On the non–existence of some inherited ovals in Moulton planes of even order

A. Aguglia and L. Giuzzi*

Abstract

No oval contained in a regular hyperoval of the Desarguesian plane $PG(2,q^2)$, q even, is inherited by a Moulton plane of order q^2.

1 Introduction

The existence problem of ovals in finite non–Desarguesian planes is still open in general and appears to be difficult. There are some planes of order 16 without ovals, found by a computer aided search; see [10]. On the other hand, ovals have been constructed in many finite planes, mostly using the idea of an inherited oval due to Korchmáros [4, 5]. Korchmáros' idea relies on the fact that any two planes π_1 and π_2 of the same order have the same number of points and lines; hence the points, as well as the lines, of the two planes may be identified. If Ω is an oval of π_1, it may be that Ω is also an oval of π_2, although π_1 and π_2 differ for some (in general many) point–line incidences; in this case Ω is called an *inherited oval* of π_2 from π_1; see also [2, Page 728]. In practice, π_1 is usually taken to be the Desarguesian plane of order q. The case where π_2 is the Hall plane $H(q)$ of order q was investigated in [4], and inherited ovals in $H(q)$ were found. For q odd, this also shows the existence of inherited ovals in the dual plane of $H(q)$, called also the Moulton plane $M(q)$ of order q. In this paper the even order case is addressed. Our main result is the following theorem.

Theorem 1. *No (hyper)oval $\mathcal{A}$ contained in a regular hyperoval Ω of the Desarguesian plane is inherited by $\mathcal{M}(q^2)$.*

Here the hypothesis on $\mathcal{A}$ being contained in a regular hyperoval cannot be dropped; see [10] for examples of hyperovals in the Moulton plane of order 16. We also obtain that the largest arc of $\mathcal{M}(q^2)$ contained in Ω has size q^2 and it is complete.

*Research supported by the Italian Ministry MIUR, Strutture geometriche, combinatoria e loro applicazioni.

2 Preliminaries

Let q be a power of 2 and denote by $||\cdot||$ the norm function

$$||\cdot|| : \begin{cases} \mathrm{GF}(q^2) \to \mathrm{GF}(q) \\ x \mapsto x^{q+1} \end{cases}$$

Following [4], take a proper subset U of $\mathrm{GF}(q)^\star$ and consider the following operation defined over the set $\mathrm{GF}(q^2)$

$$a \odot b = \begin{cases} ab & \text{if } ||b|| \notin U \\ a^q b & \text{if } ||b|| \in U. \end{cases}$$

The set $(\mathrm{GF}(q^2), +, \odot)$ is a quasifield with nucleus containing $\mathrm{GF}(q)$; see [3]. Let $\mathcal{A} = (\mathcal{P}, \mathcal{L})$ be the affine plane defined as follows: the point–set $\mathcal{P}$ is the same as that of $AG(2, q^2)$, the lines of $\mathcal{L}$ are either of the form

$$[c] = \{P(x, y) : x = c, y \in \mathrm{GF}(q^2)\}$$

or

$$[m, n] = \{P(x, y) : y = m \odot x + n\}.$$

The Moulton plane $\mathcal{M}_U(q^2)$ associated with U is the projective plane obtained by completing $\mathcal{A}$ with its points at infinity; see [9].

Write $\Phi = \{P(x, y) : ||x|| \notin U\}$ and $\Psi = \{P(x, y) : ||x|| \in U\}$. Clearly, $\mathcal{P} = \Phi \cup \Psi$; furthermore, incidence in Φ is the same as incidence in the Desarguesian affine geometry $AG(2, q^2)$.

Any hyperoval obtained from a conic by adding its nucleus is called *regular*. Let now Ω be a regular hyperoval of $PG(2, q^2)$. If $\Omega \subseteq \Phi$, that is for each point $P(x, y) \in \Omega$ the norm of x is an element of $\mathrm{GF}(q) \setminus U$, then Ω is an inherited hyperoval of $\mathcal{M}_U(q^2)$. In order to show that this case cannot occur we shall use the notion of *conic blocking set*; see [6].

A conic blocking set $\mathcal{B}$ is a set of lines in a Desarguesian projective plane met by all conics; a conic blocking set $\mathcal{B}$ is *irreducible* if for any line of $\mathcal{B}$ there is a conic intersecting $\mathcal{B}$ in just that line.

Lemma 2 (Theorem 4.4,[6])**.** *The line–set*

$$\mathcal{B} = \{y = mx : m \in \mathrm{GF}(q)\} \cup \{x = 0\}$$

is an irreducible conic blocking set in $\mathrm{PG}(2, q^2)$*, q even.*

3 Proof of Theorem 1

Lemma 3. *Let Ω be a regular hyperoval of $PG(2, q^2)$, with q an even prime power. Then, there are at least two points $P(x, y)$ in Ω such that $||x|| \in U$.*

Proof. To prove the lemma we show that the set Ψ, introduced above, is a conic blocking set. We observe that the conic blocking set of Lemma 2 is the degenerate Hermitian curve of $\mathrm{PG}(2, q^2)$ with equation $x^q y - xy^q = 0$. Since all degenerate Hermitian curves are projectively equivalent, this implies that any such a curve is a conic blocking set. On the other hand, Ψ is the union of degenerate Hermitian curves of equation $x^{q+1} = c$, as c varies in U. Thus, Ψ is also a conic blocking set. Suppose that $\Omega = \mathcal{C} \cup N$, where $\mathcal{C}$ is a conic and N its nucleus. Now, either Ψ contains at least two points of $\mathcal{C}$, or $\Psi \cap \mathcal{C} = \{P(x, y)\}$, with $x \in U$. In this case, the line $[x]$ is tangent to $\mathcal{C}$; hence, the nucleus N of $\mathcal{C}$ is on the line $[x] \subseteq \Psi$. The result follows. $\square$

In [1, Theorem 1.1], it is proven that for $q > 5$ an odd prime power, any arc of the Moulton plane $\mathcal{M}$ obtained as $\mathcal{C}^\star = \mathcal{C} \cap \Phi$, where $\mathcal{C}$ is a conic of the related Desarguesian plane, is complete. In fact, this result also holds when q is even and $\mathcal{C}$ is a hyperoval; thus, in this case, the theorem might be restated as follows.

Lemma 4 ([1, Theorem 1.1])**.** *Let Ω be a hyperoval of $PG(2, q^2)$, with $q > 2$ an even prime power. Then, no point in Ψ may be aggregated to $\Omega^\star = \Omega \cap \Phi$ in order to get an arc of $\mathcal{M}_U(q^2)$.*

Proof. Consider first the usual construction of a Hall plane $H(q^2)$ as a derived affine plane from $AG(2, q^2)$; see [7, Chapter X]. A line of $H(q^2)$ is either a line of $AG(2, q^2)$ or an affine Baer subplane. The plane $\mathcal{M}_U(q^2)$ is the dual of the projective closure of $H(q^2)$; thus, a pencil of lines of $\mathcal{M}_U(q^2)$ with centre $P(x_0, y_0)$ either consists of lines of a Baer subplane of $AG(2, q^2)$, or is the pencil with centre P in $AG(2, q^2)$, according as $||x_0|| \in U$ or not.

Let now Ω be a hyperoval and $\mathcal{B}$ a Baer subplane of $AG(2, q^2)$. Assume $Y \in \mathcal{B}$ and denote by $\mathcal{L}(Y)$ the pencil of lines in $\mathcal{B}$ with centre Y. Take Δ as the set of all points of Ω not covered by a line in $\mathcal{L}(Y)$ and let $n = |\Delta|$. Write $m = q^2 + 2 - n$. Observe that the lines of $\mathcal{L}(Y)$ cover at most $2(q + 1)$ points of Ω; thus, $q^2 - 2q \le n \le q^2 + 2$. We shall show that there is at least a line in $\mathcal{B}$ meeting Δ in two points. This implies that for any point $P(x_0, y_0)$ with $||x_0|| \in U$ there is at least a 2–secant to $\Omega^\star$ in $\mathcal{M}_U(q^2)$; thus, no point with $||x_0|| \in U$ may be aggregated to $\Omega^\star$ in order to obtain an arc.

Let $T \in \Delta$; since $T \notin \mathcal{B}$, there is a unique line ℓ_T of $\mathcal{B}$ through T. Every point $Q \in \Omega \setminus \Delta$ lies on at most $q + 1 - (m - 1) = q - m + 2$ lines ℓ_T with $T \in \Delta$. Suppose by contradiction that for every $T \in \Delta$,

$$\ell_T \cap \Omega = \{T, Q\}, \text{with } Q \in \Omega \setminus \Delta.$$

The total number of lines obtained as Q varies in $\Omega \setminus \Delta$ does not exceed $m(q - m + 2)$. So,

$$n = q^2 - m + 2 \le m(q - m + 2).$$

As m is a non–negative integer, this is possible only for $q = 2$. $\square$

From Lemma 3, we know that $\Omega^\star$ contains at most q^2 points; furthermore, $\Omega^\star$ is a complete arc of the linear space with support Φ contained in $\mathcal{M}_U(q^2)$;

hence, by Lemma 4, $\Omega^\star$ is a complete arc of $\mathcal{M}_U(q^2)$. As every oval of $PG(2,q^2)$ is contained in a hyperoval, we obtain Theorem 1.

We have seen that the largest arc of $\mathcal{M}_U(q^2)$ contained in a regular hyperoval of $PG(2,q^2)$ has at most q^2 points; for an actual example of a q^2–arc of $\mathcal{M}_U(q^2)$ coming from a regular hyperoval of the Desarguesian plane see [8]. This also shows that the result of [4] cannot be extended to even q.

References

[1] V. Abatangelo, B. Larato, *Canonically Inherited Arcs in Moulton Planes of Odd Order*, preprint (2008).

[2] C.J. Colbourn, J.H. Diniz, *Handbook of Combinatorial Designs*, Chapman & Hall (2007).

[3] P. Dembowski, *Finite Geometries*, Springer (1968).

[4] G. Korchmáros, *Ovali nei piani di Moulton d'ordine dispari*, Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II, 395–398. Atti dei Convegni Lincei, No. **17**, Accad. Naz. Lincei, Rome, (1976).

[5] G. Korchmáros, *Ovali nei piani di Hall di ordine dispari*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **56** (1974), no. 3, 315–317.

[6] L. D. Holder, *Conic blocking sets in Desarguesian projective planes*, J. Geom. 80 (2004), no. **1-2**, 95–105.

[7] D.Hughes, F. Piper, *Projective planes*, Springer (1973).

[8] G. Menichetti, *k–archi completi in piani di Moulton d'ordine q^m*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 60 (1976), no. **6**, 775–781.

[9] F.R. Moulton, *A Simple Non–Desarguesian Plane Geometry*, Trans. Amer. Math. Soc. (3) no. **2** (1902), 192–195.

[10] T. Penttila, G.F. Royle, M.K. Simpson, *Hyperovals in the known projective planes of order* 16, J. Combin. Des. 4 (1996), no. **1**, 59–65.

Angela Aguglia and Luca Giuzzi
Dipartimento di Matematica
Politecnico di Bari
Via Orabona 4
I–70125 Bari
Italy
e-mail: a.aguglia@poliba.it
l.giuzzi@poliba.it