{2\ell+2}] \mid \ell = 0, \ldots, h-1\}. \tag{4}$$

Obviously, $E(P_{2h+1}) = E(\mathcal{P}_3(P_{2h+1}))$. Also, for a path with an odd number of edges, say $P_{2h} = [a_0, a_1, \ldots, a_{2h-1}]$, let

$$\mathcal{P}_3(P_{2h}) = \{[a_{2\ell+1}, a_{2\ell+2}, a_{2\ell+3}] \mid \ell = 0, \ldots, h-2\}. \tag{5}$$

Note that $E(P_{2h}) = E(\mathcal{P}_3(P_{2h})) \cup [a_0, a_1]$.

Theorem 7.1. *Suppose k even. For any admissible v,*

$$\{n \mid n \geq v-1,\ n \equiv 0, 1 (\mathrm{mod}\, 4)\} \subseteq \mathcal{LC}_2^k(v) \subseteq \mathcal{LC}_1^k(v).$$

Proof. Take a (K_v, C_k)-design $\mathcal{B}$ and fix a vertex $x \in K_v$. Write $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 = \{C_k \in \mathcal{B} : x \notin C_k\},$$
$$\mathcal{B}_2 = \{C_k \in \mathcal{B} : x \in C_k\}.$$

We want to down-link $\mathcal{B}$ to a (K_{v-1}, P_3)-design. Thus, construct the set

$$\mathcal{B}' = \{\mathcal{P}_3(C_k) \mid C_k \in \mathcal{B}_1\} \cup \{\mathcal{P}_3(C_k^x) \mid C_k \in \mathcal{B}_2\}.$$

Since $\mathcal{B}$ is a design, $\mathcal{B}'$ is a (K_{v-1}, P_3)-design with vertex-set $V(K_v) \setminus \{x\}$. It is easy to see that $f : \mathcal{B} \to \mathcal{B}'$, mapping any cycle $C_k \in \mathcal{B}_1$ to an element of $\mathcal{P}_3(C_k)$ and any cycle $C_k \in \mathcal{B}_2$ to an element of $\mathcal{P}_3(C_k^x)$, is a down-link. The result follows from Remark 4.2. □

Theorem 7.2. *Suppose $v \equiv 1 \pmod{8}$, $v > 1$. Then,*

$$\mathcal{LC}_2^4(v) = \mathcal{LC}_1^4(v) = \{n \mid n \geq v-1,\ n \equiv 0, 1 (\mathrm{mod}\, 4)\}.$$

Proof. We have only to prove that $\mathcal{LC}_1^4(v) \subseteq \{n \mid n \geq v-1,\ n \equiv 0, 1(\mathrm{mod}\, 4)\}$. Let $\mathcal{B}$ be a (K_v, C_4)-design and let $\mathcal{B}'$ be a (K_n, P_3)-design. Suppose $f : \mathcal{B} \to \mathcal{B}'$ to be a down-link. For every $C \in \mathcal{B}$, its image $f(C)$ contains 3 of its vertices. Since $\mathcal{B}$ is a design, any two vertices of $V(K_v)$ are contained together in at least one block of $\mathcal{B}$. It easily follows that $V(K_n)$ contains all the vertices of $V(K_v)$ apart from, at most, one. □

The case of k-cycle systems with k odd requires some further preliminaries. we recall some well known results about matchings of graphs; for further references, see [17]. A *matching* $\mathcal{M}$ in a graph Γ is a set of edges of Γ, no two of which are incident. A *maximum matching* is a matching that contains the largest possible number of edges. For any graph Γ and any set $X \subseteq V(\Gamma)$, write $\Delta(X)$ for the set of all vertices of Γ adjacent to at least one vertex of X. We recall Philip Hall's Theorem on matchings for bipartite graphs.

Theorem 7.3 (P. Hall)**.** *Let $\Gamma = A \cup B$ be a bipartite graph. Then, Γ has a matching of A into B if, and only if, $|\Delta(X)| \geq |X|$, for any $X \subseteq A$.*

The matching of Theorem 7.3 has $|A|$ edges and, thus, is maximum.

Lemma 7.4. *Suppose k odd and let $v \equiv 1 (\mathrm{mod}\ 4)$ be admissible. Any (K_v, C_k)-design can be down-linked to a (K_v, P_3)-design.*

Proof. Let $\mathcal{B}$ be a (K_v, C_k)-design, and take $V(K_v) = \{0, 1, \ldots, v-1\} \subseteq \mathbb{Z}$. Write $\mathcal{B} = \bigcup_{i=0}^{v-k} \mathcal{B}_i$, where

$$\mathcal{B}_i = \{C \in \mathcal{B} \mid \min V(C) = i\}.$$

We can assume without loss of generality $c_0 = i$ for any $C \in \mathcal{B}_i$. We want to deal first with the sets $\mathcal{B}_i$ for $i \geq 1$. If $|\mathcal{B}_i|$ is even, let $\mathcal{P}_3(\mathcal{B}_i) = \{\mathcal{P}_3(C) \mid C \in \mathcal{B}_i\}$ and note that $E(\mathcal{B}_i) \setminus E(\mathcal{P}_3(\mathcal{B}_i))$ is a star S^i of center i and with an even number of external vertices. For any i where $|\mathcal{B}_i|$ is odd, fix a $C^i \in \mathcal{B}_i$. Observe that $|\mathcal{B}_i \setminus \{C^i\}|$ is even, so it is possible to construct a set of P_3's as above from $\mathcal{B}_i \setminus \{C^i\}$. Write $\mathcal{C}$ for the set of all C^i's extracted from $\mathcal{B}_i$'s with $|\mathcal{B}_i|$ odd. Clearly, $|\mathcal{C}| \leq v - k$.

Introduce the graph G with $V(G) = \mathcal{C}$ and $[C^i, C^j] \in E(G)$ if, and only if, $V(C^i) \cap V(C^j) \neq \emptyset$; let $\mathcal{M}$ be one of its maximum matchings. For any edge $[C^i, C^j]$ of $\mathcal{M}$, write the cycles C^i and C^j in such a way as to have $c_0 \in V(C^i) \cap V(C^j)$ and define

$$\mathcal{P}_3(C^i, C^j) = \mathcal{P}_3(C^i) \cup \mathcal{P}_3(C^j) \cup [c^i_{k-1}, c_0, c^j_{k-1}]. \tag{6}$$

Let now $\mathcal{D} = \mathcal{C} \setminus V(\mathcal{M})$. This set is possibly empty. In any case $\mathcal{D}$ is a set of cycles which are pairwise disjoint; hence, $|\mathcal{D}| \leq \lfloor \frac{v-1}{k} \rfloor$. Since $v \equiv 1 (\mathrm{mod}\ 4)$, both $|\mathcal{D}|$ and $|\mathcal{B}_0|$ are even. Construct now the bipartite graph $G' = \mathcal{D} \cup \mathcal{B}_0$, where $[D, B] \in E(G')$ if, and only if, $V(D) \cap V(B) \neq \emptyset$. It is not hard to verify that G' satisfies the hypotheses of P. Hall's Theorem 7.3 and $|\mathcal{D}| \leq |\mathcal{B}_0|$; thus, G' has a maximum matching $\mathcal{M}'$ covering all the vertices of $\mathcal{D}$. For any $[D, B] \in \mathcal{M}'$, construct the set $\mathcal{P}_3(D, B)$ as in (6). Consider now $\mathcal{D}_0 = \mathcal{B}_0 \setminus (\mathcal{B}_0 \cap V(\mathcal{M}'))$. Since $|\mathcal{D}|$ is even, $|\mathcal{D}_0|$ is also even. We can write $\mathcal{D}_0 = \{D_0, \ldots, D_{2m-1}\}$ and, for any $\ell = 0, \ldots, m-1$, construct $\mathcal{P}_3(D_{2\ell}, D_{2\ell+1})$ as in (6). The union of all the P_3's obtained as above, that is the set

$$\mathcal{B}' = \left(\bigcup_{i=1}^{v-k} \mathcal{P}_3(\mathcal{B}_i) \setminus \bigcup_{C \in \mathcal{C}} \mathcal{P}_3(C) \right) \cup \bigcup_{i=1}^{v-k} \mathcal{P}_3(S^i)$$

$$\cup \bigcup_{\substack{[X,Y] \in \\ E(\mathcal{M}) \cup E(\mathcal{M}')}} \mathcal{P}_3(X, Y) \quad \cup \quad \bigcup_{\ell=0}^{m-1} \mathcal{P}_3(D_{2\ell}, D_{2\ell+1}) \,, \tag{7}$$

is a (K_v, P_3)-design. Any map $f : \mathcal{B} \to \mathcal{B}'$ sending $B \in \mathcal{B}$ to $P_3 \in \mathcal{P}_3(B)$ is clearly a down-link. □

Lemma 7.5. *Suppose k odd and let $v \equiv 3 \pmod 4$ be admissible. Any (K_v, C_k)-design can be down-linked to a (K_{v+1}, P_3)-design.*

Proof. Proceed as in the proof of Lemma 7.4. Since $v \equiv 3 \pmod 4$, the number $|\mathcal{D}_0|$ is odd. Thus, at the end of the construction, there remain to arrange the edges of a cycle $D_{2m} = (0, d_1, \ldots, d_{k-1}) \in \mathcal{D}_0$. Write $V(K_v) = \{0, \ldots, v-1\}$ and $V(K_{v+1}) = V(K_v) \cup \{\alpha\}$ and consider

$$\mathcal{P} = \mathcal{P}_3(D_{2m}) \cup \mathcal{P}_3(S^\alpha) \cup \{[d_{k-1}, 0, \alpha]\},$$

where S^α is the star of center α and external vertices $\{1, \ldots, v-1\}$. The set $\mathcal{B}'$ of (7) in the proof of Lemma 7.4 together $\mathcal{P}$ gives a (K_{v+1}, P_3)-design. The result follows as usual. □

Theorem 7.6. *Take k odd. For any admissible v,*

$$\{n \mid n \geq v,\ n \equiv 0, 1 \pmod 4\} \subseteq \mathcal{LC}_2^{2k+1}(v) \subseteq \mathcal{LC}_1^{2k+1}(v).$$

Proof. The theorem follows directly from Lemma 7.4, Lemma 7.5 and Remark 4.2. □

8 Balanced path-designs

In this section we investigate down-links from balanced path-designs to P_3-designs. Problems (I) and (II) are also completely solved for $k = 4$.

Tarsi [26] proved that the necessary conditions for the existence of a (K_v, P_k)-design, namely $v \geq k$ and $v(v-1) \equiv 0 (\bmod 2(k-1))$, are also sufficient. In [15], Hung and Mendelsohn proved that a *balanced* (K_v, P_{2k+1})-design ($k \geq 1$) exists if, and only if, $v \equiv 1 (\bmod 4k)$, and that a *balanced* (K_v, P_{2k})-design ($k \geq 2$) exists if, and only if, $v \equiv 1 (\bmod 2k-1)$. Balanced path-designs are also known as handcuffed designs; see [15].

For each admissible v, denote by $\mathcal{LP}_1^k(v)$ the set of all the integers $n \equiv 0, 1 (\bmod 4)$ such that there exists *some* balanced P_k-design of order v down-linked to a P_3-design of order n. Write $\mathcal{LP}_2^k(v)$ for the set of all the integers $n \equiv 0, 1 (\bmod 4)$ such that *every* balanced P_k-design of order v can be down-linked to a P_3-design of order n.

We shall extensively use the notation (4) for paths, as introduced in Section 7.

Theorem 8.1. *Let $v \equiv 1 (\bmod 2k-1)$, with $v, k > 1$.*

$$\{n \mid n \geq v,\ n \equiv 0, 1 (\bmod 4)\} \subseteq \mathcal{LP}_2^{2k}(v) \subseteq \mathcal{LP}_1^{2k}(v).$$

Proof. Let $\mathcal{B}$ be a balanced (K_v, P_{2k})-design. It is always possible, for any path P and vertex $a \in P$ to remove an edge $[a, b]$ from P so that each of the remaining connected components P^{a+} and P^{a-} contains an even number of edges. For any two paths $P, Q \in \mathcal{B}$ with $a \in V(P) \cap V(Q) \neq \emptyset$, consider the components P^{a+}, P^{a-}, Q^{a+}, Q^{a-} and let $[a, b]$ and $[a, c]$ be respectively the edge removed from P and that removed from Q. Define

$$\mathcal{P}_3(P, Q) = \mathcal{P}_3(P^{a+}) \cup \mathcal{P}_3(P^{a-}) \cup \mathcal{P}_3(Q^{a+}) \cup \mathcal{P}_3(Q^{a-}) \cup \{[c, a, b]\}. \quad (8)$$

Write $V(K_v) = \{0, 1, \ldots, v-1\} \subseteq \mathbb{Z}$ and let

$$\mathcal{B}_i = \{\, B \in \mathcal{B} \mid \min V(B) = i \,\};$$

it is immediate to see that $|\mathcal{B}_0| = \frac{(v-1)k}{2k-1}$ and $\mathcal{B} = \bigcup_{i=0}^{v-2k} \mathcal{B}_i$. For $i > 0$, fix a maximal set $\mathcal{C}_i$ of disjoint pairs of $\mathcal{B}_i$ and construct, for any of these pairs, P_3's as in (8), thus determining partial decompositions $\mathcal{P}_3(\mathcal{C}_i)$. Let now $\mathcal{C}$ be the set of all paths in $\mathcal{B}_i$ with $i > 0$ not in any of the elements of $\mathcal{C}_i$. Clearly, $|\mathcal{C}| \leq v - 2k$. Proceed as in Lemma 7.4, using (8) instead of (6) and, determine $\mathcal{D} = \mathcal{C} \setminus V(\mathcal{M})$. It is immediate to see that $|\mathcal{D}| \leq \lfloor \frac{v-1}{2k} \rfloor$. Construct, as in the aforementioned lemma the graph G' and determine a maximum matching $\mathcal{M}'$ and the set $\mathcal{D}_0 = \mathcal{B}_0 \setminus (\mathcal{B}_0 \cap V(\mathcal{M}'))$. Let $\mathcal{C}_0$ be a maximal set of disjoint pairs of elements of $\mathcal{D}_0$. There are now two possibilities:

1. $v \equiv 0, 1 (\bmod 4)$; in this case $|\mathcal{B}|$ also $|\mathcal{D}_0|$ are even and $\mathcal{C}_0$ covers all the elements of $\mathcal{D}_0$. Thus, we obtain a (K_v, P_3)-design $\mathcal{B}'$ given by

$$\mathcal{B}' = \bigcup_{i=0}^{v-2k} \mathcal{P}_3(\mathcal{C}_i) \cup \bigcup_{\substack{[X,Y] \in \\ E(\mathcal{M}) \cup E(\mathcal{M}')}} \mathcal{P}_3(X, Y). \quad (9)$$

2. $v \equiv 2, 3 (\mathrm{mod}\, 4)$; in this case $|\mathcal{B}|$ is odd, as well as $|\mathcal{D}_0|$. In particular, there is a path $P \in \mathcal{D}_0$ not covered by $\mathcal{C}_0$. Let $[0, p]$ be the edge of P neither in P^{0+} nor in P^{0-}.

   If $v \equiv 2 (\mathrm{mod}\, 4)$, we construct a (K_{v+2}, P_3)-design $\mathcal{B}'$. Write $V(K_{v+2}) = V(K_v) \cup \{\alpha, \beta\}$ and denote by S^α and S^β respectively the stars of center α and β and external vertices $\{1, 2, \ldots, v-2\}$. Take then

$$\mathcal{B}' = \bigcup_{i=0}^{v-2k} \mathcal{P}_3(\mathcal{C}_i) \cup \bigcup_{\substack{[X,Y]\in \\ E(\mathcal{M})\cup E(\mathcal{M}')}} \mathcal{P}_3(X, Y) \cup \mathcal{P}_3(P^{0+}) \cup \mathcal{P}_3(P^{0-}) \cup \mathcal{P}_3(S^\alpha) \cup \mathcal{P}_3(S^\beta) \cup \{[\alpha, 0, p], [\alpha, v-1, \beta], [0, \beta, \alpha]\}. \qquad (10)$$

   If $v \equiv 3 (\mathrm{mod}\, 4)$ we construct a (K_{v+1}, P_3)-design $\mathcal{B}'$ as follows. Let $V(K_{v+1}) = V(K_v) \cup \{\alpha\}$ and define

$$\mathcal{B}' = \bigcup_{i=0}^{v-2k} \mathcal{P}_3(\mathcal{C}_i) \cup \bigcup_{\substack{[X,Y]\in \\ E(\mathcal{M})\cup E(\mathcal{M}')}} \mathcal{P}_3(X, Y) \cup \mathcal{P}_3(P^{0+}) \cup \mathcal{P}_3(P^{0-}) \cup \mathcal{P}_3(S^\alpha) \cup \{[\alpha, 0, p]\}, \qquad (11)$$

   where S^α is the star with center α and external vertices $\{1, \ldots, v-1\}$.

It is easy to see that in all of the above cases there is a down-link $f : \mathcal{B} \to \mathcal{B}'$. The main result follows from Remark 4.2. □

Example 8.2. *Consider the* (K_{16}, P_4)*-design* $\mathcal{B}$ *whose blocks are given by the union of the sets*

$$\begin{aligned}
\mathcal{B}_0 =& \{[0,4,15,11],[0,10,5,11],[0,15,10,3],[0,1,3,4],[0,7,12,13],\\
&[12,0,2,5],[9,0,8,11],[3,0,5,8],[6,0,11,7],[13,0,14,10]\};\\
\mathcal{B}_1 =& \{[1,8,6,7],[1,5,7,8],[1,14,12,8],[1,4,9,15],[1,11,13,14],\\
&[2,1,12,15],[7,1,6,15],[13,1,9,5],[10,1,15,8]\};\\
\mathcal{B}_2 =& \{[2,9,14,7],[2,12,10,6],[2,15,7,3],[2,6,4,10],[3,2,4,5],\\
&[14,2,10,11],[8,2,7,13],[11,2,13,10]\};\\
\mathcal{B}_3 =& \{[3,13,8,14],[15,3,14,4],[9,3,5,6],[6,3,8,9],[12,3,11,14]\};\\
\mathcal{B}_4 =& \{[4,11,9,10],[4,8,10,7],[4,7,9,12],[13,4,12,5]\};\\
\mathcal{B}_5 =& \{[5,15,13,6],[14,5,13,9]\};\\
\mathcal{B}_6 =& \{[9,6,11,12],[12,6,14,15]\}.
\end{aligned}$$

Here only $\mathcal{B}_1$ *and* $\mathcal{B}_3$ *contain an odd number of blocks. Observe that* $\mathcal{B}_i = \emptyset$ *for* $i > 6$. *We may apply the construction of Theorem* 8.1. *Consider, for instance the set* $\mathcal{B}_3$. *In this case*

$$\mathcal{C}_3 = \{\{[3,13,8,14],[15,3,14,4]\},\{[9,3,5,6],[6,3,8,9]\}\}$$

and, consequently

$$\mathcal{P}_3(\mathcal{C}_3) = \{[13, 8, 14], [3, 14, 4], [13, 3, 15], [3, 5, 6], [3, 8, 9], [9, 3, 6]\}$$

Hence $[12, 3, 11, 14] \in \mathcal{D}$. *An analogous argument for* $\mathcal{B}_1$ *gives*

$$\mathcal{D} = \{[10, 1, 15, 8], [12, 3, 11, 14]\}.$$

A matching $\mathcal{M}'$ *is given by*

$$\mathcal{M}' = \{([12, 3, 11, 14], [0, 4, 15, 11]), ([10, 1, 15, 8], [0, 10, 5, 11])\}$$

and $|\mathcal{C}_0| = 8$. *We may now construct a* (K_{16}, P_3)*-design, according to* (9)*.*

Example 8.3. *Consider the balanced* (K_6, P_6)*-design*

$$\mathcal{B} = \{[0, 5, 1, 4, 2, 3], [1, 0, 2, 5, 3, 4], [5, 4, 0, 3, 1, 2]\}.$$

In this case $\mathcal{B} = \mathcal{B}_0$. *We have*

$$\mathcal{P}_3([0, 5, 1, 4, 2, 3], [1, 0, 2, 5, 3, 4]) = \{[5, 1, 4], [4, 2, 3], [0, 2, 5], [5, 3, 4], [5, 0, 1]\}.$$

Let $P = [5, 4, 0, 3, 1, 2]$*; thus,*

$$P^{0-} = [5, 4, 0]; \qquad P^{0+} = [3, 1, 2].$$

Then,

$$\begin{aligned}\mathcal{B}' = &\{[5, 1, 4], [4, 2, 3], [0, 2, 5], [5, 3, 4], [5, 0, 1], [5, 4, 0], [3, 1, 2]\}\\ &\cup \{[1, \alpha, 2], [3, \alpha, 4]\} \cup \{[1, \beta, 2], [3, \beta, 4]\}\\ &\cup \{[\alpha, 0, 3], [\alpha, 5, \beta], [0, \beta, \alpha]\}.\end{aligned}$$

Note that, in this case, Theorem 8.1 *does not provide a down-link from* $\mathcal{B}$ *to a design with the minimum number of vertices; indeed,* $\mathcal{B}$ *can also be down-linked to the* (K_5, P_3)*-design*

$$\mathcal{B}'' = \{[5, 1, 4], [4, 2, 3], [5, 3, 4], [3, 1, 2], [2, 5, 4]\}.$$

Theorem 8.4. *Let* $v \equiv 1 (\text{mod } 3)$, $v > 1$

$$\mathcal{LP}_2^4(v) = \mathcal{LP}_1^4(v) = \{n \mid n \geq v,\ n \equiv 0, 1 (\text{mod}\, 4)\}.$$

Proof. By Theorem 8.1, it remains only to prove that $\mathcal{LP}_1^4(v) \subseteq \{n \mid n \geq v,\ n \equiv 0, 1 (\text{mod}\, 4)\}$. Let $\mathcal{B}$ be a balanced (K_v, P_4)-design and $\mathcal{B}'$ a (K_n, P_3)-design. Suppose a down-link $f : \mathcal{B} \to \mathcal{B}'$ exists. For any $P = [a, b, c, d] \in \mathcal{B}$, $f(P)$ contains the internal vertices of $\mathcal{B}$. On the other hand, since $\mathcal{B}$ is balanced, each vertex is contained in $\frac{2(v-1)}{3}$ blocks and, in particular, it is internal in exactly $\frac{v-1}{3} > 0$ of these. Hence, $n \geq v$ follows. □

Theorem 8.5. *Let* $v \equiv 1 (\text{mod}\, 4k)$ *with* $v > 1$,

$$\{n \geq v \mid n \equiv 0, 1 (\text{mod}\, 4)\} \subseteq \mathcal{LP}_2^{2k+1} \subseteq \mathcal{LP}_1^{2k+1}$$

Proof. It is sufficient to observe that every P_{2k+1} can be decomposed into k edge-disjoint P_3's. The result then follows from Remark 4.2. □

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