Minimum distance of Symplectic Grassmann codes

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Abstract

In this paper we introduce Symplectic Grassmann codes, in analogy to ordinary Grassmann codes and Orthogonal Grassmann codes, as projective codes defined by symplectic Grassmannians. Lagrangian–Grassmannian codes are a special class of Symplectic Grassmann codes. We describe all the parameters of line Symplectic Grassmann codes and we provide the full weight enumerator for the Lagrangian–Grassmannian codes of rank 2 and 3.

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1. Introduction

Grassmann codes have been introduced in [19, 20] as generalizations of Reed–Muller codes of the first order; they have been extensively investigated ever since. Their parameters, as well as some of their higher weights have been fully determined in [17]. These are projective codes, arising from the Plücker embedding of a k–Grassmannian. A further point of interest is that the weight distribution provides some interesting insight on the geometry of the embedding itself.

Codes arising from the Plücker embedding of the k–Grassmannian of an orthogonal polar space have been introduced in a recent series of papers [4, 6, 5]. In [4], we computed the minimum distance for the codes arising from orthogonal dual polar spaces of rank 2 and 3 and provided a general bound on the minimum distance. More recently, in [6], for q odd the minimum distance for all Line Polar Grassmann codes of orthogonal type has been determined. In [5] an encoding scheme, as well as strategies for decoding and error correction, has been proposed for Line Polar Grassmann codes. We point out that, even if the parameters of the codes under consideration arise from the Grassmann embedding, neither the encoding scheme we considered nor the error correction strategy we proposed make direct use of Plücker coordinates.

The aim of the present paper is to provide results analogous to those of [4, 6] for codes arising from the Plücker embedding of k–Grassmannians of symplectic type.

More in detail, we shall denote by $\mathcal{W}(n, k)$, the projective code defined by the image under the Plücker embedding of the k–symplectic Grassmannian $\Lambda_{n,k}$ defined by a non–degenerate alternating bilinear form σ on a vector space $V := V(2n, q)$ of dimension $2n$ over a finite field $\mathbb{F}_q$. This will be referred as a *Symplectic Grassmann Code*.

The paper is organized as follows: in Section 2 some basic notions about projective codes and symplectic Grassmannians are recalled; Section 3 is dedicated to Line Symplectic Grassmann

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codes and contains our main results for $k = 2$; Section 4 is dedicated to the case of rank $k = 3$. Overall, in these sections we prove the following.

Main Theorem. *The code $\mathcal{W}(n, k)$ has parameters*

$$N = \prod_{i=0}^{k-1} (q^{2n-2i} - 1)/(q^{i+1} - 1), \qquad K = \binom{2n}{k} - \binom{2n}{k-2}.$$

Furthermore,

- *For $k = 2$, its minimum distance is $q^{4n-5} - q^{2n-3}$;*

- *For $n = k = 3$, its minimum distance is $q^6 - q^4$.*

Finally, in Section 5 we discuss some further bounds for the minimum distance in the general case of Symplectic Grassmann codes arising from higher weights of Grassmann codes.

We point out that the code $\mathcal{W}(n, n)$ where $k = n$, corresponding to the so called *dual polar space*, has already been introduced under the name of Lagrangian-Grassmannian code of rank n in [7], where some bounds on the parameters have been obtained.

2. Preliminaries

A $[N, K, d_{\min}]$ projective system $\Omega \subseteq \mathrm{PG}(K - 1, q)$ is just a set of N distinct points in $\mathrm{PG}(K - 1, q)$ whose span is $\mathrm{PG}(K - 1, q)$ and such that for any hyperplane Σ of $\mathrm{PG}(K - 1, q)$

$$\#(\Omega \setminus \Sigma) \geq d_{\min}.$$

It is well known that existence of a $[N, K, d_{\min}]$ projective system is equivalent to that of a projective linear code $\mathcal{C}$ with the same parameters. Indeed, several codes can be obtained by taking as generator matrix G the matrix whose columns are the coordinates of the points of Ω normalized in some way. As the order of the points, the choice of coordinates as well as the normalization adopted change, we obtain potentially different codes arising from Ω, but all of these turn out to be equivalent. As such, in the following discussion, they will be silenty identified and we shall write $\mathcal{C} = \mathcal{C}(\Omega)$. The spectrum of the intersections of Ω with the hyperplanes of $\mathrm{PG}(K - 1, q)$ provides the list of the weights of $\mathcal{C}$; we refer to [22] for further details.

Let now and throughout the paper $V := V(2n, q)$ be a $2n$-dimensional vector space equipped with a non–degenerate bilinear alternating form σ. Denote by $\mathcal{G}_{2n,k}$ the k-Grassmannian of the projective space $\mathrm{PG}(V)$, that is the point–line geometry whose points are the k-dimensional subspaces of V and whose lines are the sets

$$\ell_{W,T} := \{X : W \leq X \leq T, \dim X = k\}$$

with $\dim W = k - 1$ and $\dim T = k + 1$. A projective embedding of $\mathcal{G}_{2n,k}$ is a function $e : \mathcal{G}_{2n,k} \to \mathrm{PG}(U)$ such that $\langle e(\mathcal{G}_{2n,k}) \rangle = \mathrm{PG}(U)$ and each line of $\mathcal{G}_{2n,k}$ is mapped onto a line of $\mathrm{PG}(U)$. The dimension of U is called *dimension of the embedding*. It is well known that the geometry $\mathcal{G}_{2n,k}$ affords a projective embedding $e_k^{gr} : \mathcal{G}_{2n,k} \to \mathrm{PG}(\bigwedge^k V)$ by means of Plücker coordinates. In particular, e_k^{gr} maps an arbitrary k–dimensional subspace $\langle v_1, v_2, \ldots, v_k \rangle$ of V to the point $\langle v_1 \wedge v_2 \wedge \cdots \wedge v_k \rangle$. The image $e_k^{gr}(\mathcal{G}_{2n,k})$ is a projective variety of $\mathrm{PG}(\bigwedge^k V)$, usually denoted by the symbol $\mathbb{G}(2n - 1, k - 1)$, see [15, Lecture 6], called the *Grassmann variety*.

The symplectic Grassmannian $\Lambda_{n,k}$ induced by σ, is defined for $k = 1, \ldots, n$ as the subgeometry of $\mathcal{G}_{2n,k}$ having as points the totally σ–isotropic subspaces of V of dimension k and as lines

- for $k < n$, the sets of the form

$$\ell_{W,T} := \{X : W \le X \le T, \dim X = k\}$$

with T totally isotropic and $\dim W = k - 1$, $\dim T = k + 1$.

- for $k = n$, the sets of the form

$$\ell_W := \{X : W \le X, \dim X = n\}$$

with $\dim W = n - 1$, W totally isotropic.

For $k = n$, $\Lambda_{n,n}$ is usually called *dual polar space of rank n* or *Lagrangian Grassmannian*.

The image of $\Lambda_{n,k}$ under the Plücker embedding e_k^{gr} is a subvariety $\mathbb{L}(n-1, k-1)$ of the Grassmann variety $\mathbb{G}(2n-1, k-1)$.

Let $\Sigma = \langle \mathbb{L}(n-1, k-1) \rangle < \mathrm{PG}(\bigwedge^k V)$. It is well known, see [9], that

$$\dim \Sigma = \binom{2n}{k} - \binom{2n}{k-2};$$

indeed, the variety $\mathbb{L}(n-1, k-1)$ is the full intersection of $\mathbb{G}(2n-1, k-1)$ with a suitable subspace of $\bigwedge^k V$ of codimension $\binom{2n}{k-2}$.

The following formula provides the length of $\mathcal{W}(n, k)$:

$$\#\mathbb{L}(n-1, k-1) = \#\Lambda_{n,k} = \prod_{i=0}^{k-1} (q^{2n-2i} - 1)/(q^{i+1} - 1). \tag{1}$$

As pointed out before, the pointset of $\mathbb{L}(n-1, k-1)$ is a projective system of $\mathrm{PG}(\Sigma)$; thus this determines an associated projective code which we shall denote by $\mathcal{W}(n, k)$ and call it a *Symplectic Grassmann code*. When $n = k$ Symplectic Grassmann codes are known in the literature also as *Lagrangian-Grassmannian codes*. A straightforward consequence of the remarks presented above is the following lemma.

Lemma 2.1. *The code $\mathcal{W}(n, k)$ has length $N = \#\mathbb{L}(n-1, k-1)$ and dimension $K = \dim \Sigma$.*

3. Line Symplectic Grassmann Codes

Throughout this section $\mathcal{S} := W(2n-1, q)$ denotes a non-degenerate symplectic polar space defined by the non–degenerate alternating bilinear form σ on V of rank n. By θ we shall denote a different (possibly degenerate) alternating bilinear form. We shall also write $\perp_\sigma$ and $\perp_\theta$ for the orthogonality relations induced by σ and θ respectively. Finally, recall that the *radical* of θ is the set

$$\mathrm{Rad}\,\theta := \{x \in \mathcal{S} : x^{\perp_\theta} = \mathrm{PG}(V)\} = \{x \in \mathcal{S} : \forall y \in \mathcal{S}, \theta(x, y) = 0\}.$$

For $k = 2$ the expression (1) becomes

$$N := \#\Lambda_{n,2} = \frac{(q^{2n} - 1)(q^{2n-2} - 1)}{(q - 1)(q^2 - 1)}; \qquad K := 2n^2 - n - 1.$$

It is well known that any bilinear alternating form θ determines an hyperplane of $\bigwedge^2 V$ and conversely; hence, the minimum distance of $\mathcal{W}(n, 2)$ can be deduced from the maximum number of lines which are simultaneously totally isotropic for both the forms θ and σ, under the assumption $\theta \ne \sigma$. In order to determine this number we follow an approach similar to that of [6, Lemma 3.2].

Lemma 3.1. *Suppose M is the matrix representing σ and S the matrix representing θ with respect to a given reference system. For any $p \in \mathcal{S}$ we have $p^{\perp_\sigma} \subseteq p^{\perp_\theta}$ if, and only if, p is an eigenvector of the matrix $M^{-1}S$.*

Proof. Since M is non–singular, if $M^{-1}Sp = \mathbf{0}$, then $Sp = 0$, that is to say $p \in \operatorname{Rad}\theta$, i.e. $p^{\perp_\theta} = \operatorname{PG}(V)$. In this case, obviously, $p^{\perp_\sigma} \subseteq p^{\perp_\theta}$.

When $p \notin \operatorname{Rad}\theta$, we have $\dim p^{\perp_\sigma} = \dim p^{\perp_\theta}$. Thus, $p^{\perp_\sigma} \subseteq p^{\perp_\theta}$ if, and only if, $p^{\perp_\sigma} = p^{\perp_\theta}$, that is to say the systems of equations $x^T M p = 0$ and $x^T S p = 0$ are equivalent. This yields $Sp = \lambda M p$ for some $\lambda \neq 0$, whence p is an eigenvector of eigenvalue λ for $M^{-1}S$. $\square$

Write now

$$N_0 := \#\{p \in \mathcal{S} : p^{\perp_\sigma} \not\subseteq p^{\perp_\theta}\}, \qquad N_1 := \#\{p \in \mathcal{S} : p^{\perp_\sigma} \subseteq p^{\perp_\theta}\}.$$

Clearly, $N_0 = \frac{q^{2n}-1}{q-1} - N_1$.

For any $p \in \mathcal{S}$, a line ℓ through p is both totally σ–isotropic and θ–isotropic if, and only if, $\ell \in p^{\perp_\sigma} \cap p^{\perp_\theta}$. In particular,

- if $p^{\perp_\sigma} \subseteq p^{\perp_\theta}$, then $\frac{q^{2n-2}-1}{q-1}$ lines through p are both σ– and θ–isotropic;

- if $p^{\perp_\sigma} \not\subseteq p^{\perp_\theta}$, then $p^{\perp_\sigma} \cap p^{\perp_\theta}$ is a subspace of codimension 2 in $\operatorname{PG}(V)$ and the number of lines which are both σ– and θ–isotropic is $\frac{q^{2n-3}-1}{q-1}$.

Denote now by η the number of lines of $\mathcal{S}$ which are simultaneously totally σ– and θ–isotropic. As each line contains $(q+1)$ points, we have

$$(q+1)\eta = N_0 \frac{q^{2n-3}-1}{q-1} + N_1 \frac{q^{2n-2}-1}{q-1} = q^{2n-3}N_1 + \frac{(q^{2n}-1)(q^{2n-3}-1)}{(q-1)^2}. \tag{2}$$

Clearly, η is maximum when N_1 is maximum. In the remainder of this section we shall determine exactly how large N_1 can be.

Lemma 3.2. *If the matrix $M^{-1}S$ has has just two eigenspaces, one of dimension $2n-2$, the other of dimension 2, then the number of eigenvectors of $M^{-1}S$ is maximum.*

Proof. In order for the number of eigenvectors of $M^{-1}S$ to be maximum we need $M^{-1}S$ to be diagonalizable. Suppose that there are at least 3 distinct eigenspaces, say V_α, V_β and V_γ of dimensions respectively a, b, c with $a \leq b \leq c$. Then, the number of eigenvectors is

$$\#V_\alpha + \#V_\beta + \#V_\gamma - 3 = q^a + q^b + q^{2n-a-b} - 3.$$

Clearly $a, b > 1$ and $a + b < 2n$. In particular, this number is maximum for $a = b = 1$, in which case we get

$$\#V_\alpha + \#V_\beta + \#V_\gamma - 3 = 2q + q^{2n-2} - 3 < q^2 + q^{2n-2} - 2.$$

Thus, the maximum number of eigenvectors which can be obtained with just 2 eigenspaces, say V_α and V_β is larger than that possible with 3 distinct eigenspaces.

Observe that since S is antisymmetric, its rank is necessarily even; in particular, the rank of $M^{-1}S$ is also even. Suppose now $\lambda \neq 0$ to be an eigenvalue of $M^{-1}S$ and suppose than the corresponding eigenspace is maximum and has dimension g. The number of simultaneously σ– and θ–isotropic lines $\ell = \langle v_1, v_2 \rangle$ is the same as the number of lines which are simultaneously σ– and $(\theta - \lambda\sigma)$–isotropic, as $\sigma(v_1, v_2) = 0$ and $\theta(v_1, v_2) = 0$ yields $(\theta - \lambda\sigma)(v_1, v_2) = \theta(v_1, v_2) -$

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$\lambda\sigma(v_1, v_2) = 0$. The latter alternating form, say $\theta' = \theta - \lambda\sigma$ is represented the matrix $S' = S - \lambda M$. In particular, we can replace S with S' and we get

$$M^{-1}S' = M^{-1}(S - \lambda M) = M^{-1}S - \lambda I.$$

For this new matrix, 0 is an eigenvalue with eigenspace of dimension g. Thus, g must be even.

We conclude that the maximum number of eigenvectors might occur when $g = 2n - 2$ and there is a further eigenspace of dimension 2, that is

$$\#V_\lambda + \#V_\mu - 2 = q^{2n-2} + q^2 - 2.$$

$\square$

The case of the previous lemma corresponds to

$$N_1 = \frac{q^{2n-2} - 1}{q - 1} + \frac{q^2 - 1}{q - 1}.$$

Plugging in this value in (2) we obtain

$$\eta = \frac{q^{4n-3} + q^{4n-4} - q^{4n-5} - q^{2n-1} - 2q^{2n-2} + q^{2n-3} + 1}{(q - 1)(q^2 - 1)}.$$

whence we get the following lemma.

Lemma 3.3.
$$d_{\min}(\mathcal{W}(n, 2)) \geq q^{4n-5} - q^{2n-3}.$$

We are now ready to prove our main theorem for Line Symplectic Grassmann codes.

Theorem 3.4. *The minimum distance of the code $\mathcal{W}(n, 2)$ is $d_{\min}(\mathcal{W}(n, 2)) = q^{4n-5} - q^{2n-3}$.*

Proof. We shall show that, given a non-degenerate alternating form σ represented by a matrix M, it is always possible to define an alternating form θ represented by a matrix S such that $M^{-1}S$ has only two eigenspaces, one of dimension $2n - 2$ and the other of dimension 2.

In order to prove this, let $\ell = \langle v_1, v_2 \rangle$ be a line of PG$(2n - 1, q)$ which is not σ–isotropic and define an alternating form θ such that $\theta(v_1, v_2) = \sigma(v_1, v_2)$ and Rad $\theta = \ell^{\perp_\sigma}$.

Take $B = B_1 \cup B_2$ to be an ordered basis of V where $B_1 = (v_1, v_2)$ and B_2 is an ordered basis of $\ell^{\perp_\sigma}$.

Let M be the matrix representing σ with respect to B. We can suppose $M = \text{diag}(M_{11}, M_{22})$ to be a block diagonal matrix where

$$M_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } M_{22} = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}$$

with O_n the null $(n \times n)$–matrix and I_n the $(n \times n)$–identity matrix. By construction, the matrix S representing θ with respect to B is also block diagonal $S = \text{diag}(S_{11}, O_{2n-2})$ with $S_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and O_{2n-2} the null $(2n - 2) \times (2n - 2)$–matrix.

Hence $M^{-1}S$ is the block diagonal matrix $M^{-1}S = \text{diag}(I_2, O_{2n-2})$. Clearly, $M^{-1}S$ has only two eigenspaces, one of dimension $2n - 2$ and the other of dimension 2. The thesis now follows from Lemma 3.2 and Lemma 3.3.

$\square$

The proof of Theorem 3.4 holds also for the code $\mathcal{W}(2,2)$ arising from the dual polar space $\Lambda_{2,2}$. However, in this case we can easily provide the full weight enumerator.

Proposition 3.5. *The code $\mathcal{W}(2,2)$ has exactly 3 nonzero weights, namely $q^3 + q$, $q^3 - q$ and q^3 and the following weight enumerator*

Weight	# Codewords
$q^3 - q$	$q^2(q^2+1)(q-1)/2$
q^3	$q^4 - 1$
$q^3 + q$	$q^2(q^2-1)(q-1)/2$

Proof. The Lagrangian-Grassmmannian $\mathbb{L}(1,1)$ is a non–singular hyperplane section of the ordinary line–Grassmannian $\mathbb{G}(3,1)$ of $\mathrm{PG}(3,q)$. In particular $\mathbb{L}(1,1) = \mathbb{G}(3,1) \cap \Sigma = Q(4,q)$, where Σ is a suitable hyperplane of $\mathrm{PG}(4,q)$, depending only on σ. Thus, the code $\mathcal{W}(2,2)$ is the same as the code determined by the projective system of $Q(4,q)$ in a $\mathrm{PG}(4,q)$. Let μ be the orthogonal polarity induced on Σ by $Q(4,q)$. The three weights of the code correspond to 3–spaces which are polar (with respect to μ) of points either on $Q(4,q)$ or internal or external to it. In particular, there are $q^3 + q^2 + q + 1$ points on $Q(4,q)$ and their polar hyperplane meets $Q(4,q)$ in a cone consisting of $q^2 + q + 1$ points; the number of internal points is $q^2(q^2-1)/2$ and that of external points is $q^2(q^2+1)/2$. Observe that each hyperplane corresponds to $(q-1)$ words; this provides the complete enumerator. $\square$

4. Lagrangian-Grassmannian codes of rank 3

In this section we shall provide the full weight enumerator for the Lagrangian–Grassmannian code $\mathcal{W}(3,3)$, and discuss some codes arising from different embeddings of the Symplectic Grassmannian $\Lambda_{3,3}$.

Theorem 4.1. *For $k = n = 3$, the minimum distance of the code $\mathcal{W}(3,3)$ is $q^6 - q^4$. The enumerator is as follows*

Weight	# Codewords
$q^6 - q^4$	$\frac{1}{2}q^2(q^2+1)(q^2+q+1)(q^3+1)(q-1)$
q^6	$(q+1)^2(q^2-q+1)(q^2+1)(q^6-q^3+1)(q-1)$
$q^6 + q^3$	$q^9(q^4-1)(q-1)$
$q^6 + q^4$	$\frac{1}{2}q^2(q+1)(q^6-1)(q-1)$

Furthermore, all codewords of minimum weight lie in the same orbit.

Proof. The theorem is a direct consequence of the classification of the classical (geometric) hyperplanes of the dual polar space $\Lambda_{3,3}$ of rank 3 arising from the Grassmann embedding, as provided in [10, 11]. We refer, in particular, to [10, Tables 1,2,3] for the exact numbers of hyperplanes and the cardinalities of their intersection. $\square$

Remark 4.2. We point out that for $q = 2^h$ even, the Lagrangian-Grassmannian $\Lambda_{n,n}$ always affords the spin embedding in $\mathrm{PG}(2^n - 1, q)$; see [2]. In particular, $\Lambda_{3,3}$ can also be embedded in $\mathrm{PG}(7,q)$. Such an embedding gives rise to a projective system with parameters $N = q^6 + q^5 + q^4 + 2q^3 + q^2 + q + 1$ and $K = 8$. The corresponding code has just two weights, see [11], namely

Weight	# Codewords
q^6	$(q^2-1)(q^2+1)(q^3+1)$
$q^6 + q^3$	$(q^7-q^3)(q-1).$

We observe that for $q = 2$, this determines a $[135, 8, 64]$ code, and the best known code with length $N = 135$ and dimension $K = 8$ has minimum distance $d = 65$ (see [14]).

Likewise, $\Lambda_{4,4}$ can be embedded in $\mathrm{PG}(15, q)$ and there it also determines a 2–weight code of parameters $[N, K] = [(q^4 + 1)(q^3 + 1)(q^2 + 1)(q + 1), 16]$ and weights q^{10} and $q^{10} + q^7$; see [3].

Remark 4.3. For $q = 2$, the universal embedding of $\Lambda_{3,3}$ is different from the Grassmann embedding and it spans a $\mathrm{PG}(14, 2)$; see [1, 16]. As such it determines a code of length $N = 135$, dimension $K = 15$ with weight enumerator as follows, see [11, 18]:

Weight	# Codewords	Weight	# Codewords
30	36	70	10368
48	630	72	7680
54	1120	78	1080
62	3780	80	378
64	7695		

In particular, the minimum distance in this case is 30.

5. Further bounds on the minimum distance

As $\mathbb{L}(n - 1, k - 1)$ is a section of $\mathbb{G}(2n - 1, k - 1)$ with a subspace of codimension $\binom{2n}{k-2}$, it is possible to provide a bound on the minimum distance of $\mathcal{W}(n, k)$ in terms of higher weights of the projective Grassmann code induced by the projective system $\mathbb{G}(n - 1, k - 1)$. Recall that the r–th higher weight of a code $\mathcal{C}$ induced by a projective system Ω consisting of N points is

$$d_r := N - \max\{\#(\Omega \cap \Pi) : \Pi \text{ projective subspace of codimension } r \text{ in } \langle \Omega \rangle\};$$

see [21] for the definition and some properties. Higher weights for Grassmann codes have been extensively studied; see [17, 12, 13]. In general, we have for the minimum distance $d_{\min}(\mathcal{W}(n, k))$ of $\mathcal{W}(n, k)$

$$d_{\min}(\mathcal{W}(n, k)) \geq d_s,$$

where d_s is the $\binom{2n}{k-2} + 1$ higher weight of the Grassmann code arising from $\mathcal{G}_{2n,k}$. However, this bound is not sharp, in general. This can be seen directly by considering the case of the code $\mathcal{W}(n, 2)$ investigated before. Indeed, using the the second highest weight of the Grassmann code, we see that

$$d_{\min}(\mathcal{W}(n, 2)) \geq \begin{bmatrix} 2n \\ 2 \end{bmatrix}_q - q^{2(2n-2)-1}(q + 1) = \frac{2q^{4n-3} - q^{4n-5} - q^{2n} - q^{2n-1} + 1}{(q^2 - 1)(q - 1)} \approx q^{4n-6}.$$

This, however, is quite far away from the correct value for line symplectic Grassmann codes, namely $d_{\min}(\mathcal{W}(n, 2)) = q^{4n-5} - q^{2n-3}$, as we have determined in Section 3.

We point out that in [7, Proposition 5], an upper bound on the minimum distance for Lagrangian-Grassmannian codes is given in terms of the dimension of the Lagrangian-Grassmannian variety, that is

$$d_{\min}(\mathcal{W}(n, n)) \leq q^{n(n+1)/2}.$$

By Section 4, we see that this bound is not sharp for $n = 2$ and $n = 3$.

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