Line Polar Grassmann Codes of Orthogonal Type

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Abstract

Polar Grassmann codes of orthogonal type have been introduced in [1]. They are subcodes of the Grassmann code arising from the projective system defined by the Plücker embedding of a polar Grassmannian of orthogonal type. In the present paper we fully determine the minimum distance of line polar Grassmann Codes of orthogonal type for q odd.

1 Introduction

Codes $\mathcal{C}_{m,k}$ arising from the Plücker embedding of the k–Grassmannians of m–dimensional vector spaces have been widely investigated since their first introduction in [12, 13]. They are a remarkable generalization of Reed–Muller codes of the first order and their monomial automorphism groups and minimum weights are well understood, see [10, 5, 6, 4].

Recently, in [1], the first two authors of the present paper introduced some new codes $\mathcal{P}_{n,k}$ arising from embeddings of orthogonal Grassmannians $\Delta_{n,k}$. These codes correspond to the projective system determined by the Plücker embedding of the Grassmannian $\Delta_{n,k}$ representing all totally singular k–spaces with respect to some non-degenerate quadratic form η defined on a vector space $V(2n+1, q)$ of dimension $2n+1$ over a finite field $\mathbb{F}_q$. An orthogonal Grassmann code $\mathcal{P}_{n,k}$ can be obtained from the ordinary Grassmann code $\mathcal{C}_{2n+1,k}$ by just deleting all the columns corresponding to k–spaces which are non-singular with respect to η; it is thus a punctured version of $\mathcal{C}_{2n+1,k}$. For q odd, the dimension of $\mathcal{P}_{n,k}$ is the same as that of $\mathcal{G}_{2n+1,k}$, see [1]. The minimum distance $d_{\min}$ of $\mathcal{P}_{n,k}$ is always bounded away from 1. Actually, it has been shown in [1] that for q odd, $d_{\min} \geq q^{k(n-k)+1} + q^{k(n-k)} - q$. By itself, this proves that the redundancy of these codes is somehow better than that of $\mathcal{C}_{2n+1,k}$.

In the present paper we prove the following theorem, fully determining all the parameters for the case of line orthogonal Grassmann codes (that is polar Grassmann codes with $k = 2$) for q odd.

Main Theorem. *For q odd, the minimum distance $d_{\min}$ of the orthogonal Grassmann code $\mathcal{P}_{n,2}$ is*

$$d_{\min} = q^{4n-5} - q^{3n-4}.$$

Furthermore, all words of minimum weight are projectively equivalent.

Hence, we have the following.

Corollary 1.1. *For q odd, line polar Grassmann codes of orthogonal type are $[N, K, d_{\min}]$-projective codes with*

$$N = \frac{(q^{2n-2} - 1)(q^{2n} - 1)}{(q^2 - 1)(q - 1)}, \quad K = \binom{2n+1}{2}, \quad d_{\min} = q^{4n-5} - q^{3n-4}.$$

1.1 Organization of the paper

In Section 2 we recall some well–known facts on projective systems and related codes, as well as the notion of polar Grassmannian of orthogonal type. In Section 3 we prove our main theorem; as some long, yet straightforward, computations are required, here we present in full detail the arguments for two of the main cases to be considered, while we simply summarize the results, which can be obtained in an analogous way, for the remaining two.

2 Preliminaries

2.1 Projective systems and Grassmann codes

An $[N, K, d_{\min}]_q$ projective system $\Omega \subseteq \mathrm{PG}(K - 1, q)$ is a set of N points in $\mathrm{PG}(K - 1, q)$ such that for any hyperplane Σ of $\mathrm{PG}(K - 1, q)$,

$$|\Omega \setminus \Sigma| \geq d_{\min}.$$

Existence of $[N, K, d_{\min}]_q$ projective systems is equivalent to that of projective linear codes with the same parameters; see, for instance, [14]. Indeed, let Ω be a projective system and denote by G a matrix whose columns $G_1, \ldots, G_N$ are the coordinates of representatives of the points of Ω with respect to some fixed reference system. Then, G is the generator matrix of an $[N, K, d_{\min}]$ code over $\mathbb{F}_q$, say $\mathcal{C} = \mathcal{C}(\Omega)$. The code $\mathcal{C}(\Omega)$ is not, in general, uniquely determined, but it is unique up to code equivalence. We shall thus speak, with a slight abuse of language, of *the* code defined by Ω.

As any word c of $\mathcal{C}(\Omega)$ is of the form $c = mG$ for some row vector $m \in \mathbb{F}_q^K$, it is straightforward to see that the number of zeroes in c is the same as the number of points of Ω lying on the hyperplane of equation $m \cdot x = 0$ where $m \cdot x = \sum_{i=1}^K m_i x_i$ and $m = (m_i)_1^K$, $x = (x_i)_1^K$. The minimum distance $d_{\min}$ of $\mathcal{C}$ is thus

$$d_{\min} = |\Omega| - f_{\max}, \quad \text{where} \quad f_{\max} = \max_{\substack{\Sigma \leq \mathrm{PG}(K-1,q) \\ \dim \Sigma = K-2}} |\Omega \cap \Sigma|. \tag{1}$$

We point out that any projective code $\mathcal{C}(\Omega)$ can also be regarded, equivalently, as an evaluation code over Ω of degree 1. In particular, when Ω spans the whole of $\mathrm{PG}(K - 1, q) = \mathrm{PG}(W)$, where W is the underlying vector space, then there is a bijection, induced by the standard inner product of W, between the elements of the dual vector space W^* and the codewords c of $\mathcal{C}(\Omega)$.

Let $\mathcal{G}_{2n+1,k}$ be the Grassmannian of the k–subspaces of a vector space $V := V(2n + 1, q)$, with $k \leq n$ and let $\eta : V \to \mathbb{F}_q$ be a non-degenerate quadratic form over V. Denote by $\varepsilon_k : \mathcal{G}_{2n+1,k} \to \mathrm{PG}(\bigwedge^k V)$ the usual Plücker embedding

$$\varepsilon_k : \langle v_1, \ldots, v_k \rangle \to \langle v_1 \wedge \cdots \wedge v_k \rangle.$$

The orthogonal Grassmannian $\Delta_{n,k}$ is a geometry having as points the k–subspaces of V totally singular for η. Let $\varepsilon_k(\mathcal{G}_{2n+1,k}) := \{\varepsilon_k(X_k) : X_k \text{ is a point of } \mathcal{G}_{2n+1,k}\}$ and $\varepsilon_k(\Delta_{n,k}) =$

$\{\varepsilon_k(\bar{X}_k)\colon \bar{X}_k$ is a point of $\Delta_{n,k}\}$. Clearly, we have $\varepsilon_k(\Delta_{n,k}) \subseteq \varepsilon_k(\mathcal{G}_{2n+1,k}) \subseteq \mathrm{PG}(\bigwedge^k V)$. Throughout this paper we shall denote by $\mathcal{P}_{n,k}$ the code arising from the projective system $\varepsilon_k(\Delta_{n,k})$. By [2, Theorem 1.1], if $n \geq 2$ and $k \in \{1, \ldots, n\}$, then $\dim\langle\varepsilon_k(\Delta_{n,k})\rangle = \binom{2n+1}{k}$ for q odd, while $\dim\langle\varepsilon_k(\Delta_{n,k})\rangle = \binom{2n+1}{k} - \binom{2n+1}{k-2}$ when q is even.

We recall that for $k < n$, any line of $\Delta_{n,k}$ is also a line of $\mathcal{G}_{2n+1,k}$. For $k = n$, the lines of $\Delta_{n,n}$ are not lines of $\mathcal{G}_{2n+1,n}$; indeed, in this case $\varepsilon_n|_{\Delta_{n,n}} \colon \Delta_{n,n} \to \mathrm{PG}(\bigwedge^n V)$ maps the lines of $\Delta_{n,n}$ onto non-singular conics of $\mathrm{PG}(\bigwedge^n V)$.

Thus, for q odd, the projective system identified by $\varepsilon_k(\Delta_{n,k})$ determines a code of length $N = \prod_{i=0}^{k-1} \frac{q^{2(n-i)}-1}{q^{i+1}-1}$ and dimension $K = \binom{2n+1}{k}$; if q is even the projective system identified by $\varepsilon_k(\Delta_{n,k})$ determines instead a code of length $N = \prod_{i=0}^{k-1} \frac{q^{2(n-i)}-1}{q^{i+1}-1}$ and dimension $K = \binom{2n+1}{k} - \binom{2n+1}{k-2}$.

The following universal property provides a well-known characterization of alternating multilinear forms; see for instance [11, Theorem 14.23].

Theorem 2.1. *Let V and U be vector spaces over the same field. A map $f : V^k \longrightarrow U$ is alternating k-linear if and only if there is a linear map $\overline{f} : \bigwedge^k V \longrightarrow U$ with $\overline{f}(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = f(v_1, v_2, \ldots, v_k)$. The map $\overline{f}$ is uniquely determined.*

In general, the dual space $(\bigwedge^k V)^*$ of $\bigwedge^k V$ is isomorphic to the space of all k-linear alternating forms of V. Observe that when $\dim V = 2n + 1$, we can also write $(\bigwedge^k V)^* \cong \bigwedge^{2n+1-k} V$.

In this paper we are concerned with line Grassmannians, that is we assume $k = 2$. The above argument shows that for any hyperplane π of $\mathrm{PG}(\bigwedge^2 V)$, induced by a linear functional in $(\bigwedge^2 V)^*$, there is an alternating bilinear form $\varphi_\pi : V \times V \to \mathbb{F}_q$ such that $p \wedge q \in \pi$ for $p, q \in V$ if and only if $\varphi_\pi(p, q) = 0$. In particular, when one considers the set of totally singular lines of V with respect to a given quadratic form η, the image of a totally singular line $\ell = \langle p, q \rangle$ of V belongs to the hyperplane π if and only if ℓ is also totally isotropic for φ_π, that is to say $\varphi_\pi(p, q) = 0$.

Denote by $\mathcal{L}_\varphi$ the set of all totally isotropic lines for the alternating form $\varphi := \varphi_\pi$ corresponding to a hyperplane π of $\mathrm{PG}(\bigwedge^2 V)$. The number of points in $\varepsilon_2(\Delta_{n,2}) \cap \pi$ is the same as the number of lines of $\mathrm{PG}(V)$ simultaneously totally singular for the quadratic form η defining $\Delta_{n,2}$ and totally isotropic for the alternating form φ_π. Hence, by (1),

$$d_{\min}(\mathcal{P}_{n,2}) = \#\{\text{points of } \Delta_{n,2}\} - \max_\varphi \#\{\text{points of } \Delta_{n,2} \cap \mathcal{L}_\varphi\}.$$

In other words, in order to determine the minimum distance of $\mathcal{P}_{n,2}$ we need to find the maximum number of lines which are simultaneously totally singular for a fixed non-degenerate quadratic form η on V and totally isotropic for a (necessarily degenerate) alternating form φ.

Recall that the *radical* of φ is the set

$$\mathrm{Rad}(\varphi) := \{v \in V : \forall w \in V, \varphi(v, w) = 0\}.$$

This is always a vector space and its codimension in V is even. As $\dim V$ is odd, $2n - 1 \geq \dim \mathrm{Rad}(\varphi) \geq 1$.

We point out that for the line projective Grassmann code $\mathcal{C}_{2n+1,2}$, it has been proven in [10] that minimum weight codewords correspond to points of $\varepsilon_{2n-1}(\mathcal{G}_{2n+1,2n-1})$; these can be regarded as bilinear alternating forms of V of maximum radical.

In the case of orthogonal line Grassmannians, not all points of $\mathcal{G}_{2n+1,2n-1}$ yield codewords of $\mathcal{P}_{n,2}$ of minimum weight. However, as a consequence of the proof of our main result, we shall show in Proposition 3.18 that all the codewords of minimum weight of $\mathcal{P}_{n,2}$ do indeed correspond to some $(2n - 1)$-dimensional subspaces of V, that is to say, to bilinear alternating forms of V of maximum radical.

3

2.2 Generalities on quadrics

Let $Q := Q(2t, q)$ be a non-singular parabolic quadric of rank t in $\mathrm{PG}(2t, q)$ and write $\kappa^0 = (q^{2t} - 1)/(q - 1)$ for the number of its points. The points of $\mathrm{PG}(2t, q)$ are partitioned in three orbits under the action of the stabilizer $\mathrm{PO}(2t + 1, q)$ of Q in $\mathrm{PGL}(2t + 1, q)$; namely the points of Q, those whose polar hyperplane cuts on Q an elliptic quadric $Q^-(2t - 1, q)$ of rank $t - 1$ and those whose polar hyperplane meets Q in a hyperbolic quadric $Q^+(2t - 1, q)$ of rank t. As customary, call the former points *internal* and the latter *external to Q*. Write κ_0^- for the number of the internal points and κ_0^+ for that of the external ones. Then, see e.g. [7],

$$\kappa_0^- = \frac{1}{2}q^t(q^t - 1), \qquad \kappa_0^+ = \frac{1}{2}q^t(q^t + 1).$$

If $Q := Q^+(2t-1, q)$ is a non-singular hyperbolic quadric in $\mathrm{PG}(2t-1, q)$ or $Q := Q^-(2t-1, q)$ is a non-singular elliptic quadric in $\mathrm{PG}(2t - 1, q)$, then the polar hyperplane of a point p not in Q always cuts a parabolic section of rank $t - 1$ on Q. There are still two orbits of $\mathrm{PO}(2t, q)$ on the non-singular points of $\mathrm{PG}(2t - 1, q)$; they have the same size, but can be distinguished by the value (either square or non-square) assumed by the quadratic form defining Q on vectors representing their points. Denote by κ_+ and κ_- the size of these orbits, in the hyperbolic and elliptic case respectively. We have

$$\kappa_+ = \frac{1}{2}q^{t-1}(q^t - 1), \qquad \kappa_- = \frac{1}{2}q^{t-1}(q^t + 1).$$

Fix now a quadratic form η on V inducing a quadric Q and let the symbols $\square$ and $\boxslash$ stand for the set of non-null square elements and the set of non-square element of $\mathbb{F}_q$.

With a slight abuse of notation, we shall say that a (projective) point p is *square* and, consequently, write $p \in \square$, when $\eta(v_p)$ is a square for any (non-null) vector v_p representing the projective point $p = \langle v_p \rangle$. Note that $\eta(v_p) \in \square$ if and only if $\forall \lambda \in \mathbb{F}_q \setminus \{0\}$, $\eta(\lambda v_p) \in \square$; so, the above definition is well posed. Analogously, we say that a (projective) point p is a *non-square* and write $p \in \boxslash$, when $\eta(v_p)$ is a non-square, for v_p a (non-null) vector representing $p = \langle v_p \rangle$. Recall the quadratic character of a point is constant on the orbits of the orthogonal group. In particular, in the parabolic case, the external points are either all squares or non-squares. For the internal points the opposite behavior holds.

3 Proof of the Main Theorem

In order to simplify the notation, throughout this section, whenever no ambiguity might arise, we shall usually denote by the same symbol a point $p \in \mathrm{PG}(2n + 1, q)$ and any non-null vector v_p representing p with respect to a suitably chosen basis. This slight lack of rigour will however be harmless.

For $n = k = 2$, by [1, Main Result 2], the minimum distance of the code $\mathcal{P}_{2,2}$ is $d_{\min} = q^3 - q^2$ and there is nothing to prove. Suppose henceforth $n > 2$, $k = 2$ and q odd.

As

$$\#\{\text{points of } \Delta_{n,2}\} = \frac{(q^{2n} - 1)(q^{2n-2} - 1)}{(q - 1)(q^2 - 1)},$$

by Theorem 2.1, the part on the minimum distance in the Main Theorem is equivalent to the following statement.

Theorem 3.1. *Let $V := V(2n+1, q)$, q odd. The maximum number of lines totally singular for a given non-singular quadratic form η defined on V and simultaneously totally isotropic for a (degenerate) alternating form φ over V is*

$$f_{\max} = \frac{(q^{n-1} - 1)}{(q-1)(q^2-1)}(q^{3n-2} + q^{3n-3} - q^{3n-4} + q^{2n} - q^{n-1} - 1).$$

This section is fully devoted to the proof of Theorem 3.1. In Section 3.1 we shall introduce some preliminary lemmata. Under a somehow further technical assumption we shall see that four cases will need to be analyzed; they depend on the dimension and position of the radical R of a generic alternating form φ with respect to the quadric Q. In Section 3.4 we shall perform a detailed analysis of the first of these cases, the one yielding the actual minimum distance. The outcome of our investigation for the next two cases will be outlined in Section 3.5; there we shall just describe in what measure they differ from the case of Section 3.4 and summarize the results obtained. The fourth case will be dealt with in Section 3.6. In Section 3.7 we shall drop the assumption used in sections 3.3–3.6 and see that the theorem holds in full generality. Finally, in Section 3.8 the projective equivalence of all the words of minimum weight is proved.

3.1 Some linear algebra

Throughout the remainder of the paper, we shall always denote by η a fixed non-singular quadratic form on V and by φ an arbitrary alternating form defined on the same space. We shall also write M and S for the matrices representing respectively η and φ with respect to a given, suitably chosen, basis B of V; write also $\perp_Q$ for the orthogonal polarity induced by η and $\perp_W$ for the (degenerate) symplectic polarity induced by φ. In particular, for $v \in V$, the symbols $v^{\perp_Q}$ and $v^{\perp_W}$ will respectively denote the space orthogonal to v with respect to η and φ. Likewise, when X is a subspace of V, the notations $X^{\perp_Q}$ and $X^{\perp_W}$ will be used to denote the spaces orthogonal to X with respect to η and φ. We shall say that a subspace X is *totally singular* if $X \leq X^{\perp_Q}$ and *totally isotropic* if $X \leq X^{\perp_W}$. Let also $R := \mathrm{Rad}(\varphi)$ and $r := \dim R$.

Lemma 3.2. *1. For any $v \in V$, $v^{\perp_Q} = v^{\perp_W}$ if and only if v is an eigenvector of non-zero eigenvalue of $M^{-1}S$.*

2. The radical R of φ corresponds to the eigenspace of $M^{-1}S$ of eigenvalue 0.

Proof.   1. Observe that $v^{\perp_Q} = v^{\perp_W}$ if and only if the equations $x^T M v = 0$ and $x^T S v = 0$ are equivalent for any $x \in V$. This means that there exists an element $\lambda \in \mathbb{F}_q \setminus \{0\}$ such that $Sv = \lambda M v$. As M is non-singular, the latter says that v is an eigenvector of non-zero eigenvalue λ for $M^{-1}S$.

2. Let v be an eigenvector of $M^{-1}S$ of eigenvalue 0. Then $M^{-1}Sv = 0$, hence $Sv = 0$ and $x^T S v = 0$ for every $x \in V$, that is $v^{\perp_W} = V$. This means $v \in R$. □

We can now characterize the eigenspaces of $M^{-1}S$.

Lemma 3.3. *Let μ be a non-zero eigenvalue of $M^{-1}S$ and V_μ be the corresponding eigenspace. Then,*

1. $\forall v \in V_\mu$ and $r \in R$, $r \perp_Q v$. Hence, $V_\mu \leq R^{\perp_Q}$.

2. The eigenspace V_μ is both totally isotropic for φ and totally singular for η.

3. *Let $\lambda, \mu \neq 0$ be two not necessarily distinct eigenvalues of $M^{-1}S$ and u, v be two corresponding eigenvectors. Then either of the following holds:*

   (a) *$u \perp_Q v$ and $u \perp_W v$.*

   (b) *$\mu = -\lambda$.*

Proof. 1. Take $v \in V_\mu$. As $M^{-1}Sv = \mu v$ we also have $\mu v^T = v^T S^T M^{-T}$. So, $v^T M^T = \mu^{-1} v^T S^T$. Let $r \in R$. Then, as $S^T = -S$, $v^T M r = \mu^{-1} v^T S^T r$ and $v^T S r = 0$ for any v, we have $v^T M r = 0$, that is $r \perp_Q v$.

2. Let $v \in V_\mu$. Then $M^{-1}Sv = \mu v$, which implies $Sv = \mu Mv$. Hence, $v^T Sv = \mu v^T Mv$. Since $v^T Sv = 0$ and $\mu \neq 0$, we also have $v^T Mv = 0$, for every $v \in V_\mu$. Thus, V_μ is totally singular for η. Since V_μ is totally singular, for any $u \in V_\mu$ we have $u^T Mv = 0$; so, $u^T Sv = \mu u^T Mv = 0$, that is V_μ is also totally isotropic.

3. Suppose that either $u \not\perp_Q v$ or $u \not\perp_W v$. Since by Lemma 3.2 $u^{\perp_Q} = u^{\perp_W}$ and $v^{\perp_Q} = v^{\perp_W}$, we have $Mu = \lambda^{-1}Su$ and $Mv = \mu^{-1}Sv$. So, $u \not\perp_Q v$ or $u \not\perp_W v$ implies $v^T Mu \neq 0 \neq v^T Su$. Since $M^{-1}Su = \lambda u$ and $M^{-1}Sv = \mu v$, we have

$$v^T Su = v^T S(\lambda^{-1}M^{-1}Su) = \lambda^{-1}(-M^{-1}Sv)^T Su = -(\lambda^{-1}\mu)v^T Su;$$

hence, $-\lambda^{-1}\mu = 1$. $\qquad\square$

Corollary 3.4. *Let V_λ and V_μ be two eigenspaces of non-zero eigenvalues $\lambda \neq -\mu$. Then, $V_\lambda \oplus V_\mu$ is both totally singular and totally isotropic.*

Proposition 3.5. *If $x \in V_\lambda$, then $\forall y \in V$, $\lambda y^T Mx = y^T Sx$.*

Proof. If $x \in V_\lambda$, then $M^{-1}Sx = \lambda x$; hence, $\forall y \in V$, $y^T Sx = \lambda y^T Mx$. $\qquad\square$

Lemma 3.6. *The maximum number of eigenvectors for $M^{-1}S$ of non-zero eigenvalue is obtained when a complement H_0 of $R \cap R^{\perp_Q}$ in $R^{\perp_Q}$ contains a direct sum $V_\mu \oplus V_\lambda$ of two eigenspaces of $M^{-1}S$, each of dimension m, where m is the rank of the non-singular quadric $Q_0 := Q \cap H_0$.*

Proof. By Claim 2 of Lemma 3.3, any maximal eigenspace V_μ of $M^{-1}S$ with non-zero eigenvalue is both totally singular for η and totally isotropic for φ. By Claim 1 of Lemma 3.3, V_μ is contained in a complement H_0 of $R \cap R^{\perp_Q}$ in $R^{\perp_Q}$. In particular, V_μ is contained in a generator of the quadric Q_0, so $\dim V_\mu \leq m$. If there were at least three distinct eigenspaces $V_\lambda, V_\mu, V_\theta$ with $\lambda = -\mu$, then, obviously, $\theta \neq \pm\lambda, \pm\mu$. Let $c = \dim V_\theta \geq 1$. By Corollary 3.4, both $V_\theta \oplus V_\lambda$ and $V_\theta \oplus V_\mu$ are totally singular for η; hence they are contained in two generators, say G_+ and G_- of Q_0, with $V_\theta \leq G_+ \cap G_-$ and $c < m$, $\dim V_\lambda, V_\mu \leq m - c$. Thus, we have the following upper bond on the number of eigenvectors of non-zero eigenvalue:

$$|V_\lambda| + |V_\theta| + |V_\mu| - 3 \leq 2q^{m-c} + q^c - 3 < 2q^m - 2 = |G_+| + |G_-| - 2.$$

This is to say that the possible maximum number of eigenvectors of non-zero eigenvalue attained when there are at least three distinct non-zero eigenvalues is strictly less than the number of vectors contained in two vector spaces of dimension m.

We now show that there actually are alternating forms φ inducing two eigenspaces of dimension m; this yields that the number $2(q^m - 1)$ of eigenvectors can be achieved and, consequently, this is the maximum possible. Let G_+ and G_- be two trivially intersecting generators of Q_0 with bases respectively $\{b_i^+\}_{i=1}^m$ and $\{b_i^-\}_{i=1}^m$. We can suppose without loss of generality that the

quadratic form $\eta|_{V'}$, restriction of η to $V' := G_+ \oplus G_-$, is represented with respect to the basis $B' = \{b_i^+\}_{i=1}^m \cup \{b_i^-\}_{i=1}^m$ by the matrix

$$M' = \begin{pmatrix} \mathbf{0_m} & I_m \\ I_m & \mathbf{0_m} \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix and $\mathbf{0_m}$ stands for the null matrix of order m. Choose also η such that V decomposes as $V = V_0 \perp_Q V' \perp_Q R$, where V_0 is an orthogonal complement of $V' \oplus R$ with respect to $\perp_Q$. Define now an alternating form φ with radical R represented on V' with respect to B' by the matrix

$$S' = \begin{pmatrix} \mathbf{0_m} & -I_m \\ I_m & \mathbf{0_m} \end{pmatrix}$$

and such that we also have $V = V' \perp_W V_0 \perp_W R$. This is always possible, as $\dim(V' + V_0)$ is even. For any $v \in G_+ \cup G_- \subseteq V'$,

$$v^{\perp_W} = v^{\perp'_W} + R + V_0 = v^{\perp'_Q} + R + V_0 = v^{\perp_Q},$$

where by $\perp'_W$ and $\perp'_Q$ we denote the orthogonality relations defined by the restriction of the forms η and φ to respectively V' and $V' \times V'$. By Lemma 3.2, v is an eigenvector of $M^{-1}S$. Thus, G_+, G_- are eigenspaces of $M^{-1}S$ of dimension m. $\qquad\square$

By Lemma 3.6, the alternating forms φ inducing a maximum number of eigenvectors of $M^{-1}S$, determine two eigenspaces V_λ and V_μ with $\dim V_\lambda = \dim V_\mu = m$. In this case, Lemma 3.3, point 3 shows that, for V_λ and V_μ to be both maximal, $\lambda = -\mu$ is also required.

3.2 Sketch of the proof and setup

As outlined before our aim is to count the maximum number f of lines totally isotropic for φ and totally singular for η.

Let p be a point of Q and consider the spaces $p^{\perp_Q}$ and $p^{\perp_W}$. Since Q is non-singular, $p^{\perp_Q}$ is a hyperplane of $\mathrm{PG}(V)$ for any $p \in Q$, while $p^{\perp_W}$ is a hyperplane of $\mathrm{PG}(V)$ if and only if $p \notin R$.

Let now Q_p be the orthogonal geometry induced by η on $p^{\perp_W}$ and denote by $\mathrm{Res}_{Q_p}p$ the geometry having as elements the (singular) subspaces (with respect to η) through p contained in Q_p.

As each line contains $q + 1$ points and each line through p in $p^{\perp_W} \cap p^{\perp_Q}$ corresponds to a point in $\mathrm{Res}_{Q_p}p$, the number of lines simultaneously totally isotropic for φ and totally singular for η is

$$f = \frac{1}{q+1} \sum_{p \in Q} \tau(p), \quad \text{where } \tau(p) := \#\{\text{points of } \mathrm{Res}_{Q_p}p\}. \tag{2}$$

We distinguish two main cases.

- **Case A:** $p^{\perp_Q} \subseteq p^{\perp_W}$
  Let $\mathfrak{P} := \mathfrak{P}_a \cup \mathfrak{P}_b$ and $A := |\mathfrak{P}|$, where

  $$\mathfrak{P}_a := \{p \in Q : p^{\perp_W} = \mathrm{PG}(V)\}, A_R := |\mathfrak{P}_a| \quad \text{and}$$

  $$\mathfrak{P}_b := \{p \in Q : p^{\perp_W} = p^{\perp_Q}\}, A_V := |\mathfrak{P}_b|.$$

For any $p \in \mathfrak{P}$, $\mathrm{Res}_{Q_p} p \cong Q(2n-2, q)$ (where $Q(2n-2, q)$ is a non-singular parabolic quadric of rank $n-1$). Thus, we have

$$p \in \mathfrak{P} \Rightarrow \tau(p) = \frac{q^{2n-2} - 1}{q - 1} =: A^0.$$

The points in $\mathfrak{P}_a$ are the points of Q contained in R; by Lemma 3.2 the points in $\mathfrak{P}_b$ correspond to eigenvectors of $M^{-1}S$ of non-zero eigenvalue. In particular,

$$A_R = \#\{\text{points of } Q \cap R\}, \qquad A_V = \frac{|\bigcup_{\lambda \neq 0} V_\lambda| - 1}{q - 1}$$

where V_λ are the eigenspaces of $M^{-1}S$ as λ varies among all of its non-null eigenvalues. Clearly, $A = A_R + A_V$.

- **Case B:** $\mathrm{codim}_{p^{\perp_W}} p^{\perp_W} \cap p^{\perp_Q} = 1$
  Three possibilities can occur for $\mathrm{Res}_{Q_p} p$:

  1. $\mathrm{Res}_{Q_p} p \cong Q^+(2n-3, q)$ is a non-singular hyperbolic quadric of rank $n-1$ in the $(2n-3)$-dimensional projective space $p^{\perp_Q} \cap p^{\perp_W}$; let

     $$\mathfrak{P}^+ := \{p \in Q \colon \mathrm{Res}_{Q_p} p \cong Q^+(2n-3, q)\} \text{ and } N^+ = |\mathfrak{P}^+|.$$

     In particular,
     $$p \in \mathfrak{P}^+ \Rightarrow \tau(p) = \frac{(q^{n-1} - 1)(q^{n-2} + 1)}{q - 1} =: B^+.$$

  2. $\mathrm{Res}_{Q_p} p \cong Q^-(2n-3, q)$ is a non-singular elliptic quadric of rank $n-2$ in the $(2n-3)$-dimensional projective space $p^{\perp_Q} \cap p^{\perp_W}$; define

     $$\mathfrak{P}^- := \{p \in Q \colon \mathrm{Res}_{Q_p} p \cong Q^-(2n-3, q)\} \text{ and } N^- = |\mathfrak{P}^-|.$$

     Then,
     $$p \in \mathfrak{P}^- \Rightarrow \tau(p) = \frac{(q^{n-1} + 1)(q^{n-2} - 1)}{q - 1} =: B^-.$$

  3. $\mathrm{Res}_{Q_p} p$ is isomorphic to a cone $TQ(2n-4, q)$ having a point T as vertex and a non-singular parabolic quadric $Q(2n-4, q)$ of rank $n-2$ as base; put

     $$\mathfrak{P}^0 := \{p \in Q \colon \mathrm{Res}_{Q_p} p \cong TQ(2n-4, q)\} \text{ and } N^0 = |\mathfrak{P}^0|.$$

     Then,
     $$p \in \mathfrak{P}^0 \Rightarrow \tau(p) = \frac{(q^{2n-3} - 1)}{q - 1} =: B^0,$$

Clearly, as pointsets, $Q \setminus \mathfrak{P} = \mathfrak{P}^+ \cup \mathfrak{P}^0 \cup \mathfrak{P}^-$.

By replacing the aforementioned numbers in (2), we obtain

$$f = \frac{1}{q+1}(AA^0 + N^0 B^0 + N^+ B^+ + N^- B^-). \tag{3}$$

The aim of the remainder of the current paper is to determine the quantities A, N^0, N^+ and N^- in such a way as to compute the maximum $f_{\max}$ of f.

Write $Q_R := R \cap Q$ for the quadric induced by η on R and take D as the radical of Q_R; this is to say $D = \text{Rad}(\eta|_R)$; write also $d = \dim D$. Observe that, in general, $R \leq D^{\perp_Q}$ and the space V decomposes as follows

$$V = H \oplus D^{\perp_Q}; \qquad D^{\perp_Q} = H_0 \oplus R; \qquad R = D_0 \oplus D,$$

where D_0 is a direct complement of D in R, H is a direct complement of $D^{\perp_Q}$ in V and H_0 is a direct complement of R in $D^{\perp_Q}$. Thus,

$$V = H \oplus H_0 \oplus D_0 \oplus D.$$

Let also

$$R_0 := Q \cap D_0, \qquad Q_0 := Q \cap H_0$$

be the quadrics induced by η in respectively D_0 and H_0. As Q is non-singular we have

$$\dim H = \dim D = d, \qquad \dim H_0 = 2n + 1 - (r + d), \qquad \dim D_0 = r - d.$$

Denote by m the rank of Q_0; since for any generator X of Q_0 we have $X + D \subseteq Q$, then $d + m \leq n$. The function f in (3) is then dependent on r and d. The possible ranks of R_0 and Q_0 are outlined in Table 1. These correspond to four cases to investigate. In particular, we shall denote by $f^i(r, d)$, $1 \leq i \leq 4$, the function providing the values of f in a given case i and by $f^i_{\max}$ its corresponding maximum.

3.3   Forms for M and S

In this section we shall determine suitable forms for the matrix M and S which should provide the maximum possible values for f under the following Assumption 3.7; this shall be silently used in Sections 3.4–3.6 and removed in Section 3.7.

Assumption 3.7. *The maximum of the function f can be attained only if for a given radical R the number of eigenvectors A_V for $M^{-1}S$ is maximum.*

It is always possible to fix an ordered basis $B = B_H \cup B_{H_0} \cup B_{D_0} \cup B_D$ of V such that

$$\begin{aligned}
&B_H = \{b_1 \ldots b_d\} \text{ is an ordered basis of } H; \\
&B_{H_0} = \{b_{d+1} \ldots b_{2n+1-r}\} \text{ is an ordered basis of } H_0; \\
&B_{D_0} = \{b_{2n+2-r}, \ldots, b_{2n+1-d}\} \text{ is an ordered basis of } D_0; \\
&B_D = \{b_{2n+2-d} \ldots b_{2n+1}\} \text{ is an ordered basis of } D.
\end{aligned} \tag{4}$$

As all parabolic quadrics of given rank are projectively equivalent, the matrix M representing Q with respect to B may be taken of the form

$$M = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & I_d \\ \mathbf{0} & Q_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R_0 & \mathbf{0} \\ I_d & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{5}$$

where R_0 and Q_0 are given by Table 2, according to the cases of Table 1. Here, with a slight abuse of notation, as no ambiguity may arise, we use R_0 and Q_0 to denote both the matrices and the corresponding quadrics.

Observe that there is always a vector $x = (0, \ldots, 0, 1, 0 \ldots 0) \in V$ such that $x^T M x = 1$; the exact x to be chosen according varies to the case being considered as described in Table 3. It can be seen directly that the hyperplane $x^{\perp_Q}$ cuts Q in a section which is hyperbolic for cases 1 and 3 and elliptic otherwise; thus, the correspondence of Table 3 between square/non-square points and internal/external points to Q is determined using the remarks of Section 2.2.

case	parity of d	type of R_0	rank R_0	type of Q_0	rank Q_0
1	odd	hyperbolic	$(r-d)/2$	parabolic	$n-(r+d)/2$
2	odd	elliptic	$(r-d)/2-1$	parabolic	$n-(r+d)/2$
3	even	parabolic	$(r-d-1)/2$	hyperbolic	$n-(r+d-1)/2$
4	even	parabolic	$(r-d-1)/2$	elliptic	$n-(r+d+1)/2$

Table 1: Decomposition of the quadric Q

Case	R_0	Q_0	Case	R_0	Q_0
1	$R_0^+ := \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} & I & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$	3	$\begin{pmatrix} \mathbf{0} & I & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$	$Q_0^+ := \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}$
2	$R_0^- := \begin{pmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & -\xi \end{pmatrix}$	$\begin{pmatrix} \mathbf{0} & I & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$	4	$\begin{pmatrix} \mathbf{0} & I & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}$	$Q_0^- := \begin{pmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & -\xi \end{pmatrix}$

Table 2: Matrices for R_0 and Q_0

In this table ξ is a non-square in $\mathbb{F}_q$; the order of the identity matrices I and the null matrices $\mathbf{0}$ is provided by Table 1.

Lemma 3.8. *Let $p \notin R$ be a singular point with respect to η. If the points external to Q are squares, then*

- *$p \in \mathfrak{P}^+$ if and only if $p^T S M^{-1} S p \in -\square$;*

- *$p \in \mathfrak{P}^0$ if and only if $p^T S M^{-1} S p = 0$;*

- *$p \in \mathfrak{P}^-$ if and only if $p^T S M^{-1} S p \in -\boxslash$.*

When the points external to Q are non-squares, the classes $\mathfrak{P}^+$ and $\mathfrak{P}^-$ are exchanged.

Proof. Let $W_p := p^{\perp_W}$ and write $a_p := W_p^{\perp_Q}$ for the point orthogonal with respect to η to the hyperplane W_p. Then, $a_p^{\perp_Q} = p^{\perp_W}$. In particular, the following two equations are equivalent for any x:
$$x^T S p = 0, \qquad x^T M a_p = 0.$$
In other words, there exists $\rho \in \mathbb{F}_q \setminus \{0\}$ such that $\rho S p = M a_p$ and, consequently, $a_p = \rho M^{-1} S p$. Observe that the point a_p belongs to $p^{\perp_Q}$, as $p^T M a_p = \rho p^T M M^{-1} S p = \rho p^T S p = 0$. Clearly, p is an eigenvector of $M^{-1} S$ if and only if $p = a_p$. In this case $p \in \mathfrak{P}$.

Suppose now p not to be an eigenvector of $M^{-1} S$ and consider the quadric
$$Q_p = W_p \cap Q = a_p^{\perp_Q} \cap Q.$$

Observe that the the residue at p of Q_p is either an hyperbolic, elliptic or degenerate quadric (more precisely, in the latter case, a cone with vertex a point and base a parabolic quadric) according as a_p is external, internal or contained in $Q_p \cap p^{\perp_Q}$. Thus, the three cases above are determined by the value assumed by the quadratic form η on a_p, that is by
$$a_p^T M a_p = \rho^2 p^T S^T M^{-T} M M^{-1} S p = -\rho^2 p^T S M^{-1} S p.$$

The result now follows. □

10

case	x	$x^{\perp_Q} \cap Q$	internal points	external points
1	$(0_{2n-r}, 1, 0_r)$	hyperbolic	non-square	square
2	$(0_{2n-r}, 1, 0_r)$	elliptic	square	non-square
3	$(0_{2n-d}, 1, 0_d)$	hyperbolic	non-square	square
4	$(0_{2n-d}, 1, 0_d)$	elliptic	square	non-square

Table 3: Internal/external points for Q

Case	S_{22}	ν	Non-null entries in U
1, 2	$\begin{pmatrix} \mathbf{0} & I & \mathbf{0} \\ -I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix}$	$n - (r+d)/2$	1 in the position $U_{1,2n+1-(r+d)}$
3	$\begin{pmatrix} \mathbf{0} & I \\ -I & \mathbf{0} \end{pmatrix}$	$n - \frac{1}{2}(r+d-1)$	None
4	$\begin{pmatrix} \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & \alpha \\ \mathbf{0} & \mathbf{0} & -\alpha & 0 \end{pmatrix}$	$n - \frac{1}{2}(r+d+1)$	• if $\alpha = 0$, a non-singular 2×2 minor U_I contained in the last two columns of U. • if $\alpha \neq 0$, possibly a 2×2 minor U_I contained in the last two columns of U.

Table 4: Structure of the matrix S

For any $p \in R$ we have $Sp = 0$; if $p \in V_\lambda$, where V_λ is an eigenspace of $M^{-1}S$, then $p^T S^T M^{-1} S p = \lambda p^T S^T p = 0$. In particular, the coordinates of all the points of $\mathfrak{P}$ (see Section 3.2) satisfy the system

$$\begin{cases} p^T S^T M^{-1} S p = 0 \\ p^T M p = 0. \end{cases} \tag{6}$$

Lemma 3.9. *Suppose φ to be an alternating form with a maximum number of totally isotropic lines which are also totally singular for the quadric Q. Then φ can be represented with respect to the basis B by an antisymmetric matrix of the form*

$$S = \begin{pmatrix} S_{11} & U & \mathbf{0} & \mathbf{0} \\ -U^T & S_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{7}$$

where $S_{11} = -S_{11}^T$ and U, S_{22} are given by Table 4.

Proof. The generic matrix of an antisymmetric form with radical R is of the form (7), with S_{11} and S_{22} antisymmetric and U arbitrary. By Lemma 3.6 and Assumption 3.7, if the number of totally isotropic lines which are also totally singular is maximum, then there are two maximal subspaces of dimension m contained in a complement H_0 of $R \cap R^{\perp_Q}$ in $R^{\perp_Q}$ which are both totally singular and totally isotropic. Thus, we may take the first two blocks of columns of S_{22} as described in Table 4. Observe that the linear transformation induced by

$$D = \begin{pmatrix} Z & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Z^{-T} \end{pmatrix},$$

with Z a non-singular $d \times d$ matrix, acts on M and S as follows

$$D^T M D = M, \qquad D^T S D = \begin{pmatrix} Z^T S_{11} Z & Z^T U & \mathbf{0} & \mathbf{0} \\ -U^T Z & S_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

In particular, by a suitable choice of Z, the matrix U can be assumed to be of the form

$$\begin{pmatrix} \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where I is either the 1×1 or 2×2 identity matrix, according as the case being considered is 1, 2 or 4. $\qquad \square$

3.4 First case

Throughout this section we shall write the coordinates of a generic point p with respect to the basis B given by (4) as

$$p = (\mathbf{x_1}, \mathbf{z_1}, \mathbf{z_2}, y, \mathbf{x_2}, \mathbf{y_2})$$

where $y \in \mathbb{F}_q$, $\mathbf{z_1}, \mathbf{z_2} \in \mathbb{F}_q^{n-(r+d)/2}$, $\mathbf{x_1}, \mathbf{y_2} \in \mathbb{F}_q^d$ and $\mathbf{x_2} \in \mathbb{F}_q^{(r-d)}$. Furthermore, $z \in \mathbb{F}_q$ is taken to be the first component of the vector $\mathbf{x_1}$. By Tables 1 and 2, we have $R_0 = R_0^+ := \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}$ with I the identity matrix of order $(r-d)/2$. Then,

$$p^T S^T M^{-1} S p = -z^2 + 2\mathbf{z_2}^T \mathbf{z_1} \quad \text{and} \quad p^T M p = 2\mathbf{z_2}^T \mathbf{z_1} + y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2} + 2\mathbf{x_1}^T \mathbf{y_2}.$$

We need a preliminary technical lemma.

Lemma 3.10. *The following properties hold.*

1. *For any given $\beta \in \mathbb{F}_q \setminus \{0\}$, the number of solutions $(y, \mathbf{x_2})$ of the equation $y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2} = \beta^2$ is $q^{(r-d)/2}(q^{(r-d)/2} + 1)$.*

2. *Consider the quadratic form $\theta(z, \mathbf{z_1}, \mathbf{z_2}) = -z^2 + 2\mathbf{z_2}^T \mathbf{z_1}$. Then the number of vectors $(z, \mathbf{z_1}, \mathbf{z_2})$ with $z \neq 0$ such that $\theta(z, \mathbf{z_1}, \mathbf{z_2}) \in -\square$ is*

$$\frac{q-1}{2} \left(q^{2n-(r+d)} - q^{2n-(r+d)-1} + q^{n-(r+d)/2} + q^{n-(r+d)/2-1} \right). \tag{8}$$

Proof. 1. Let $\xi(y, \mathbf{x_2}) = y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2}$ be a quadratic form defined on $J := \langle b_{2n+1-r}, B_{D_0} \rangle$, where the coordinates of vectors are taken with respect to the basis $\{b_{2n+1-r}\} \cup B_{D_0}$. Then ξ induces a parabolic quadric R' of rank $(r-d)/2$ and the polar hyperplane of the vector $(1, 0, \ldots, 0) = (1, \mathbf{0}) \in J$ cuts a hyperbolic section on R'. So, $(1, \mathbf{0})$ is external to R' and, consequently, its orbit has size $\frac{1}{2} q^{(r-d)/2}(q^{(r-d)/2} + 1)$. Furthermore, $\xi(1, \mathbf{0}) = 1 \in \square$; thus the points external to R' are always squares and they number to $\frac{1}{2} q^{(r-d)/2}(q^{(r-d)/2} + 1)$. Note that for each square point p there are exactly 2 vectors v_1, v_2 representing $p = \langle v_1 \rangle = \langle v_2 \rangle$ such that $\xi(v_1) = \xi(v_2) = \beta^2$; thus, the overall number of solutions of the equation is $q^{(r-d)/2}(q^{(r-d)/2} + 1)$.

2. Consider the quadratic form $\theta(z, \mathbf{z_1}, \mathbf{z_2}) = -z^2 + 2\mathbf{z_2}^T\mathbf{z_1}$ defined on the space $J' := \langle b_1, B_{H_0} \setminus \{b_{2n+1-r}\}\rangle$. Observe that the point $(1, 0, \ldots, 0) = (1, \mathbf{0}) \in J'$ is always external to the parabolic quadric of equation $\theta(z, \mathbf{z_1}, \mathbf{z_2}) = 0$, as its polar hyperplane cuts a hyperbolic section of equation $2\mathbf{z_2}^T\mathbf{z_1} = 0$. Note that $\theta(1, \mathbf{0}) = -1$ — namely, $\theta(1, \mathbf{0})$ is a square if $-1 \in \square$ and a non-square otherwise. This gives that the number of vectors on which θ assumes a square value when $-1 \in \square$ is the same as the number of values on which θ assumes a non-square value for $-1 \notin \square$. In particular, for $-1 \in \square$, the number of such vectors is the number of vectors $(z, \mathbf{z_1}, \mathbf{z_2})$ corresponding to external points to $\theta(z, \mathbf{z_1}, \mathbf{z_2}) = 0$, excluding those lying in the hyperplane $z = 0$. This gives (8). The same number is obtained for $-1 \notin \square$.

$\square$

Proposition 3.11. *Suppose we have a form φ yielding the maximum possible number of totally singular and totally isotropic lines in Case 1. Then,*

$$A = 2\frac{q^{n-(r+d)/2} - 1}{q-1} + \frac{q^{r-1} - 1}{q-1} + q^{(r+d-2)/2}; \qquad N^0 = \frac{q^{2n-1} - 1}{q-1} - A;$$

$$N^+ = \frac{1}{2}\left(q^{2n-1} + q^{2n-(r+d)/2-1} + q^{n+(r+d)/2-1} - q^{n-1}\right);$$

$$N^- = \frac{1}{2}\left(q^{2n-1} - q^{2n-(r+d)/2-1} - q^{n+(r+d)/2-1} + q^{n-1}\right).$$

Proof. By Assumption 3.7, in order for the number of totally singular, totally isotropic vectors to be maximum we need $M^{-1}S$ to have two eigenspaces V_λ, V_μ of non-zero eigenvalues $\lambda, \mu = -\lambda$, both of maximal dimension $m = n - (r+d)/2$; thus, $A_V = 2\frac{q^{n-(r+d)/2}-1}{q-1}$. As the quadric Q_R induced by η on $\mathrm{PG}(R)$ can be seen as the product of a hyperbolic quadric of rank $(r-d)/2$ with a subspace of dimension d, we have

$$A_R = \#\{\text{points of } Q_R\} = \frac{q^{r-1} - 1}{q-1} + q^{(r+d-2)/2}.$$

It is now straightforward to retrieve A.

For any $p \in Q$, we have $p \in \mathfrak{P}^0$ if and only if $p \notin \mathfrak{P}$ and the coordinates of p are solution of System (6), that is

$$\begin{cases} -z^2 + 2\mathbf{z_2}^T\mathbf{z_1} = 0 \\ 2\mathbf{z_2}^T\mathbf{z_1} + y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2} + 2\mathbf{x_1}^T\mathbf{y_2} = 0. \end{cases} \tag{9}$$

To determine the number of solutions of (9) we distinguish three cases:

- $\mathbf{x_1} = \mathbf{0}$; consequently we also have $z = 0$. Under this assumption the first equation in (9) is $\mathbf{z_2}^T\mathbf{z_1} = 0$; it has

$$(q^{n-(r+d)/2} - 1)(q^{n-(r+d)/2-1} + 1) + 1$$

  solutions in $(\mathbf{z_1}, \mathbf{z_2})$, that is the number of singular *vectors* for the hyperbolic quadratic form $\mathbf{z_2}^T\mathbf{z_1}$ of rank $n - (r+d)/2$. Given $\mathbf{z_1}$ and $\mathbf{z_2}$, we can choose $\mathbf{y_2}$ in an arbitrary way; thus it can assume q^d values. Finally, the second equation in (9) is fulfilled when the vector $(y, \mathbf{x_2})$ is singular for the parabolic form $y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2}$ of rank $(r-d)/2$; that is to say there are q^{r-d} possibilities for it.

  Thus, the number of (projective) points whose coordinates satisfy (9) with $\mathbf{x_1} = \mathbf{0}$ is

$$N_1^0 = \frac{q^r(q^{2n-r-d-1} + q^{n-(r+d)/2} - q^{n-(r+d)/2-1}) - 1}{q-1}.$$

13

- Assume now $z = 0$ and $\mathbf{x_1} \neq \mathbf{0}$. The first equation in (9) is the same as before; thus the vector $(\mathbf{z_1}, \mathbf{z_2})$ can assume $(q^{n-(r+d)/2} - 1)(q^{n-(r+d)/2-1} + 1) + 1$ distinct values. As $z = 0$ and $\mathbf{x_1} \neq \mathbf{0}$, the vector $\mathbf{x_1}$ can be chosen in $q^{d-1} - 1$ ways, while y and $\mathbf{x_2}$ are arbitrary — thus there are respectively q and q^{r-d} possibilities for these. Observe that given $\mathbf{z_1}, \mathbf{z_2}, y, \mathbf{x_1}, \mathbf{x_2}$ the second equation in (9) is a non-null linear equation in $\mathbf{y_2}$; thus there are q^{d-1} possible solutions $\mathbf{y_2}$. Overall we get that the number of projective points satisfying (9) with $z = 0$ and $\mathbf{x_1} \neq \mathbf{0}$ is

$$N_2^0 = \frac{q^r}{q-1}(q^{d-1} - 1)(q^{2n-r-d-1} + q^{n-(r+d)/2} - q^{n-(r+d)/2-1}).$$

- Finally, suppose $z \neq 0$. Clearly, there are $(q-1)$ possible choices for z and, consequently, $q^{d-1}(q-1)$ choices for $\mathbf{x_1}$. The first equation in (9) becomes $\mathbf{z_2}^T \mathbf{z_1} = z^2$. Observe that the hyperbolic form $\mathbf{z_2}^T \mathbf{z_1}$ assumes a given square value z^2 for exactly $(\frac{q^{2n-r-d}-1}{q-1} - \frac{(q^{n-(r+d)/2}-1)(q^{n-(r+d)/2-1}+1)}{q-1})\frac{1}{2} \cdot 2$ choices of $(\mathbf{z_1}, \mathbf{z_2})$. The values of y and $\mathbf{x_2}$ can now be chosen arbitrarily; that is to say, there are respectively q and q^{r-d} possibilities. Finally, as in the previous case, the vector $\mathbf{y_2}$ must be the solution of a non-null linear equation in d unknowns; thus, it can assume q^{d-1} distinct values. So, for $z \neq 0$, then the number of projective points being solutions of (9) is

$$N_3^0 = q^{2n-2} - q^{n+(r+d)/2-2}.$$

In particular,

$$N^0 = N_1^0 + N_2^0 + N_3^0 - A = \frac{q^{2n-1} - 1}{q-1} - A. \tag{10}$$

By Lemma 3.8 and Table 3, $p \in \mathfrak{P}^+$ if and only if $p^T S M^{-1} S p \in -\square$. Thus, the coordinates of the points of $\mathfrak{P}^+$ satisfy

$$\begin{cases} -z^2 + 2\mathbf{z_2}^T \mathbf{z_1} \in -\square \\ 2\mathbf{z_2}^T \mathbf{z_1} + y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2} + 2\mathbf{x_1}^T \mathbf{y_2} = 0 \end{cases} \tag{11}$$

We argue as above.

- Suppose $\mathbf{x_1} = \mathbf{0}$; hence $z = 0$. The vector $\mathbf{y_2}$ can be chosen arbitrarily; thus, it may assume q^d values. The first equation in (11) gives $2\mathbf{z_2}^T \mathbf{z_1} = -\beta^2$ for some element $\beta \in \mathbb{F}_q \setminus \{0\}$. As the quadric induced by $2\mathbf{z_2}^T \mathbf{z_1}$ is hyperbolic, there are

$$(\frac{q^{2n-r-d} - 1}{q-1} - \frac{(q^{n-(r+d)/2} - 1)(q^{n-(r+d)/2-1} + 1)}{q-1})\frac{1}{2} \cdot (q-1)$$

possible vectors of $V(2n - r - d, q)$ on which the quadratic form $2\mathbf{z_2}^T \mathbf{z_1}$ assumes a value opposite of a square. Observe that for any of these choices of $(\mathbf{z_1}, \mathbf{z_2})$, an element $-\beta^2$ is determined by the first equation. Given such β^2, the second equation becomes

$$\beta^2 = y^2 + \mathbf{x_2}^T R_0^+ \mathbf{x_2}.$$

By Claim 1 of Lemma 3.10, the number of solutions of this equation is $q^{(r-d)/2}(q^{(r-d)/2} + 1)$. So, the contribution of this case to the number of points fulfilling (11) is

$$N_1^+ = \frac{(q^{2n-r-d-1} - q^{n-(r+d)/2-1})(q^r + q^{(r+d)/2})}{2}.$$

14

- Suppose now $\mathbf{x_1} \neq \mathbf{0}$ and $z = 0$. The vector $\mathbf{x_1}$, clearly, can assume $q^{d-1} - 1$ distinct non-null values. The analysis of the first equation in (11) is exactly as before and gives that the vector $(\mathbf{z_1}, \mathbf{z_2})$ can assume $(\frac{q^{2n-r-d}-1}{q-1} - \frac{(q^{n-(r+d)/2}-1)(q^{n-(r+d)/2-1}+1)}{q-1})\frac{1}{2} \cdot (q-1)$ values. The values of y and $\mathbf{x_2}$ may be assigned arbitrarily, thus there are respectively q and q^{r-d} possibilities for them. Finally, $\mathbf{y_2}$ can assume q^{d-1} different values, this being the number of solutions of a linear equation in d unknowns. So, for $z = 0$ and $\mathbf{x_1} \neq \mathbf{0}$, the number of projective points solution of (11) is

$$N_2^+ = \frac{(q^{2n-r-d-1} - q^{n-(r+d)/2-1})(q^{d-1} - 1)q^r}{2}.$$

- Assume now $z \neq 0$. By Lemma 3.10, Claim 2, $-z^2 + 2\mathbf{z_2}^T\mathbf{z_1} \in -\square$ has

$$\frac{q-1}{2}\left(q^{2n-(r+d)} - q^{2n-(r+d)-1} + q^{n-(r+d)/2} + q^{n-(r+d)/2-1}\right)$$

solutions in $(z, \mathbf{z_1}, \mathbf{z_2})$. For $\mathbf{x_1}$ there remain q^{d-1} possibilities since the first coordinate z has already been taken into account in the first equation. As before, the values of y and $\mathbf{x_2}$ can be assigned arbitrarily, that is in respectively q and q^{r-d} ways and $\mathbf{y_2}$ is a solution of a linear equation in d unknowns; thus there are q^{d-1} possibilities for the latter. The contribution of solutions in terms of projective points to the system (11) for $z \neq 0$ is

$$N_3^+ = \frac{q^{2n-1} + q^{n+(r+d)/2-1} - q^{2n-2} + q^{n+(r+d)/2-2}}{2}.$$

In particular,

$$N^+ = N_1^+ + N_2^+ + N_3^+ = \frac{q^{2n-1} + q^{2n-(r+d)/2-1} + q^{n+(r+d)/2-1} - q^{n-1}}{2}. \tag{12}$$

The value of N^- can now be recovered either with a similar argument, or by just observing that $N^- = \#\{\text{points of } Q(2n, q)\} - (A + N^0 + N^+)$. $\qquad\square$

Proposition 3.12. *The function $f^1(r, d)$ attains its maximum $f_{\max}^1$ for $r = 2n - 1$ and $d = 1$, where it assumes the value*

$$f_{\max}^1 := f^1(2n - 1, 1) = \frac{(q^{n-1} - 1)(q^{3n-2} + q^{3n-3} - q^{3n-4} + q^{2n} - q^{n-1} - 1)}{(q-1)^2(q+1)}.$$

Proof. By plugging the values of A, N^0, N^+ and N^- into (3), we get

$$f^1(r, d)(q+1)(q-1)^2 = q^{2n-3}(q-1)(q^{r-1} + q^{(r+d)/2+1} - q^{(r+d)/2-1} + q^{n-(r+d)/2+1} + q^{n-(r+d)/2}) +$$
$$q^{4n-3} - q^{2n-1} - q^{2n-2} + q^{2n-3} - q^{2n} + 1. \tag{13}$$

Recall that from the last paragraph of Section 3.2, $1 \leq d \leq \min\{r, 2n - r\}$.

Let $s = r + d$. It is straightforward to see that $f(r, d)$ is maximum if and only if $g(r, s)$ is maximum, where

$$g(r, s) = q^{n-s/2+1} + q^{n-s/2} + q^{s/2+1} - q^{s/2-1} + q^{r-1}, \tag{14}$$

with the constraints $1 \leq r \leq 2n - 1$ and

$$r + 1 \leq s \leq \min\{2r, 2n\}. \tag{15}$$

In order to determine the maximum of $g(r, s)$ in its domain, we regard it as a continuous function defined over $\mathbb{R}^2$ and then we reinterpret its behavior over $\mathbb{Z}^2$. So, we can consider the derivative

$$\frac{\partial}{\partial s} g(r, s) = \frac{\log q}{2} \left(q^{s/2+1} - q^{n-s/2+1} - q^{s/2-1} - q^{n-s/2} \right). \tag{16}$$

This is positive for

$$q^{s/2} \left(q - \frac{1}{q} \right) > q^{n-s/2}(q + 1),$$

that is

$$s > n + \log_q \frac{q}{q - 1}. \tag{17}$$

As $1 < \frac{q}{q-1} < q$, also $0 < \log_q \frac{q}{q-1} < 1$, and (17) gives $s \geq n + 1$. Hence, for $s \geq n + 1$ the function $g(r, s)$ is increasing in s, while for $s \leq n$ it is decreasing. Define

$$h(r) := \max_s g(r, s),$$

where s varies in all allowable ways for any given r. The following cases are possible:

- for $s \geq n + 1$ the maximum of $g(r, s)$ is attained when s is maximum, that is

$$h(r) = g(r, \max s),$$

where by $\max s$ we denote the maximum value s may assume, subject to the constraints of (15). This leads to the following two subcases:

1. if $r > n$, then $\max s = 2n$ and

$$h(r) := \max_s g(r, s) = g(r, 2n).$$

By (15), $r < s \leq 2n$ is odd and by

$$\frac{\partial}{\partial r} h(r) = \frac{\partial}{\partial r} g(r, 2n) = q^{r-1} \log q > 0, \tag{18}$$

the value of $h(r) = g(r, 2n)$ is maximum for r maximum, that is $r = 2n - 1$. Since $s = r + d$ by definition, as $r = 2n - 1$ and $s = 2n$, we have $d = 1$. So, $g(r, s)$ is maximum for $r = 2n - 1$ and $s = 2n$. The value assumed in this case is

$$g(2n - 1, 2n) = q^{2n-2} + q^{n+1} - q^{n-1} + q + 1. \tag{19}$$

2. if $r \leq n$ and also $s \geq n + 1$, then $n + 1 \leq s \leq 2r$ implies $r \geq (n + 1)/2$. Since $s \leq 2r \leq 2n$, by (15), we have $\max s = 2r$. Thus,

$$h(r) = g(r, 2r) = q^{r+1} + q^{n-r+1} + q^{n-r}.$$

Then,

$$\frac{\partial}{\partial r} h(r) = (\log q) \left(q^{r+1} - q^{n-r+1} - q^{n-r} \right).$$

We have $\frac{\partial}{\partial r} h(r) > 0$ if and only if $(r + 1) > (n - r) + \log_q(q + 1)$, that is $2r > n - 1 + \log_q(q + 1)$, i.e. $2r \geq n + 1$. In particular, for $2r \geq n + 1$, the function $h(r)$ is increasing and it attains its maximum for $r = n$, where

$$h(n) = q^{n+1} + q + 1 = g(n, 2n).$$

This is smaller than (19); so in the range $\frac{n+1}{2} \leq r \leq 2n - 1$ the maximum is given by (19).

16

- Suppose now $2 \leq s < n+1$; then, as $d \geq 1$, we have $r \leq s-1$ and

$$\begin{cases} r-1 \leq s-2 < n-1 \\ n-s/2+1 \leq n \\ n-s/2 \leq n-1 \\ s/2+1 \leq n/2+3/2 \\ s/2-1 \geq 0. \end{cases} \qquad . \qquad (20)$$

Using the estimates of (20) in (14) we get

$$g(r,s) < q^n + 2q^{n-1} + q^{n/2+3/2} =: g_0. \qquad (21)$$

Observe that the value g_0 from (21) is always smaller than that of $g(r,s)$ given by (19).

The above argument proves that the maximum of $g(r,s)$ is always attained in (19); consequently, the maximum for $f^1(r,d)$ is $f^1_{\max} := f^1(2n-1,1)$. □

In particular,
$$\frac{q^{2n}-1}{q-1}\frac{q^{2n-2}-1}{q-1}\frac{1}{q+1} - f^1_{\max} = q^{4n-5} - q^{3n-4}.$$

Thus, this case correspond to words of minimum weight and these words are alternating bilinear forms with a radical of dimension $2n-1$, that is to say, they correspond to some points of $\mathcal{G}_{2n+1,2n-1}$.

We now show that in the three remaining cases $f(r,d)$ cannot ever be larger than $f^1_{\max}$.

3.5 Cases 2 and 3

Cases 2 and 3 can be carried out in close analogy to Section 3.4. The values they yield for $f^2_{\max}$ and $f^3_{\max}$ turn out to be always lower than $f^1_{\max}$. The following proposition summarizes the results; its proof is quite analogous to that of Proposition 3.11.

Proposition 3.13. *Suppose we have a maximum number of totally singular totally isotropic lines and Assumption 3.7 holds. Then,*

- *In case 2,*

$$A = 2\frac{q^{n-(r+d)/2}-1}{q-1} + \frac{q^{r-1}-1}{q-1} - q^{(r+d-2)/2}; \qquad N_0 = \frac{q^{2n-1}-1}{q-1} - A;$$

$$N^+ = \frac{q^{2n-1} + q^{2n-(r+d)/2-1} - q^{n+(r+d)/2-1} - q^{n-1}}{2};$$

$$N^- = \frac{q^{2n-1} - q^{2n-(r+d)/2-1} + q^{n+(r+d)/2-1} + q^{n-1}}{2}.$$

- *In case 3,*

$$A = 2\frac{q^{n-(r+d-1)/2}-1}{q-1} + \frac{q^{r-1}-1}{q-1}; \qquad N_0 = \frac{q^{2n-1} + q^{n+(r+d-1)/2} - q^{n+(r+d-1)/2-1} - 1}{q-1} - A;$$

$$N^{\pm} = \frac{q^{2n-r-d} - q^{n-(r+d+1)/2}}{2(q-1)}(q^{r+d-1} \pm q^{(r+d-1)/2}).$$

Using arguments similar to those of Proposition 3.12, we can show that for $n > 3$ the maximum number of totally singular totally isotropic lines is attained when the radical R of the alternating form φ is as large as possible. Our results are described by the following proposition.

Proposition 3.14. *For $1 \leq i \leq 3$ denote by $f^i(r, d)$ the function $f(r, d)$ obtained in case i and by $f^i_{\max}$ its maximum. The values of r, d where f^i attains its maximum $f^i_{\max}$ for $i = 1, 2, 3$ are those outlined in Table 5.*

Case	$n = 3$		$n > 3$	
1	$r = 5,$	$d = 1$	$r = 2n - 1,$	$d = 1$
2	$r = 1,$	$d = 1$	$r = 2n - 1,$	$d = 1$
3	$r = 1,$	$d = 0$	$r = 2n - 1,$	$d = 0$

Table 5: Maxima for the functions $f^i(r, d)$.

3.6 Fourth case

As Q_0 is elliptic, we have from Table 4

$$A = 2\frac{q^{n-(r+d+1)/2} - 1}{q - 1} + \frac{q^{r-1} - 1}{q - 1}.$$

Recall that $A^0 > B^+ > B^0 > B^-$; thus,

$$f^4_{\max} < \frac{1}{q+1}\left(AA_0 + (\frac{q^{2n} - 1}{q - 1} - A)B^+\right) = \frac{A_0 - B^+}{q + 1}A + \frac{q^{2n} - 1}{q^2 - 1}B^+.$$

Some straightforward algebraic manipulations show that

$$f^4_{\max}(r, d)(q - 1)^2(q + 1) < \tau(r, s),$$

where $s = r + d$ and

$$\tau(r, s) := q^n(q^{n-1} - 1)(q - 1)(q^{r-3} + 2q^{n-(s+5)/2}) +$$
$$q^{4n-3} + q^{3n-1} - q^{3n-2} - 3q^{2n-2} + 2q^{2n-3} - q^{2n} + 2q^{n-1} - 2q^{n-2} + 1.$$

Regarding $\tau(r, s)$ as a function defined over $\mathbb{R}^2$,

$$\frac{\partial}{\partial s}\tau(r, s) = \log q\left(-q^{3n-(s+5)/2} + q^{3n-(s+7)/2} + q^{2n-(s+3)/2} - q^{2n-(s+5)/2}\right) < 0.$$

In particular, the maximum of $\tau(r, s)$ is attained for s minimum, that is $s = r$. Thus,

$$h(r) := \max_s \tau(r, s) = \tau(r, r).$$

By computing $\frac{\partial}{\partial r}h(r)$, we see that the function $h(r)$ has one critical point in the range $1 < r < 2n - 1$ and this critical point is a minimum. Thus, the maximum of $h(r)$ is for either $r = 1$ or $r = 2n - 1$. We have

$$h(1) = q^{4n-3} + q^{3n-1} - q^{3n-2} + 2q^{3n-3} - 2q^{3n-4} - 4q^{2n-2} + 3q^{2n-3} - q^{2n} + q^{n-1} - q^{n-2} + 1,$$

$$h(2n - 1) = q^{4n-3} + q^{4n-4} - q^{4n-5} + q^{3n-1} - q^{3n-2} - q^{3n-3} + q^{3n-4} - q^{2n-2} - q^{2n} + 1,$$

and $h(1) < h(2n - 1)$. In any case, $f^4_{\max} < \frac{h(2n-1)}{(q-1)^2(q+1)} = f^1_{\max}$.

Proposition 3.15. *Under Assumption 3.7, the maximum of the function $f(r, d)$ is $f^1_{\max}$, attained in case 1, for $r = 2n - 1$ and $d = 1$; consequently, the minimum distance of the orthogonal Grassmann code $\mathcal{P}_{n,2}$ is $q^{4n-5} - q^{3n-4}$.*

18

Type of ℓ	$\#(\ell \cap \mathfrak{P}^+)$	$\#(\ell \cap \mathcal{W})$	$\#(\ell \cap \mathfrak{P}^-)$	Corresponding set
0	0	$q+1$	0	Δ^0
$+$	q	1	0	Δ^+
α	$\frac{q+1}{2}$	0	$\frac{q+1}{2}$	Δ^α
β	$\frac{q-1}{2}$	2	$\frac{q-1}{2}$	Δ^β
$-$	0	1	q	Δ^-

Table 6: Types of lines in $\Delta_{n,2}$ and corresponding subsets

3.7 Removal of Assumption 3.7

We are now ready to drop Assumption 3.7. Let $\mathcal{W}$ be the quadric induced by the matrix $W = SM^{-1}S$, as studied in Lemma 3.8. We say that a line $\ell \in \Delta_{n,2}$ is of type 0, $+$, α, β or $-$ according to the conditions in Table 6; observe that actually $\ell \cap \mathcal{W} = \ell \cap (\mathfrak{P} \cup \mathfrak{P}^0)$.

Lemma 3.16. *For any choice of M and S we have*

$$(N^+ - N^-)\frac{q^{2n-2}-1}{q-1} = q(\#\Delta^+ - \#\Delta^-).$$

Proof. We count the number of flags of type (p, ℓ) with $p \in \mathfrak{P}^-$ or $p \in \mathfrak{P}^+$ and $\ell \in \Delta_{2,n}$ in two different ways. Let

$$\mathfrak{S}^- = \{(p, \ell) : p \in \mathfrak{P}^-, p \in \ell, \ell \in \Delta_{n,2}\}.$$

As there are exactly $\frac{q^{2n-2}-1}{q-1}$ lines of $\Delta_{2,n}$ through any point $p \in Q$ we have

$$\#\mathfrak{S}^- = N^- \frac{q^{2n-2}-1}{q-1}.$$

On the other hand, only lines of type α, β or $-$ are incident with points of $\mathfrak{P}^-$. Using Table 6 we get

$$\#\mathfrak{S}^- = q\#\Delta^- + \frac{q+1}{2}\#\Delta^\alpha + \frac{q-1}{2}\#\Delta^\beta.$$

So,

$$N^- \frac{q^{2n-2}-1}{q-1} = q\#\Delta^- + \frac{q+1}{2}\#\Delta^\alpha + \frac{q-1}{2}\#\Delta^\beta.$$

By the same counting argument on $\mathfrak{S}^+ := \{(p, \ell) : p \in \mathfrak{P}^+, p \in \ell, \ell \in \Delta_{n,2}\}$, we have

$$N^+ \frac{q^{2n-2}-1}{q-1} = q\#\Delta^+ + \frac{q+1}{2}\#\Delta^\alpha + \frac{q-1}{2}\#\Delta^\beta.$$

Consequently,

$$(N^+ - N^-)\frac{q^{2n-2}-1}{q-1} = q(\#\Delta^+ - \#\Delta^-).$$

$\square$

Proposition 3.17. *The maximum of the function f is $f^1_{\max}$, as described in Proposition 3.15.*

Proof. Both the lines of Δ^+ and those of Δ^- are simultaneously tangent to Q and 1–secant to $\mathcal{W}$. In particular, all of them are tangent to both Q and $\mathcal{W}$ at some point $p \in Q \cap \mathcal{W}$. Thus,

$$\#\Delta^+ + \#\Delta^- \leq \frac{q^{2n-2}-1}{q-1}\#(Q \cap \mathcal{W}).$$

A fortiori,

$$\#\Delta^+ - \#\Delta^- \leq \frac{q^{2n-2}-1}{q-1}\#(Q \cap \mathcal{W});$$

consequently, by Lemma 3.16,

$$\delta := (N^+ - N) \leq q\#(Q \cap \mathcal{W}) < q\frac{q^{2n}-1}{q-1}. \tag{22}$$

Since $B^+ + B^- = 2B^0$, $B^+ - B^0 = q^{n-2}$, $A^0 - B^0 = q^{2n-3}$, we can rewrite (3) in the form

$$(q+1)f = A(A^0 - B^0) + B^0\frac{q^{2n}-1}{q-1} + \delta(B^+ - B^0) = Aq^{2n-3} + \delta q^{n-2} + \frac{q^{2n-3}-1}{q-1}\frac{q^{2n}-1}{q-1}. \tag{23}$$

Using the estimate of (22) this becomes

$$f < \frac{Aq^{2n-3}(q-1)^2 + q^{4n-3} - q^{3n-1} - q^{2n-3} + q^{3n} - q^{2n} - q^n + q^{n-1} + 1}{(q+1)(q-1)^2}. \tag{24}$$

Suppose now $f \geq f_{\max}^1 = \frac{q^{4n-3}+q^{4n-4}+\cdots}{(q+1)(q-1)^2}$. Then, $Aq^{2n-3}(q-1)^2 \geq q^{4n-4}$; thus, $A \approx q^{2n-3}$. Observe now that $A = A_V + A_R$ with $A_V \leq 2\frac{q^n-1}{q-1}$. In particular, we have $A_R \approx q^{2n-3}$. As $A_R = \#(Q \cap R)$ this gives $r = \dim R \geq 2n - 1$. As $R \leq D^{\perp_Q}$, we obtain $d \leq 2$. For $d = 1$ there are no possible eigenspaces contributing to A_V and we end up in either case 1 or 2, according to the nature of R_0; thus, $f = f_{\max}^1$. Likewise, for $d = 2$ there are also no possible eigenspaces contributing to A_V and we are done. Finally, for $d = 0$ then $M^{-1}S$ has necessarily two eigenspaces of dimension 1, that is of maximal dimension; thus we end up in either case 3 or 4. In particular, all possible configurations have been already investigated and we can conclude $f = f_{\max}^1$. $\qquad\square$

3.8 Minimum weight codewords

Proposition 3.18. *All minimum weight codewords are projectively equivalent.*

Proof. By Proposition 3.17 a minimum weight codeword corresponds to a configuration in which $\dim R = 2n - 1$, $d = 1$ and $\dim V_\lambda = \dim V_\mu = 0$. The form of the matrices M and S with respect to the basis B of Section 3.3 is as dictated by Case 1. In particular, we have with respect to B

$$M = \begin{pmatrix} 0 & 0 & \mathbf{0} & 1 \\ 0 & 1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & R_0^+ & \mathbf{0} \\ 1 & 0 & \mathbf{0} & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 1 & \mathbf{0} & 0 \\ -1 & 0 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & \mathbf{0} & 0 \end{pmatrix}.$$

This means that, up to projective equivalence, M and S are uniquely determined. The result follows. $\qquad\square$

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